On the use of evolutionary methods in metric theories of gravity VIb.

István Rácz

istvan.racz@fuw.edu.pl & racz.istvan@wigner.mta.hu

Faculty of Physics, University of Warsaw, Warsaw, Poland

Wigner Research Center for Physics, Budapest, Hungary

Supported by the POLONEZ programme of the National Science Centre of Poland which has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 665778.

National Science Centre Poland

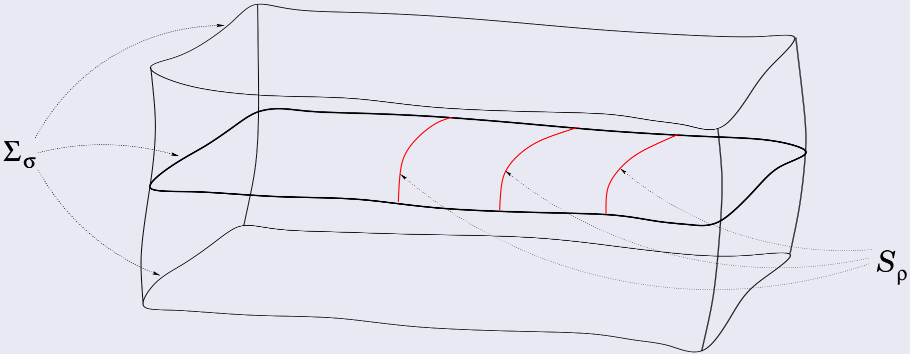
European Commission | Horizon 2020 European Union funding for Research & Innovation

Institute of Theoretical Physics, University of Warsaw

Warsaw, 6 December 2018

The constraint surface:

- Last time our **considerations were restricted to a specific** $\sigma = const$ **hypersurface**



- metric of Euclidean signature will be involved
- no gauge condition

  ... arbitrary choice of foliations & "time evolution" vector field

The program for the previous and for the present lecture is:

Our slogan to remember:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

Plans (were and) are:

- **Constraints as evolutionary systems**
  - open any textbook on GR: "the constrains are elliptic PDEs"
  - parabolic-hyperbolic system

    ... global solution to the involved parabolic equation
  - strongly hyperbolic system

    ... study of near Kerr configurations

References:

- I. Rácz: *Constraints as evolutionary systems*, CQG **33** 015014 (2016)
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, CQG **33** 015006 (2015)
- I. Rácz and J. Winicour: *Black hole initial data without elliptic equations*, Phys. Rev. D **91**, 124013 (2015)
- I. Rácz and J. Winicour: *Toward computing gravitational initial data without elliptic solvers*, CQG **35** 135002 (2018)
- I. Rácz: *On the evolutionary form of the constraints in electrodynamics*, arXiv:1811.06873 [gr-qc] (2018)
- A. Nakonieczna, L. Nakonieczny, I. Rácz: *Black hole initial data by numerical integration of the parabolic-hyperbolic form of the constraints*, arXiv:1712.00607

The spaces:

- **The primary space:** (M, g_{ab})
  - M : $n+1$-dimensional ($n \geq 3$), smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,\ldots,+)}$ or Riemannian$_{(+,\ldots,+)}$ metric
- **Einstein's equations:** restricting the geometry

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0}$$

with source term $\mathscr{G}_{ab}$ having a vanishing divergence $\nabla^a \mathscr{G}_{ab} = 0$

The explicit form of the constraints:

- The projections of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$ determine the constraint expressions:

$$\boxed{\begin{aligned} E^{(\mathcal{H})} &= n^e n^f E_{ef} = \tfrac{1}{2}\,\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0 \\ E^{(\mathcal{M})}_a &= \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\,[D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a] = 0 \end{aligned}}$$

- where D_a denotes the covariant derivative operator associated with h_{ab} and

$$\boxed{\mathfrak{e} = n^e n^f \mathscr{G}_{ef}\,, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f \mathscr{G}_{ef}}$$

- it is an underdetermined system: $n+1$ equations for the $n\,(n+1)$ variables

$$\boxed{(h_{ij}, K_{ij})}$$

New variables by applying $(n-1)+1$ decompositions:

Splitting of the metric h_{ij}:

assume: $\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$

Σ is smoothly foliated by a one-parameter family of codimension-one surfaces $\mathscr{S}_\rho$: $\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\Longrightarrow \quad \boxed{\widehat{n}_i = \widehat{N}\, \partial_i \rho \ \ldots \ \& \ldots \ h^{ij} \ \longrightarrow \ \widehat{n}^i = h^{ij}\widehat{n}_j \ \longrightarrow \ \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i \widehat{n}_j}$$

- choose ρ^i to be a vector field on Σ : the integral curves... & $\boxed{\rho^i \partial_i \rho = 1}$
- 'lapse' and 'shift' of ρ^i

$$\boxed{\rho^i = \widehat{N}\, \widehat{n}^i + \widehat{N}^i\,, \quad \text{where} \quad \widehat{N} = \rho^j \widehat{n}_j \quad \text{and} \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\, \rho^j}$$

- induced metric, extrinsic curvature and acceleration of the $\mathscr{S}_\rho$ level surfaces:

$$\boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i\, \widehat{\gamma}^l{}_j\, h_{kl}} \qquad \boxed{\widehat{K}_{ij} = \tfrac{1}{2}\, \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij}} \qquad \boxed{\dot{\widehat{n}}_i := \widehat{n}^e \nabla_e \widehat{n}_i = -\widehat{D}_i \ln \widehat{N}}$$

- the metric h_{ij} can then be given as

$$\boxed{h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j} \quad \Longleftrightarrow \quad \boxed{\{\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}\}}$$

The new variables:

