On the use of evolutionary methods in metric theories of gravity X.

István Rácz

istvan.racz@fuw.edu.pl & racz.istvan@wigner.mta.hu

Faculty of Physics, University of Warsaw, Warsaw, Poland

Wigner Research Center for Physics, Budapest, Hungary

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Institute of Theoretical Physics, University of Warsaw

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Plans and Aims:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

- **Time evolution and the degrees of freedom**
  - intimate relations between various parts of Einstein's equations
  - fully constrained evolutionary scheme
  - evolutionary-evolutionary systems

    ... gauge choices

    ... the conformal structure

    ... gravitational degrees of freedom

Based on some recent papers

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, Class. Quantum Grav. 31 (2014) 155004
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. 32 (2015) 015006
- I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, arXiv:1412.0667 (2015)
- I. Rácz: *Constraints as evolutionary systems*, Class. Quantum Grav. **33** 015014 (2016)

Assumptions:

- **The primary space:** (M, g_{ab})
  - M : $n+1$-dim. ($n \geq 3$), smooth, paracompact, connected, orientable manifold
    - smoothly foliated by a one-parameter family of homologous hypersurfaces determined by a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing $\partial_a \sigma$ gradient; a flow σ^a has also been chosen such that $\sigma^a \partial_a \sigma = 1$
    - one of these level surfaces is smoothly foliated by a one-parameter family of homologous codimension-two surfaces determined by a smooth function $\rho : M \to \mathbb{R}$ with a.e. non-vanishing $\partial_a \rho$ gradient; a "horizontal" flow ρ^a has also been chosen such that $\rho^a \partial_a \rho = 1$
    - $\Longrightarrow$ M is smoothly foliated by
      a **two-parameter family of codimension-two-surfaces**:
  - g_{ab}: smooth Lorentzian$_{(-,+,\dots,+)}$ or Riemannian$_{(+,\dots,+)}$ metric
- **Einsteinian space:** Einstein's equation restricting the geometry

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0}$$

with source term $\mathscr{G}_{ab}$ having a vanishing divergence, $\nabla^a \mathscr{G}_{ab} = 0$.

The main creatures:

- n^a the 'unit norm' vector field that is normal to the Σ_σ level surfaces

$$n^a n_a = \epsilon$$

  - ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, resp.

- the **projection operator** and the **metric induced**

$$h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b \qquad h_{ab} = h_a{}^e h_b{}^f\, g_{ef} = g_{ab} - \epsilon\, n_a n_b$$

- D_a denotes the covariant derivative operator associated with h_{ab}

$$\mathscr{G}_{ab} = n_a n_b\, \mathfrak{e} + [n_a\, \mathfrak{p}_b + n_b\, \mathfrak{p}_a] + \mathfrak{S}_{ab}$$

$$\mathfrak{e} = n^e n^f\, \mathscr{G}_{ef}\,, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f\, \mathscr{G}_{ef}\,, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\, \mathscr{G}_{ef}$$

- r.h.s. of Einstein's equation: $E_{ab} = G_{ab} - \mathscr{G}_{ab}$

$$E_{ab} = n_a n_b\, E^{(\mathcal{H})} + [n_a\, E^{(\mathcal{M})}_b + n_b\, E^{(\mathcal{M})}_a] + (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})$$

$$E^{(\mathcal{H})} = n^e n^f\, E_{ef}\,, \quad E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f\, E_{ef}\,, \quad E^{(\mathcal{EVOL})}_{ab} = h^e{}_a h^f{}_b\, E_{ef} - h_{ab}\, E^{(\mathcal{H})}$$

Relations between various parts of the basic equations:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})}\,(K^e{}_e) - 2\,\epsilon\,(\dot n^e\, E^{(\mathcal{M})}_e) - \epsilon\, K^{ae}\,(E^{(\mathcal{EVOL})}_{ae} + h_{ae}\,E^{(\mathcal{H})})\,] = 0$$

$$\mathscr{L}_n E^{(\mathcal{M})}_b + D^a(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\,E^{(\mathcal{H})}) + [\,(K^e{}_e)\,E^{(\mathcal{M})}_b + E^{(\mathcal{H})}\,\dot n_b - \epsilon\,(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\,E^{(\mathcal{H})})\,\dot n^a\,] = 0$$

Corollary

If the constraint expressions $E^{(\mathcal{H})}$ *and* $E^{(\mathcal{M})}_a$ *vanish on the* $\sigma = const$ *level surfaces then the relations*

$$K^{ab}\,E^{(\mathcal{EVOL})}_{ab} = 0$$

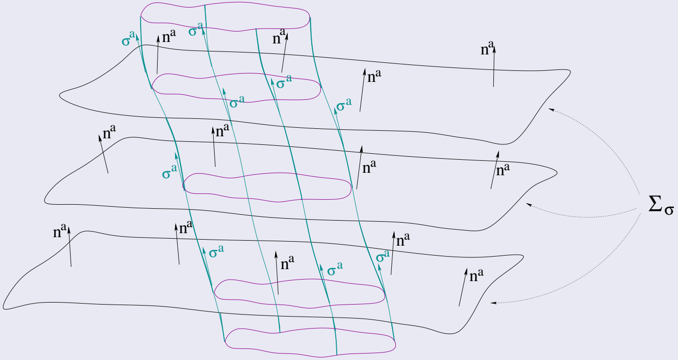
$$D^a E^{(\mathcal{EVOL})}_{ab} - \epsilon\,\dot n^a\,E^{(\mathcal{EVOL})}_{ab} = 0$$

hold for the evolutionary expression $E^{(\mathcal{EVOL})}_{ab}$.

