On the use of evolutionary methods in metric theories of gravity VI.

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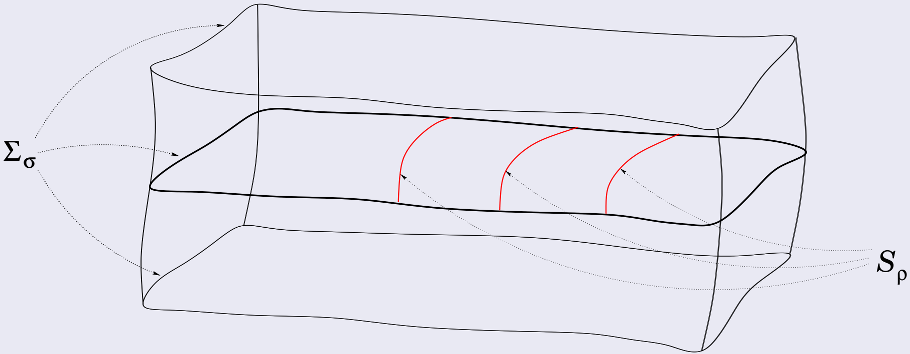
The constraint surface:

- Last time our **considerations were restricted to a specific** $\sigma = const$ **hypersurface**



- metric of Euclidean signature will be involved
- no gauge condition

... arbitrary choice of foliations & "time evolution" vector field

The program for the previous and for the present lecture is:

Our slogan to remember:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

Plans (were and) are:

- **Constraints as evolutionary systems**
  - open any textbook on GR: "the constrains are elliptic PDEs"
  - parabolic-hyperbolic system
    ... global solution to the involved parabolic equation
  - strongly hyperbolic system
    ... study of near Kerr configurations

References:

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- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, CQG **33** 015006 (2015)
- I. Rácz and J. Winicour: *Black hole initial data without elliptic equations*, Phys. Rev. D **91**, 124013 (2015)
- I. Rácz and J. Winicour: *Toward computing gravitational initial data without elliptic solvers*, CQG **35** 135002 (2018)
- I. Rácz: *On the evolutionary form of the constraints in electrodynamics*, arXiv:1811.06873 [gr-qc] (2018)
- A. Nakonieczna, L. Nakonieczny, I. Rácz: *Black hole initial data by numerical integration of the parabolic-hyperbolic form of the constraints*, arXiv:1712.00607

The spaces:

- **The primary space:** (M, g_{ab})
  - M : $n+1$-dimensional ($n \geq 3$), smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,\dots,+)}$ or Riemannian$_{(+,\dots,+)}$ metric
- **Einstein's equations:** restricting the geometry

$$G_{ab} - \mathscr{G}_{ab} = 0$$

with source term $\mathscr{G}_{ab}$ having a vanishing divergence $\nabla^a \mathscr{G}_{ab} = 0$

The explicit form of the constraints:

- The projections of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$ determine the constraint expressions:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$
$$E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\,[D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a] = 0$$

- where D_a denotes the covariant derivative operator associated with h_{ab} and

$$\mathfrak{e} = n^e n^f \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f \mathscr{G}_{ef}$$

- it is an underdetermined system: $n+1$ equations for the $n\,(n+1)$ variables

$$(h_{ij}, K_{ij})$$

New variables by applying $(n-1)+1$ decompositions:

Splitting of the metric h_{ij}:

assume: $\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$

Σ is smoothly foliated by a one-parameter family of codimension-one surfaces $\mathscr{S}_\rho$: $\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\Longrightarrow \quad \boxed{\widehat{n}_i = \widehat{N}\,\partial_i\rho \ \ldots\ \& \ldots\ h^{ij} \ \longrightarrow\ \widehat{n}^i = h^{ij}\widehat{n}_j \ \longrightarrow\ \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j}$$

- choose ρ^i to be a vector field on Σ: the integral curves... & $\boxed{\rho^i\partial_i\rho = 1}$
- 'lapse' and 'shift' of ρ^i

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i\,, \quad \text{where} \quad \widehat{N} = \rho^j\widehat{n}_j \quad \text{and} \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j}$$