The momentum constraint:

$$\boxed{E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\, [D_e K^e{}_a - D_a K^e{}_e - \epsilon\, \mathfrak{p}_a] = 0}$$

The splitting of the extrinsic curvature K_{ij}:

- $$\boxed{K_{ij} = \boldsymbol{\kappa}\, \widehat{n}_i \widehat{n}_j + [\widehat{n}_i\, \mathbf{k}_j + \widehat{n}_j\, \mathbf{k}_i] + \mathbf{K}_{ij}}$$

  where

  $$\boxed{\boldsymbol{\kappa} = \widehat{n}^k \widehat{n}^l\, K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i \widehat{n}^l\, K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j\, K_{kl}}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

  $$\boxed{\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\, \mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{n-1}\, \widehat{\gamma}_{ij} \mathbf{K}^l{}_l}$$

- the independent components of $\boxed{(h_{ij}, K_{ij})}$ are represented by the variables

  $$\boxed{(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})}$$

The momentum constraint:

First order symmetric hyperbolic system:

$$\mathscr{L}_{\hat{n}}\mathbf{k}_i - \tfrac{n-2}{n-1}\,\widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i\boldsymbol{\kappa} + \widehat{D}^l\mathring{\mathbf{K}}_{li} + (\widehat{K}^l{}_l)\,\mathbf{k}_i + \boldsymbol{\kappa}\,\dot{\hat{n}}_i - \dot{\hat{n}}^l\,\mathbf{K}_{li} - \epsilon\,\mathfrak{p}_l\,\widehat{\gamma}^l{}_i = 0 \quad (1)$$

back: str.hyp.sys.

$$\mathscr{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \boldsymbol{\kappa}\,(\widehat{K}^l{}_l) + \mathbf{K}_{kl}\widehat{K}^{kl} + 2\,\dot{\hat{n}}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\hat{n}^l = 0 \quad (2)$$

- notably, $\frac{n-1}{n-2}\,\widehat{N}\,\widehat{\gamma}^{ij}$ times of (1) and $\widehat{N}$ times of (2) when writing them out in (local) coordinates $(\rho, x^2, \dots, x^n)$, adopted to the foliation $\mathscr{S}_\rho$ and the vector field ρ^i,

$$\left\{\begin{pmatrix} \frac{n-1}{n-2}\,\widehat{\gamma}^{AB} & 0 \\ 0 & 1 \end{pmatrix}\partial_\rho + \begin{pmatrix} -\frac{n-1}{n-2}\,\widehat{N}^K\,\widehat{\gamma}^{AB} & -\widehat{N}\,\widehat{\gamma}^{AK} \\ -\widehat{N}\,\widehat{\gamma}^{BK} & -\widehat{N}^K \end{pmatrix}\partial_K\right\}\begin{pmatrix} \mathbf{k}_B \\ \mathbf{K}^E{}_E \end{pmatrix} + \begin{pmatrix} \mathscr{B}_{(\mathbf{k})}{}^A \\ \mathscr{B}_{(\mathbf{K})} \end{pmatrix} = 0$$

- indep. of ϵ: a first order **symmetric hyperbolic** system for the vector valued variable

$$(\mathbf{k}_B, \mathbf{K}^E{}_E)^T$$

  where **the 'radial coordinate'** ρ **plays the role of 'time'**.

- ... with characteristic cone (apart from the surfaces $\mathscr{S}_\rho$ with $\hat{n}^i\xi_i = 0$)

$$\left[\widehat{\gamma}^{ij} - \left(\tfrac{n-1}{n-2}\right)\hat{n}^i\hat{n}^j\right]\xi_i\xi_j = \left[h^{ij} - \left(1 + \tfrac{n-1}{n-2}\right)\hat{n}^i\hat{n}^j\right]\xi_i\xi_j = 0$$

The Hamiltonian constraint:

The Hamiltonian constraint in new dress:

- $$\boxed{E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0}$$

- using $$\boxed{{}^{(n)}\!R = \widehat{R} - \Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\Big\}}$$

  $\widehat{R}$ denotes the scalar curvature of $\widehat{\gamma}_{ij}$

- $$\boxed{\begin{aligned}-\epsilon\,\widehat{R} + \epsilon\,\Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\Big\} \\ + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0\end{aligned}}$$

Alternative choices yielding evolutionary systems:

- it is a **parabolic equation** for $\boxed{\widehat{N}}$ (the sign of $\boxed{\widehat{K}^l{}_l}$ plays a role)
- it is an **algebraic equation** for $\boxed{\boldsymbol{\kappa}}$ (what is if $\boxed{\mathbf{K}^l{}_l}$ vanishes somewhere?)

The hyperbolic-parabolic system:

The Hamiltonian constraint as a parabolic equation for $\widehat{N}$:

$$\boxed{-\epsilon\,\widehat{R} + \epsilon\left\{2\,\boxed{\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l)} + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\boxed{\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}}\right\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0}$$

- $\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\,\widehat{K}_{ij} = \widehat{N}^{-1}[\,\tfrac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j\,] = \widehat{N}^{-1}\overset{\star}{K}}$ as $\boxed{\widehat{n}^i = \widehat{N}^{-1}[\,\rho^i - \widehat{N}^i\,]}$
- $\boxed{\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) = -\widehat{N}^{-3}\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - (\widehat{N}^l\widehat{D}_l\widehat{N})\,] + \widehat{N}^{-2}[\,(\partial_\rho\overset{\star}{K}) - (\widehat{N}^l\widehat{D}_l\overset{\star}{K})\,]}$
- using $$\boxed{\begin{aligned}\mathcal{A} &= 2\,[\,(\partial_\rho\overset{\star}{K}) - \widehat{N}^l(\widehat{D}_l\overset{\star}{K})\,] + \overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl}\\ \mathcal{B} &= -\widehat{R} + \epsilon\,[\,2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e}\,]\end{aligned}}$$
- it gets to be a **Bernoulli-type parabolic partial differential equation** provided that $\overset{\star}{K}$ ...
- $$\boxed{2\,\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - \widehat{N}^l(\widehat{D}_l\widehat{N})\,] = 2\,\widehat{N}^2(\widehat{D}^l\widehat{D}_l\widehat{N}) + \mathcal{A}\,\widehat{N} + \mathcal{B}\,\widehat{N}^3}$$ & Mom. constr.
- in highly specialized cases of "quasi-spherical" foliations with $\widehat{\gamma}_{ij} = r^2\,\mathring{\gamma}_{ij}$ and assuming time symmetry, i.e. $K_{ij} \equiv 0$ R. Bartnik (1993), G. Weinstein & B. Smith (2004)