- induced metric, extrinsic curvature and acceleration of the $\mathscr{S}_\rho$ level surfaces:

$$\boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i\,\widehat{\gamma}^l{}_j\,h_{kl}} \qquad \boxed{\widehat{K}_{ij} = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}}\widehat{\gamma}_{ij}} \qquad \boxed{\dot{\widehat{n}}_i := \widehat{n}^e\nabla_e\widehat{n}_i = -\widehat{D}_i \ln \widehat{N}}$$

- the metric h_{ij} can then be given as

$$\boxed{h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i\widehat{n}_j} \quad \Longleftrightarrow \quad \boxed{\{\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}\}}$$

The new variables:

The momentum constraint:

$$E_a^{(\mathcal{M})} = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\, [D_e K^e{}_a - D_a K^e{}_e - \epsilon\, \mathfrak{p}_a] = 0$$

The splitting of the extrinsic curvature K_{ij}:

- $$K_{ij} = \boldsymbol{\kappa}\, \widehat{n}_i \widehat{n}_j + [\widehat{n}_i\, \mathbf{k}_j + \widehat{n}_j\, \mathbf{k}_i] + \mathbf{K}_{ij}$$

  where

  $$\boldsymbol{\kappa} = \widehat{n}^k \widehat{n}^l\, K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i \widehat{n}^l\, K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j\, K_{kl}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

  $$\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\, \mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{n-1}\, \widehat{\gamma}_{ij} \mathbf{K}^l{}_l$$

- the independent components of (h_{ij}, K_{ij}) are represented by the variables

  $$(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

The momentum constraint:

First order symmetric hyperbolic system:

$$\mathscr{L}_{\hat{n}}\mathbf{k}_i - \tfrac{n-2}{n-1}\,\widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i\boldsymbol{\kappa} + \widehat{D}^l\mathring{\mathbf{K}}_{li} + (\widehat{K}^l{}_l)\,\mathbf{k}_i + \boldsymbol{\kappa}\,\dot{\hat{n}}_i - \dot{\hat{n}}^l\,\mathbf{K}_{li} - \epsilon\,\mathfrak{p}_l\,\widehat{\gamma}^l{}_i = 0 \quad (1)$$

back: str.hyp.sys.

$$\mathscr{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \boldsymbol{\kappa}\,(\widehat{K}^l{}_l) + \mathbf{K}_{kl}\widehat{K}^{kl} + 2\,\dot{\hat{n}}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\hat{n}^l = 0 \quad (2)$$

- notably, $\frac{n-1}{n-2}\,\widehat{N}\,\widehat{\gamma}^{ij}$ times of (1) and $\widehat{N}$ times of (2) when writing them out in (local) coordinates $(\rho, x^2, \ldots, x^n)$, adopted to the foliation $\mathscr{S}_\rho$ and the vector field ρ^i,

$$\left\{\begin{pmatrix} \frac{n-1}{n-2}\,\widehat{\gamma}^{AB} & 0 \\ 0 & 1 \end{pmatrix}\partial_\rho + \begin{pmatrix} -\frac{n-1}{n-2}\,\widehat{N}^K\,\widehat{\gamma}^{AB} & -\widehat{N}\,\widehat{\gamma}^{AK} \\ -\widehat{N}\,\widehat{\gamma}^{BK} & -\widehat{N}^K \end{pmatrix}\partial_K\right\}\begin{pmatrix} \mathbf{k}_B \\ \mathbf{K}^E{}_E \end{pmatrix} + \begin{pmatrix} \mathscr{B}_{(\mathbf{k})}^{\;A} \\ \mathscr{B}_{(\mathbf{K})} \end{pmatrix} = 0$$

- indep. of ϵ: a first order **symmetric hyperbolic** system for the vector valued variable

$$(\mathbf{k}_B, \mathbf{K}^E{}_E)^T$$

  where **the 'radial coordinate'** ρ **plays the role of 'time'**.

- ... with characteristic cone (apart from the surfaces $\mathscr{S}_\rho$ with $\hat{n}^i\xi_i = 0$)

$$\left[\,\widehat{\gamma}^{ij} - \left(\tfrac{n-1}{n-2}\right)\hat{n}^i\hat{n}^j\,\right]\xi_i\xi_j = \left[\,h^{ij} - \left(1 + \tfrac{n-1}{n-2}\right)\hat{n}^i\hat{n}^j\,\right]\xi_i\xi_j = 0$$

The Hamiltonian constraint:

The Hamiltonian constraint in new dress:

- $$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$
- using $$^{(n)}\!R = \widehat{R} - \Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\Big\}$$

  $\widehat{R}$ denotes the scalar curvature of $\widehat{\gamma}_{ij}$
- $$-\epsilon\,\widehat{R} + \epsilon\,\Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\Big\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0$$

Alternative choices yielding evolutionary systems:

- it is a **parabolic equation** for $\widehat{N}$ (the sign of $\widehat{K}^l{}_l$ plays a role)
- it is an **algebraic equation** for $\boldsymbol{\kappa}$ (what is if $\mathbf{K}^l{}_l$ vanishes somewhere?)

The hyperbolic-parabolic system:

The Hamiltonian constraint as a parabolic equation for $\widehat{N}$:

$$-\epsilon\,\widehat{R} + \epsilon\left\{2\,\boxed{\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l)} + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\boxed{\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}}\right\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0$$

- $\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\,\widehat{K}_{ij} = \widehat{N}^{-1}[\,\tfrac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j\,] = \widehat{N}^{-1}\overset{\star}{K}$ as $\widehat{n}^i = \widehat{N}^{-1}[\,\rho^i - \widehat{N}^i\,]$
- $\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) = -\widehat{N}^{-3}\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - (\widehat{N}^l\widehat{D}_l\widehat{N})\,] + \widehat{N}^{-2}[\,(\partial_\rho\overset{\star}{K}) - (\widehat{N}^l\widehat{D}_l\overset{\star}{K})\,]$
- using
$$\mathcal{A} = 2\,[\,(\partial_\rho\overset{\star}{K}) - \widehat{N}^l(\widehat{D}_l\overset{\star}{K})\,] + \overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl}$$
$$\mathcal{B} = -\,\widehat{R} + \epsilon\,[\,2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e}\,]$$
- it gets to be a **Bernoulli-type parabolic partial differential equation** provided that $\overset{\star}{K}$ ...
- $2\,\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - \widehat{N}^l(\widehat{D}_l\widehat{N})\,] = 2\,\widehat{N}^2(\widehat{D}^l\widehat{D}_l\widehat{N}) + \mathcal{A}\,\widehat{N} + \mathcal{B}\,\widehat{N}^3$ & Mom. constr.
- in highly specialized cases of "quasi-spherical" foliations with $\widehat{\gamma}_{ij} = r^2\,\mathring{\gamma}_{ij}$ and assuming time symmetry, i.e. $K_{ij} \equiv 0$ R. Bartnik (1993), G. Weinstein & B. Smith (2004)

The parabolic-hyperbolic form of the constraints:

An evolutionary system for the constrained fields $\widehat{N}, \mathbf{k}_i$ and $\mathbf{K}^l{}_l$:

$$2\,\overset{\star}{K}\,[\,(\partial_\rho \widehat{N}) - \widehat{N}^l(\widehat{D}_l\widehat{N})\,] - 2\,\widehat{N}^2(\widehat{D}^l\widehat{D}_l\widehat{N}) - \mathcal{A}\,\widehat{N} - \mathcal{B}\,\widehat{N}^3 = 0$$

$$\mathscr{L}_{\widehat{n}}\mathbf{k}_i - \tfrac{n-2}{n-1}\,\widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i\boldsymbol{\kappa} + \widehat{D}^l\overset{\circ}{\mathbf{K}}_{li} + \widehat{N}\overset{\star}{K}\,\mathbf{k}_i + [\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,]\,\dot{\widehat{n}}_i - \dot{\widehat{n}}^l\,\overset{\circ}{\mathbf{K}}_{li} - \epsilon\,\mathfrak{p}_l\,\widehat{\gamma}^l{}_i = 0$$