The parabolic-hyperbolic form of the constraints:

An evolutionary system for the constrained fields $\widehat{N}, \mathbf{k}_i$ and $\mathbf{K}^l{}_l$:

$$
\begin{aligned}
&2\,\overset{\star}{K}\,[\,(\partial_\rho \widehat{N}) - \widehat{N}^l(\widehat{D}_l \widehat{N})\,] - 2\,\widehat{N}^2(\widehat{D}^l \widehat{D}_l \widehat{N}) - \mathcal{A}\,\widehat{N} - \mathcal{B}\,\widehat{N}^3 = 0 \\
&\mathscr{L}_{\widehat{n}}\mathbf{k}_i - \tfrac{n-2}{n-1}\,\widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i\boldsymbol{\kappa} + \widehat{D}^l\overset{\circ}{\mathbf{K}}_{li} + \widehat{N}\overset{\star}{K}\,\mathbf{k}_i + [\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,]\,\dot{\widehat{n}}_i - \dot{\widehat{n}}^l\,\overset{\circ}{\mathbf{K}}_{li} - \epsilon\,\mathfrak{p}_l\,\widehat{\gamma}^l{}_i = 0 \\
&\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \widehat{N}\overset{\star}{K}\,[\,\boldsymbol{\kappa} - \tfrac{1}{n-1}\,(\mathbf{K}^l{}_l)\,] + \widehat{N}\,\overset{\circ}{\mathbf{K}}_{kl}\overset{\star}{K}{}^{kl} + 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\widehat{n}_i = 0\,,
\end{aligned}
$$

where $\widehat{D}_i$ denotes the covariant derivative operator associated with $\widehat{\gamma}_{ij}$

$$\boxed{\overset{\star}{K} = \tfrac{1}{2}\,\widehat{\gamma}^{ij}\,\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j}$$

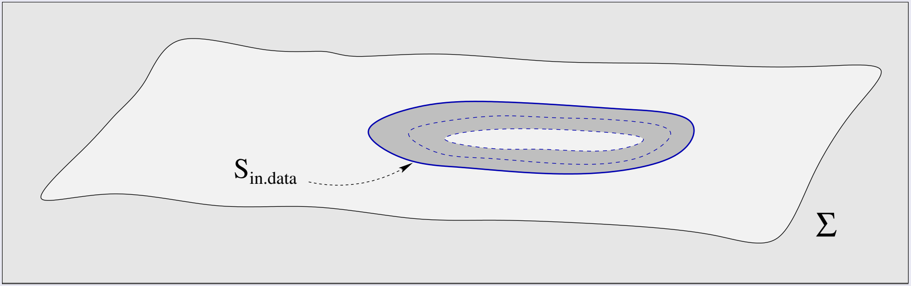
$$
\begin{aligned}
&\overset{\star}{K}_{ij} = \tfrac{1}{2}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_{(i}\widehat{N}_{j)}, \qquad \dot{\widehat{n}}_k = \widehat{n}^l D_l \widehat{n}_k = -\widehat{D}_k(\ln \widehat{N}) \\
&\mathcal{A} = 2\,[\,(\partial_\rho\overset{\star}{K}) - \widehat{N}^l(\widehat{D}_l\overset{\star}{K})\,] + \overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl} \\
&\mathcal{B} = -\widehat{R} + \epsilon\,\big[\,2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \overset{\circ}{\mathbf{K}}_{kl}\,\overset{\circ}{\mathbf{K}}{}^{kl} - 2\,\mathfrak{e}\,\big]
\end{aligned}
$$

The parabolic-hyperbolic system:

- $\boxed{(h_{ij}, K_{ij})}$ represented by the variables $\boxed{(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})}$
- the constraints comprise a **parabolic-hyperbolic** system for $\boxed{(\widehat{N}, \mathbf{k}_i, \mathbf{K}^l{}_l)}$
  - with freely specifiable variables on $\boxed{\Sigma}$ and on $\boxed{\mathrm{S}_{\mathrm{in.data}}}$:

$$\boxed{(\widehat{N}|_{\mathrm{S}_{\mathrm{in.data}}}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i|_{\mathrm{S}_{\mathrm{in.data}}}, \mathbf{K}^l{}_l|_{\mathrm{S}_{\mathrm{in.data}}}, \mathring{\mathbf{K}}_{ij})}$$

  - a fixed $(+/-)$ sign of $\boxed{\overset{\star}{K} = \frac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j}$ can be guaranteed



Theorem

Suppose that a choice had been made for the freely specifiable variables

$$\boxed{\widehat{N}^i, \widehat{\gamma}_{ij}\,; \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{ij}\,; \mathfrak{e}, \mathfrak{p}_i}$$

such that they are smooth and $\overset{\star}{K} > 0$ *throughout* Σ. *Assume that smooth initial data* $({}_0\widehat{N}, {}_0\mathbf{k}_i, {}_0\mathbf{K}^l{}_l)$, *with* ${}_0\widehat{N} > 0$, *had also been chosen to our hyperbolic-parabolic system on one of the level surfaces* $\mathscr{S}_{\rho_0}$ *in* Σ. *Then, for some* $\varepsilon > 0$, *on a short interval* $[\rho_0, \rho_0 + \varepsilon)$, *there exists a unique smooth solution*

$$\boxed{(\widehat{N}, \mathbf{k}_i, \mathbf{K}^l{}_l) : \widehat{N} > 0, \text{ such that } \widehat{N}|_{\mathscr{S}_{\rho_0}} = {}_0\widehat{N}, \mathbf{k}_i|_{\mathscr{S}_{\rho_0}} = {}_0\mathbf{k}_i \quad \& \quad \mathbf{K}^l{}_l|_{\mathscr{S}_{\rho_0}} = {}_0\mathbf{K}^l{}_l}$$

The fields h_{ij} *and* K_{ij} *which can be built up from this solution and from the freely specifiable part of the data satisfy their respective* $n+1$ *constraints in the corresponding one-sided neighborhood* $\mathscr{S}_{[\rho_0, \rho_0+\varepsilon)}$ *of* $\mathscr{S}_{\rho_0}$ *in* Σ.

Global in the time solutions:

as for certain choice of coefficients in parabolic equations or for that of the initial data classical solutions are known to blow up in finite "time" it is of obvious interest to identify those conditions which guarantee the existence of global in the time, $[\rho_0, \infty)$, solutions (these should be bounded away from zero and from infinity) for the parabolic form of the Hamiltonian constraint

The behavior of "upper" and "lower" solutions:

- assume that the $\mathscr{S}_\rho$ level surfaces are compact (or asymptotic decay conds.)
- consider a function f on $\Sigma = \mathbb{R} \times \mathscr{S}_\rho$
- define $f^\sharp$ and $f_\flat$ as

$$\boxed{f^\sharp(\rho) = \sup_{\mathscr{S}_\rho} \{ f(\rho, x^2, \dots, x^n) \}, \quad f_\flat(\rho) = \inf_{\mathscr{S}_\rho} \{ f(\rho, x^2, \dots, x^n) \}}$$

- substituting the auxiliary function $w = \widehat{N}^{-2} \, (\geq 0)$

$$\boxed{\partial_\rho w - \widehat{N}^l (\widehat{D}_l w) + \tfrac{3}{2} \overset{\star}{K}{}^{-1} w^{-2} (\widehat{D}^l w)(\widehat{D}_l w) = (\overset{\star}{K} w)^{-1} (\widehat{D}^l \widehat{D}_l w) - \mathfrak{a}\, w - \mathfrak{b}}$$

where $\mathfrak{a} = \mathcal{A}/\overset{\star}{K}$ and $\mathfrak{b} = \mathcal{B}/\overset{\star}{K}$

- if $\widehat{N}$ has its *maximum* w has its *minimum* and, vice versa, if $\widehat{N}$ has its *minimum* w has its *maximum*
- as $\overset{\star}{K} w$ is non-negative and $\widehat{\gamma}_{ij}$ is positive definite, the first term on the rhs is non-positive or non-negative at points where w attains its maximum or minimum, respectively.
- $\Longrightarrow$

$$\boxed{\partial_\rho w_\flat \geq -\mathfrak{a}^\sharp\, w_\flat - \mathfrak{b}^\sharp, \qquad \partial_\rho w^\sharp \leq -\mathfrak{a}_\flat\, w^\sharp - \mathfrak{b}_\flat}$$

The behavior of "upper" and "lower" solutions:

$$\boxed{w_\flat \geq W_\flat = \exp\left[-\int_{\rho_0}^{\rho} \mathfrak{a}^\sharp \mathrm{d}\rho'\right]\left\{w_\flat|_{\mathscr{S}_{\rho_0}} - \int_{\rho_0}^{\rho} \mathfrak{b}^\sharp \exp\left[\int_{\rho_0}^{\rho'} \mathfrak{a}^\sharp \mathrm{d}\rho''\right]\mathrm{d}\rho'\right\}}$$

$$\boxed{w^\sharp \leq W^\sharp = \exp\left[-\int_{\rho_0}^{\rho} \mathfrak{a}_\flat \mathrm{d}\rho'\right]\left\{w^\sharp|_{\mathscr{S}_{\rho_0}} - \int_{\rho_0}^{\rho} \mathfrak{b}_\flat \exp\left[\int_{\rho_0}^{\rho'} \mathfrak{a}_\flat \mathrm{d}\rho''\right]\mathrm{d}\rho'\right\}}$$

- **if** $$\boxed{\mathcal{K} = \sup_{\rho\in[\rho_0,\infty)}\left\{\int_{\rho_0}^{\rho} \mathfrak{b}^\sharp \exp\left[\int_{\rho_0}^{\rho'} \mathfrak{a}^\sharp \mathrm{d}\rho''\right]\mathrm{d}\rho'\right\}}$$