$$\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \widehat{N}\overset{\star}{K}\,[\,\boldsymbol{\kappa} - \tfrac{1}{n-1}\,(\mathbf{K}^l{}_l)\,] + \widehat{N}\,\overset{\circ}{\mathbf{K}}_{kl}\overset{\star}{K}{}^{kl} + 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\widehat{n}_i = 0\,,$$

where $\widehat{D}_i$ denotes the covariant derivative operator associated with $\widehat{\gamma}_{ij}$

$$\overset{\star}{K} = \tfrac{1}{2}\,\widehat{\gamma}^{ij}\,\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j$$

$$\overset{\star}{K}_{ij} = \tfrac{1}{2}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_{(i}\widehat{N}_{j)}, \qquad \dot{\widehat{n}}_k = \widehat{n}^l D_l\widehat{n}_k = -\widehat{D}_k(\ln\widehat{N})$$

$$\mathcal{A} = 2\,[\,(\partial_\rho\overset{\star}{K}) - \widehat{N}^l(\widehat{D}_l\overset{\star}{K})\,] + \overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl}$$

$$\mathcal{B} = -\widehat{R} + \epsilon\,[\,2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \overset{\circ}{\mathbf{K}}_{kl}\,\overset{\circ}{\mathbf{K}}{}^{kl} - 2\,\mathfrak{e}\,]$$

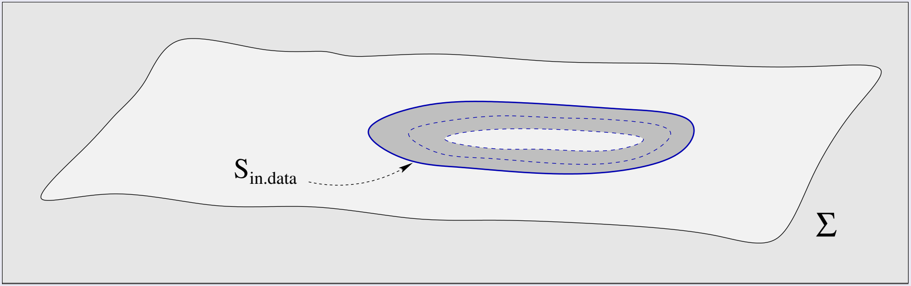
Constraints as evolutionary systems I.

The parabolic-hyperbolic system:

- (h_{ij}, K_{ij}) represented by the variables $(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$
- the constraints comprise a **parabolic-hyperbolic** system for $(\widehat{N}, \mathbf{k}_i, \mathbf{K}^l{}_l)$
  - with freely specifiable variables on Σ and on $\mathrm{S}_{\mathrm{in.data}}$:

$$(\widehat{N}|_{\mathrm{S}_{\mathrm{in.data}}}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i|_{\mathrm{S}_{\mathrm{in.data}}}, \mathbf{K}^l{}_l|_{\mathrm{S}_{\mathrm{in.data}}}, \mathring{\mathbf{K}}_{ij})$$

  - a fixed $(+/-)$ sign of $\overset{\star}{K} = \frac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j$ can be guaranteed



Theorem

Suppose that a choice had been made for the freely specifiable variables

$$\boxed{\widehat{N}^i, \widehat{\gamma}_{ij}\,; \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{ij}\,; \mathfrak{e}, \mathfrak{p}_i}$$

such that they are smooth and $\overset{\star}{K} > 0$ *throughout* Σ*. Assume that smooth initial data* $({}_0\widehat{N}, {}_0\mathbf{k}_i, {}_0\mathbf{K}^l{}_l)$*, with* ${}_0\widehat{N} > 0$*, had also been chosen to our hyperbolic-parabolic system on one of the level surfaces* $\mathscr{S}_{\rho_0}$ *in* Σ*. Then, for some* $\varepsilon > 0$*, on a short interval* $[\rho_0, \rho_0 + \varepsilon)$*, there exists a unique smooth solution*

$$\boxed{(\widehat{N}, \mathbf{k}_i, \mathbf{K}^l{}_l)\,:\ \widehat{N} > 0, \ \text{such that} \ \ \widehat{N}|_{\mathscr{S}_{\rho_0}} = {}_0\widehat{N}, \mathbf{k}_i|_{\mathscr{S}_{\rho_0}} = {}_0\mathbf{k}_i \quad \& \quad \mathbf{K}^l{}_l|_{\mathscr{S}_{\rho_0}} = {}_0\mathbf{K}^l{}_l}$$