  is positive and finite, i.e. $0 < \mathcal{K} < \infty$, **and if** for the initial data the inequality $w_\flat|_{\mathscr{S}_{\rho_0}} > \mathcal{K}$ holds $\Longrightarrow$ $W^\sharp, W_\flat > 0$, and w is bounded from below by $W_\flat > 0$
- this is equivalent to saying that $\widehat{N}$ **has to be bounded form above** as $\widehat{N} = w^{-\frac{1}{2}} \leq W_\flat{}^{-\frac{1}{2}} < \infty$, if for the initial data ${}_0\widehat{N}$ the inequality ${}_0\widehat{N} < 1/\sqrt{\mathcal{K}}$ holds [equiv. to ${}_0\widehat{N}^{-2} = w|_{\mathscr{S}_{\rho_0}} \geq w_\flat|_{\mathscr{S}_{\rho_0}} > \mathcal{K}$]
- if $\mathfrak{b} = \mathcal{B}/\overset{\star}{K}$ is negative then $\mathcal{K} < 0$ which guarantee that $w_\flat|_{\mathscr{S}_{\rho_0}} \geq 0 > \mathcal{K}$ holds **for any choice of a positive initial data** ${}_0\widehat{N} > 0 \Longrightarrow \widehat{N} < \infty$
- $\widehat{N}$ is **bounded from below**: [w is bounded from above] $w \leq w^\sharp \leq W^\sharp \Longrightarrow$ $\widehat{N}$ is positive as $0 < (W^\sharp)^{-\frac{1}{2}} \leq w^{-\frac{1}{2}} = \widehat{N}$

Theorem

Suppose that all the coefficients in the Bernoulli type parabolic equation

$$\boxed{2\,\overset{\star}{K}\,[\,(\partial_\rho \widehat N) - \widehat N^l(\widehat D_l \widehat N)\,] - 2\,\widehat N^2(\widehat D^l \widehat D_l \widehat N) - \mathcal{A}\,\widehat N - \mathcal{B}\,\widehat N^3 = 0} \qquad (*)$$

are smooth and that the freely specifiable part of the data was chosen such that $\overset{\star}{K}$ *is positive throughout* Σ*. Choose* ${}_0\widehat N$ *to be a smooth positive function on the compact level surface* $\rho = \rho_0$ *such that* ${}_0\widehat N < 1/\sqrt{\mathcal{K}}$ *if* $0 < \mathcal{K} < \infty$*, or to be arbitrary if* $\mathcal{K} \leq 0$*, where*

$$\boxed{\mathcal{K} = \sup_{\rho\in[\rho_0,\infty)} \left\{ \int_{\rho_0}^{\rho} \mathfrak{b}^{\,\sharp} \exp\left[\int_{\rho_0}^{\rho'} \mathfrak{a}^{\,\sharp} \mathrm{d}\rho''\right] \mathrm{d}\rho' \right\}\,,}$$

and $\mathfrak{a} = \mathcal{A}/\overset{\star}{K}$ *and* $\mathfrak{b} = \mathcal{B}/\overset{\star}{K}$*.*

Then $(*)$ *has a* **unique smooth global** *in the time,* $[\rho_0, \infty)$*,* **classical solution** *such that* $0 < \widehat N < \infty$*, and that* $\widehat N|_{\mathscr{S}_{\rho_0}} = {}_0\widehat N$*.*

The strongly hyperbolic system:

The Hamiltonian constraint as an algebraic equation for $\boldsymbol{\kappa}$:

$$\boxed{-\epsilon\,{}^{(n)}R + 2\,\boxed{\boldsymbol{\kappa}}\,\mathbf{K}^l{}_l + \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0}$$

with solution:

$$\boxed{\boldsymbol{\kappa} = (2\,\mathbf{K}^l{}_l)^{-1}[\,2\,\mathbf{k}^l\mathbf{k}_l - \tfrac{n-2}{n-1}\,(\mathbf{K}^l{}_l)^2 - \boldsymbol{\kappa}_0\,]\,, \quad \boldsymbol{\kappa}_0 = -\epsilon\,{}^{(n)}R - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e}}$$

- by eliminating $\boxed{\widehat{D}_i\boldsymbol{\kappa}}$ from the momentum constraint mom. constr. one gets

$$\boxed{\begin{aligned}&\mathscr{L}_{\hat{n}}\mathbf{k}_i + (\mathbf{K}^l{}_l)^{-1}[\,\boldsymbol{\kappa}\,\widehat{D}_i(\mathbf{K}^l{}_l) - 2\,\mathbf{k}^l\,\widehat{D}_i\mathbf{k}_l\,] + (2\,\mathbf{K}^l{}_l)^{-1}\,\widehat{D}_i\boldsymbol{\kappa}_0\\ &\qquad + (\widehat{K}^l{}_l)\,\mathbf{k}_i + [\,\boldsymbol{\kappa} - \tfrac{1}{n-1}\,(\mathbf{K}^l{}_l)\,]\,\dot{\hat{n}}_i - \dot{\hat{n}}^l\,\mathring{\mathbf{K}}_{li} + \widehat{D}^l\mathring{\mathbf{K}}_{li} - \epsilon\,\mathfrak{p}_l\,\widehat{\gamma}^l{}_i = 0\,,\\ &\qquad \mathscr{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \boldsymbol{\kappa}\,(\widehat{K}^l{}_l) + \mathbf{K}_{kl}\widehat{K}^{kl} + 2\,\dot{\hat{n}}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\hat{n}^l = 0\end{aligned}}$$

- the above system is a **strongly hyperbolic** one for $\boxed{(\mathbf{k}_i, \mathbf{K}^l{}_l)}$ **if** $\boxed{\boldsymbol{\kappa}\cdot\mathbf{K}^l{}_l < 0}$
- $\boxed{\boldsymbol{\kappa}}$ is **determined algebraically** once $\boxed{\mathbf{k}_i \text{ and } \mathbf{K}^l{}_l}$ are known **!!!**
- the entire three-metric $\boxed{h_{ij} = \widehat{\gamma}_{ij} + \hat{n}_i\hat{n}_j}$ is freely specifiable. **!!!**

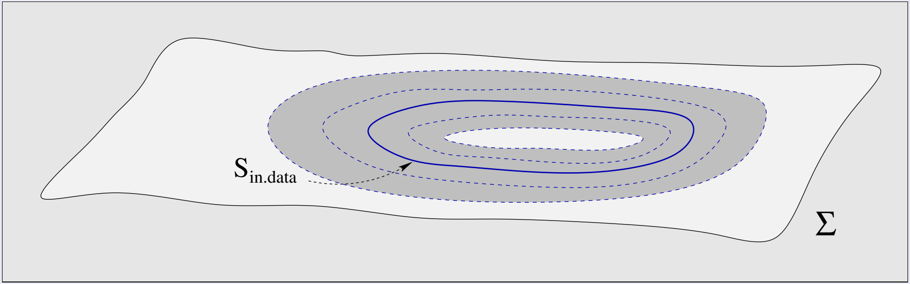
Constraints as evolutionary systems:

The strongly hyperbolic system:

- $\boxed{(h_{ij}, K_{ij})}$ represented by the variables $\boxed{(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})}$
- the constraints form a **strongly hyperbolic** system for $\boxed{(\mathbf{k}_i, \mathbf{K}^l{}_l)}$ (alg.for $\boxed{\boldsymbol{\kappa}}$)
  - with freely specifiable variables on $\boxed{\Sigma}$ and on $\boxed{\mathrm{S}_{\mathrm{in.data}}}$:

$$\boxed{(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boxed{\boldsymbol{\kappa}}, \mathbf{k}_i|_{\mathrm{S}_{\mathrm{in.data}}}, \mathbf{K}^l{}_l|_{\mathrm{S}_{\mathrm{in.data}}}, \mathring{\mathbf{K}}_{ij})}$$

  - by choosing the free data properly $\boxed{\boldsymbol{\kappa} \cdot \mathbf{K}^l{}_l < 0}$ can be guaranteed (locally!)



Strongly hyperbolic system:

$$\begin{aligned} &\mathscr{L}_{\widehat{n}}\mathbf{k}_i + (\mathbf{K}^l{}_l)^{-1}[\,\boldsymbol{\kappa}\,\widehat{D}_i(\mathbf{K}^l{}_l) - 2\,\mathbf{k}^l\widehat{D}_i\mathbf{k}_l\,] + (2\,\mathbf{K}^l{}_l)^{-1}\widehat{D}_i\boldsymbol{\kappa}_0 \\ &\qquad + (\widehat{K}^l{}_l)\,\mathbf{k}_i + [\,\boldsymbol{\kappa} - \tfrac{1}{n-1}\,(\mathbf{K}^l{}_l)\,]\,\dot{\widehat{n}}_i - \dot{\widehat{n}}^l\,\mathring{\mathbf{K}}_{li} + \widehat{D}^l\mathring{\mathbf{K}}_{li} - \epsilon\,\mathfrak{p}_l\,\widehat{\gamma}^l{}_i = 0\,, \\ &\qquad \mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \boldsymbol{\kappa}\,(\widehat{K}^l{}_l) + \mathbf{K}_{kl}\widehat{K}^{kl} + 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\widehat{n}^l = 0 \end{aligned}$$

notably, when writing out these equations in local coordinates $(\rho, x^2, \dots, x^n)$, adopted to the foliation $\mathscr{S}_\rho$ and the vector field ρ^i, the system takes the form (**HW (6)** Check the coefficients $\mathcal{B}^{(K)}$ below!)

$$\partial_\rho\mathbf{u} + \mathcal{B}^{(K)}\,\partial_K\mathbf{u} + \mathcal{C} = 0$$

for the vector valued variable $\mathbf{u} = (\mathbf{k}_B, \mathbf{K}^E{}_E)^T$, with coefficients $\mathcal{B}^{(K)}$ of ∂_K

$$\mathcal{B}^{(K)}_{ij} = -\widehat{N}^K\delta_{ij} - \widehat{N}\,\hat{\gamma}^{KL}\,\delta_{in}\,\delta_{j(L-1)} + \frac{\boldsymbol{\kappa}\,\widehat{N}}{\mathbf{K}^E{}_E}\,\delta_{i(K-1)}\,\delta_{jn} - 2\,\frac{\widehat{N}\,\mathbf{k}_L}{\mathbf{K}^E{}_E}\,\hat{\gamma}^{LM}\,\delta_{i(K-1)}\,\delta_{(j+1)M}$$

- in evaluating the term $\delta_{(j+1)K}$ above the $(n+1)$-dimensional Kronecker delta is used.
- the lowercase and uppercase Latin indices take the value $1, 2, \dots, n$ and $2, 3, \dots, n$, resp.

Strongly hyperbolic system: if for any covector ξ_K with $K = 2, \dots, n$

i) the eigenvalues of the matrix $\mathfrak{M} = \mathcal{B}^{(2)}\xi_2 + \cdots + \mathcal{B}^{(n)}\xi_n$ are all real, and

ii) $\mathfrak{M} = \mathcal{B}^{(2)}\xi_2 + \cdots + \mathcal{B}^{(n)}\xi_n$ is diagonalizable.

A square matrix $\mathfrak{M}$ is diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix S such that $S^{-1}\mathfrak{M}S$ is diagonal. The diagonal elements give the eigenvalues.