The fields h_{ij} *and* K_{ij} *which can be built up from this solution and from the freely specifiable part of the data satisfy their respective* $n+1$ *constraints in the corresponding one-sided neighborhood* $\mathscr{S}_{[\rho_0, \rho_0+\varepsilon)}$ *of* $\mathscr{S}_{\rho_0}$ *in* Σ*.*

Global in the time solutions:

as for certain choice of coefficients in parabolic equations or for that of the initial data classical solutions are known to blow up in finite "time" it is of obvious interest to identify those conditions which guarantee the existence of global in the time, $[\rho_0, \infty)$, solutions (these should be bounded away from zero and from infinity) for the parabolic form of the Hamiltonian constraint

The behavior of “upper” and “lower” solutions:

- assume that the $\mathscr{S}_\rho$ level surfaces are compact (or asymptotic decay conds.)
- consider a function f on $\Sigma = \mathbb{R} \times \mathscr{S}_\rho$
- define $f^\sharp$ and $f_\flat$ as

$$f^\sharp(\rho) = \sup_{\mathscr{S}_\rho} \{ f(\rho, x^2, \dots, x^n) \}, \quad f_\flat(\rho) = \inf_{\mathscr{S}_\rho} \{ f(\rho, x^2, \dots, x^n) \}$$

- substituting the auxiliary function $w = \widehat{N}^{-2} \, (\geq 0)$

$$\partial_\rho w - \widehat{N}^l(\widehat{D}_l w) + \tfrac{3}{2} \overset{\star}{K}{}^{-1} w^{-2} (\widehat{D}^l w)(\widehat{D}_l w) = (\overset{\star}{K} w)^{-1} (\widehat{D}^l \widehat{D}_l w) - \mathfrak{a}\, w - \mathfrak{b}$$

where $\mathfrak{a} = \mathcal{A}/\overset{\star}{K}$, $\mathfrak{b} = \mathcal{B}/\overset{\star}{K}$ and, for simplicity, assume that both $\mathfrak{a}, \mathfrak{b} \geq 0$

- if $\widehat{N}$ has its *maximum* w has its *minimum* and, vice versa, if $\widehat{N}$ has its *minimum* w has its *maximum*
- as $\overset{\star}{K} w$ is non-negative and $\widehat{\gamma}_{ij}$ is positive definite, the first term on the rhs is non-positive or non-negative at points where w attains its maximum or minimum, respectively.
- $\Longrightarrow$

$$\partial_\rho w_\flat \geq -\mathfrak{a}^\sharp\, w_\flat - \mathfrak{b}^\sharp, \qquad \partial_\rho w^\sharp \leq -\mathfrak{a}_\flat\, w^\sharp - \mathfrak{b}_\flat$$

The behavior of “upper” and “lower” solutions:

$$w_\flat \geq W_\flat = \exp\left[-\int_{\rho_0}^{\rho} \mathfrak{a}^\sharp \mathrm{d}\rho'\right]\left\{w_\flat|_{\mathscr{S}_{\rho_0}} - \int_{\rho_0}^{\rho} \mathfrak{b}^\sharp \exp\left[\int_{\rho_0}^{\rho'} \mathfrak{a}^\sharp \mathrm{d}\rho''\right]\mathrm{d}\rho'\right\}$$

$$w^\sharp \leq W^\sharp = \exp\left[-\int_{\rho_0}^{\rho} \mathfrak{a}_\flat \mathrm{d}\rho'\right]\left\{w^\sharp|_{\mathscr{S}_{\rho_0}} - \int_{\rho_0}^{\rho} \mathfrak{b}_\flat \exp\left[\int_{\rho_0}^{\rho'} \mathfrak{a}_\flat \mathrm{d}\rho''\right]\mathrm{d}\rho'\right\}$$