First order symmetrizable hyperbolic system:

$$\boxed{\partial_\rho \mathbf{u} + \mathcal{B}^{(K)}\,\partial_K \mathbf{u} + \mathcal{C} = 0}$$

for the vector valued variable $\mathbf{u} = (\mathbf{k}_B, \mathbf{K}^E{}_E)^T$

- **Friedrichs symmetrizable system**: if a common symmetrizer to $\mathcal{B}^{(K)}$ exists
- a bilinear form, $\mathcal{H}_{ij}$, is called to be a **common symmetrizer** if it is positive definite and is such that for any value of $K = 2, \dots, n$ the product $\mathcal{H}_{ij}\,\mathcal{B}^{(K)}_{jk}$ is symmetric.
- a common symmetrizer in a subset of Σ where $\boldsymbol{\kappa} \cdot \mathbf{K}^l{}_l < 0$

$$\mathcal{H}_{\,ij} = \delta_{in}\,\delta_{jn} - \frac{2}{\boldsymbol{\kappa}}\,\mathbf{k}_P\,\hat{\gamma}^{PQ}\left[\,\delta_{in}\,\delta_{(j+1)Q} + \delta_{(i+1)Q}\,\delta_{jn}\,\right] + \frac{1}{\boldsymbol{\kappa}^2}\left[4\,(\mathbf{k}_P\,\hat{\gamma}^{PQ}\,\delta_{(i+1)Q})\,(\mathbf{k}_S\,\hat{\gamma}^{ST}\,\delta_{(j+1)T}) - \boldsymbol{\kappa}\cdot\mathbf{K}^E{}_E\,\hat{\gamma}^{PQ}\,\delta_{(i+1)P}\,\delta_{(j+1)Q}\right]$$

- **First order symmetric hyperbolic system** (**HW (7)** Check this!)

$$\boxed{\mathcal{H}\,\partial_\rho \mathbf{u} + (\mathcal{H}\mathcal{B}^{(K)})\,\partial_K \mathbf{u} + \mathcal{H}\,\mathcal{C} = 0}$$

for the vector valued variable $\mathbf{u} = (\mathbf{k}_B, \mathbf{K}^E{}_E)^T$

Theorem

Assume that the freely specifiable part of data

$$\boxed{\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}\,;\mathring{\mathbf{K}}_{ij}\,;\mathfrak{e}, \mathfrak{p}_i}$$

on Σ, and the initial data $({}_0\mathbf{k}_i, {}_0\mathbf{K}^l{}_l)$ to the symmetrizable hyperbolic system

$$\boxed{\partial_\rho \mathbf{u} + \mathcal{B}^{(K)}\,\partial_K \mathbf{u} + \mathcal{C} = 0\,, \quad \text{with} \ \ \mathbf{u} = (\mathbf{k}_i, \mathbf{K}^l{}_l)^T}$$

on a $\rho = \rho_0$ level surface are chosen such that they are all smooth on their respective domains. Then, there exists a unique smooth solution $\boxed{\mathbf{k}_i, \mathbf{K}^l{}_l}$ *(at least locally) such that*

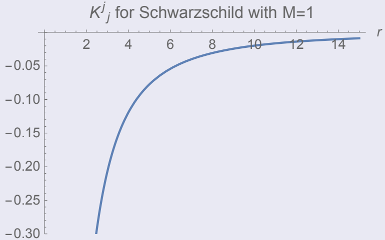
$$\boxed{\mathbf{k}_i|_{\mathscr{S}_{\rho_0}} = {}_0\mathbf{k}_i \ \ \& \ \ \mathbf{K}^l{}_l|_{\mathscr{S}_{\rho_0}} = {}_0\mathbf{K}^l{}_l}$$

This solution, along with the freely specified data, algebraically determines $\boldsymbol{\kappa}$.

The fields h_{ij} and K_{ij} which can be built up from the corresponding solution and from the freely specified data do satisfy the Hamiltonian and momentum constraints in the domain of dependence of $\mathscr{S}_{\rho_0}$ in Σ.

Could the inequality $\boldsymbol{\kappa} \cdot \mathbf{K}^l{}_l < 0$ hold globally?

- consider spaces in **Kerr-Schild form**: $\boxed{g_{ab} = \eta_{ab} + 2H\ell_a\ell_b}$
  [with Jeff Winicour (2015)]
  - H smooth**!** on $\mathbb{R}^4$, ℓ_a is null with respect to both g_{ab} and an implicit flat Minkowski background metric η_{ab}
- **Schwarzschild spacetime**: $\boxed{H = \frac{M}{r} \;\; \& \;\; \ell_\alpha = (1, \frac{x}{r}, \frac{y}{r}, \frac{y}{r})}$
  - $\mathbf{k}_l = 0,\ K^l{}_l \neq const$ already for Schwarzschild with Kerr-Schild slicing $K^l{}_l = \boldsymbol{\kappa} + \mathbf{K}^l{}_l = -\frac{2M\left[1+\frac{3M}{\rho}\right]}{\rho^2\left(\sqrt{1+\frac{2M}{\rho}}\right)^3}$

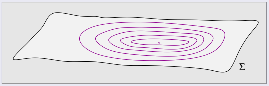


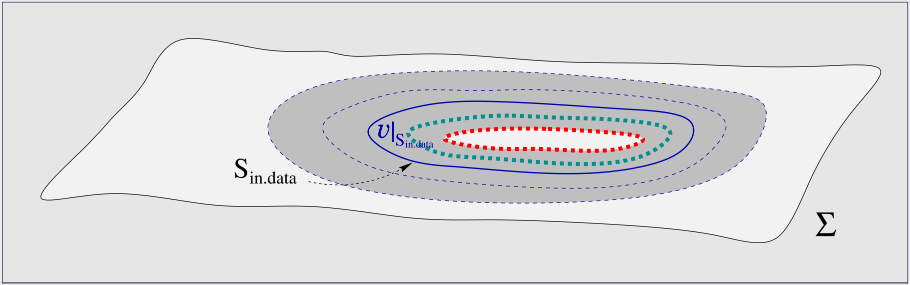
- for **near Schwarzschild** configurations: the relation $\boxed{-\frac{\mathbf{K}^l{}_l}{\boldsymbol{\kappa}} \approx \frac{2(1+2H)}{1+H}}$ holds everywhere on $t = const$ hypersurfaces

ð, $\bar{ð}$ operators with finite differencing:

near Schwarzschild configurations: [with Maciej Maliborski (2016)]

- $\Sigma \approx \mathbb{R} \times \mathbb{S}^2$
- $\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}, \mathring{\mathbf{K}}_{ij}$ are as for Schwarzschild but ...

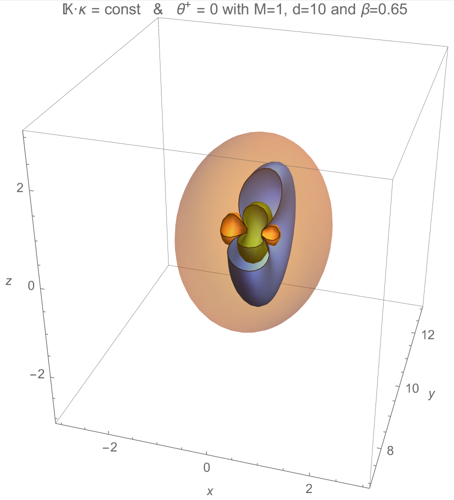




- integrating inward **singularity develops !!!** found to be **beyond trapped surfaces**: $\boldsymbol{\theta}^{\pm} < 0$
- integrating outward: the fields **decay** as: $||\mathbf{K}^l{}_l|| \sim \rho^{-2}$, $||\mathbf{k}_i|| \sim \rho^{-2}$ while $\rho \to \infty$ if the initial data for $\mathbf{K}^l{}_l = {}_{Schw}\mathbf{K}^l{}_l + \sum_{\ell=2}^{\infty} c_{\ell,m} \cdot Y_\ell{}^m$

One more note on $\boldsymbol{\kappa} \cdot \mathbf{K}^l{}_l < 0$!!! :

Boosted Schwarzschild with $v = 0.65$:



$$\boldsymbol{\kappa} \cdot \mathbf{K}^l{}_l = -3, 0, 3$$

Summary:

$n+1$-dimensional $[\, n(\geq 3)\,]$ **Riemannian and Lorentzian spaces** satisfying Einstein's equations, and some mild topological assumptions, were considered.

1. concerning the constraint equations in Einstein's theory it was shown:
   - **momentum constraint** as a **first order symmetric hyperbolic system**
   - **the Hamiltonian constraint** as a **parabolic** or an **algebraic** equation
   - **in either case** the coupled constraint equations comprise **a well-posed evolutionary system:** a **parabolic-hyperbolic** or a **strongly hyperbolic**,
   - **(local) existence and uniqueness** of C^∞ solutions is guaranteed
2. **!!! regardless** whether the metric of the ambient space is Riemannian or Lorentzian
3. **!!! no use** of gauge conditions:

   arbitrary choice of foliations $\&$ "time evolution" vector field

Outlook:

Analytic investigations I.:

- Joint work with Philippe LeFloch
  - near Schwarzschild configurations with spherical foliations
    - the parabolic-hyperbolic, and
    - in the strongly hyperbolic
  - Aims: Using energy estimates to show the global existence and proper asymptotic decay of solutions to the constrain equations in these cases

Numerical investigations: I.

- Joint work with Anna Nakonieczna and Georgios Doulis
  - Aims: to construct initial data—by integrating numerically the parabolic-hyperbolic form of the constraints (Georgios using an implicit method)—for:
    - single boosted and rotating black holes (exact and distorted ones)
    - rotating binary black holes (without restrictions in the strong field regime)
  - the first paper arXiv:1712.00607 [gr-qc]

Outlook:

Numerical investigations: II.

- Joint work with Károly Csukás
  - investigate near Kerr configurations using foliations by topological two-spheres, and all the variables are expanded in terms of spin-weighted spherical harmonics (analytic evaluation of angular derivatives)
    - the parabolic-hyperbolic system
    - plans to include strongly hyperbolic form of the constraints too

Numerical investigations: III.

- Christian Schell (PhD student of Oliver Rinne at AEI, Potsdam) investigated perturbations of Minkowski spacetime
  - parabolic-hyperbolic form of the constraints in determining initial data, and
  - hyperbolic form of momentum constraint in partly constrained evolution
  - the Σ_t time-level surfaces are foliated by topological two-spheres

the playground is open: apply these new evolutionary formulations of the constraints in solving various physically adequate situations

The roots of the evolutionary aspects

The first order symmetric hyperbolic system for $(E^{(\mathcal{H})}, E_i^{(\mathcal{M})})^T$

- Characteristic directions associated with the FOSH system governing the evolution of the constraint expressions are determined as

$$\boxed{[\,h^{ij} - n^i n^j\,]\,\xi_i \xi_j = [\,g^{ij} - (1+\epsilon)\,n^i n^j\,]\,\xi_i \xi_j = 0}$$

The momentum constraint: first order symmetric hyperbolic system

- with characteristic cone given as

$$\boxed{\left[\,\widehat{\gamma}^{ij} - \left(\tfrac{n-1}{n-2}\right)\,\widehat{n}^i \widehat{n}^j\,\right]\,\xi_i \xi_j = \left[\,h^{ij} - \left(1 + \tfrac{n-1}{n-2}\right)\,\widehat{n}^i \widehat{n}^j\,\right]\,\xi_i \xi_j = 0}$$

Deriving a Lorentzian metric from a Riemannian one

- (**HW (8)** Check this!) ... given a Riemannian metric $\mathfrak{g}_{ij}$, a unit form field $\mathfrak{n}_i$ and a positive real function α $\Longrightarrow$ a metric of Lorentzian signature can be defined as

$$\boxed{\check{\mathfrak{g}}_{ij} = \mathfrak{g}_{ij} - (1+\alpha)\,\mathfrak{n}_i \mathfrak{n}_j}$$

That is all for now...