- **if** $$\mathcal{K} = \sup_{\rho\in[\rho_0,\infty)}\left\{\int_{\rho_0}^{\rho} \mathfrak{b}^\sharp \exp\left[\int_{\rho_0}^{\rho'} \mathfrak{a}^\sharp \mathrm{d}\rho''\right]\mathrm{d}\rho'\right\}$$

  is positive and finite, i.e. $0 < \mathcal{K} < \infty$, **and if** for the initial data the inequality $w_\flat|_{\mathscr{S}_{\rho_0}} > \mathcal{K}$ holds $\Longrightarrow$ $W^\sharp, W_\flat > 0$, and w is bounded from below by $W_\flat > 0$
- this is equivalent to saying that $\widehat{N}$ **has to be bounded form above** as $\widehat{N} = w^{-\frac{1}{2}} \leq W_\flat{}^{-\frac{1}{2}} < \infty$, if for the initial data ${}_0\widehat{N}$ the inequality ${}_0\widehat{N} < 1/\sqrt{\mathcal{K}}$ holds [equiv. to ${}_0\widehat{N}^{-2} = w|_{\mathscr{S}_{\rho_0}} \geq w_\flat|_{\mathscr{S}_{\rho_0}} > \mathcal{K}$]
- if $\mathfrak{b} = \mathcal{B}/\overset{\star}{K}$ is negative then “$\mathcal{K}$”< 0 which guarantee that $w_\flat|_{\mathscr{S}_{\rho_0}} \geq 0 > \mathcal{K}$ holds **for any choice of a positive initial data** ${}_0\widehat{N} > 0 \Longrightarrow \widehat{N} < \infty$
- $\widehat{N}$ is **bounded from below**: [w is bounded from above] $w \leq w^\sharp \leq W^\sharp \Longrightarrow$ $\widehat{N}$ is positive as $0 < (W^\sharp)^{-\frac{1}{2}} \leq w^{-\frac{1}{2}} = \widehat{N}$

Theorem

Suppose that all the coefficients in the Bernoulli type parabolic equation

$$\boxed{2\,\overset{\star}{K}\,[\,(\partial_\rho \widehat N) - \widehat N^l(\widehat D_l \widehat N)\,] - 2\,\widehat N^2(\widehat D^l \widehat D_l \widehat N) - \mathcal{A}\,\widehat N - \mathcal{B}\,\widehat N^3 = 0} \qquad (*)$$

are smooth and that the freely specifiable part of the data was chosen such that $\overset{\star}{K}$ *is positive throughout* Σ. *Choose* ${}_0\widehat N$ *to be a smooth positive function on the compact level surface* $\rho = \rho_0$ *such that* ${}_0\widehat N < 1/\sqrt{\mathcal{K}}$ *if* $0 < \mathcal{K} < \infty$, *or to be arbitrary if* $\mathcal{K} \leq 0$, *where*

$$\boxed{\mathcal{K} = \sup_{\rho\in[\rho_0,\infty)} \left\{ \int_{\rho_0}^{\rho} \mathfrak{b}^{\,\sharp} \exp\left[\int_{\rho_0}^{\rho'} \mathfrak{a}^{\,\sharp} \mathrm{d}\rho'' \right] \mathrm{d}\rho' \right\}\,,}$$

and $\mathfrak{a} = \mathcal{A}/\overset{\star}{K}$ *and* $\mathfrak{b} = \mathcal{B}/\overset{\star}{K}$.

Then $(*)$ *has a* **unique smooth global** *in the time,* $[\rho_0, \infty)$, **classical solution** *such that* $0 < \widehat N < \infty$, *and that* $\widehat N|_{\mathscr{S}_{\rho_0}} = {}_0\widehat N$.

That is all for now...