$$h^e{}_a h^f{}_b\,E_{ef} = E^{(\mathcal{EVOL})}_{ab} + \cancel{h_{ab}\,E^{(\mathcal{H})}}$$

The two-parameter foliations:

The Lie drag this foliation of Σ_0 along the integral curves of the vector field σ^a yields then a two-parameter foliation $\mathscr{S}_{\sigma,\rho}$:



- the fields $\widehat{n}^i$, $\widehat{\gamma}_{ij}$ and the projection $\boxed{\widehat{\gamma}^k{}_l = h^k{}_l - \widehat{n}^k \widehat{n}_l}$, to the codimension-two surfaces $\mathscr{S}_{\sigma,\rho}$, get to be well-defined on each of the individual $\sigma = const$ hypersurfaces

The equations on the $\sigma = const$ hypersurfaces:

Some important relations we learned while studying the kinematical background:

using

$$h_b{}^e h_d{}^f R_{ef} = {}^{(n)}R_{bd} + \epsilon\,\{-\mathscr{L}_n K_{bd} - K_{bd}K_e{}^e + 2K_b{}^e K_{de} - \epsilon\, N^{-1} D_b D_d N\}$$

$$R = {}^{(n)}R + \epsilon\,\{-2\,\mathscr{L}_n(K_{bd}h^{bd}) - (K_e{}^e)^2 - K_{ef}K^{ef} - 2\,\epsilon\, N^{-1} D^e D_e N\}$$

one gets

$$\boxed{h_b{}^e h_d{}^f E_{ef}} = h_b{}^e h_d{}^f\,\{[\,R_{ef} - \tfrac{1}{2}\,g_{ef}R\,] - \mathscr{G}_{bd}\} = h_b{}^e h_d{}^f\,\{[\,R_{ef} - \tfrac{1}{2}\,h_{ef}R\,] - \mathscr{G}_{bd}\}$$
$$= [\,{}^{(n)}R_{bd} - \tfrac{1}{2}\,h_{ef}\,{}^{(n)}R\,] - {}^{(n)}\mathscr{G}_{bd} = {}^{(n)}G_{bd} - {}^{(n)}\mathscr{G}_{bd} = \boxed{{}^{(n)}E_{bd}}$$

where

$${}^{(n)}\mathscr{G}_{ab} = \mathfrak{S}_{ab} - \epsilon\Big\{ - \mathscr{L}_n K_{ab} - (K^e{}_e)K_{ab} + 2\,K_{ae}K^e{}_b - \epsilon\, N^{-1}\, D_a D_b N$$
$$+ h_{ab}\big[\mathscr{L}_n(K^e{}_e) + \tfrac{1}{2}\,(K^e{}_e)^2 + \tfrac{1}{2}\,K_{ef}K^{ef} + \epsilon\, N^{-1}\, D^e D_e N\big]\Big\}$$

The explicit forms:

Expressions in the $[n-1]+1$ decomposition:

$$ {}^{(n)}E_{ij} = \widehat{E}^{(\mathcal{H})}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\widehat{E}^{(\mathcal{M})}_j + \widehat{n}_j\widehat{E}^{(\mathcal{M})}_i] + (\widehat{E}^{(\mathcal{EVOL})}_{ij} + \widehat{\gamma}_{ij}\widehat{E}^{(\mathcal{H})}) $$

$$ \widehat{E}^{(\mathcal{H})} = \widehat{n}^e\widehat{n}^f\,{}^{(n)}E_{ef}\,,\quad \widehat{E}^{(\mathcal{M})}_i = \widehat{\gamma}^e{}_j\widehat{n}^f\,{}^{(n)}E_{ef}\,,\quad \widehat{E}^{(\mathcal{EVOL})}_{ij} = \widehat{\gamma}^e{}_i\widehat{\gamma}^f{}_j\,{}^{(n)}E_{ef} - \widehat{\gamma}_{ij}\widehat{E}^{(\mathcal{H})} $$

$$ \begin{aligned} \widehat{E}^{(\mathcal{H})} =&\ \tfrac{1}{2}\,\{-\widehat{R} + (\widehat{K}^l{}_l)^2 - \widehat{K}_{kl}\widehat{K}^{kl} - 2\,\widehat{\mathfrak{e}}\}\,, \\ \widehat{E}^{(\mathcal{M})}_i =&\ \widehat{D}^l\widehat{K}_{li} - \widehat{D}_i\widehat{K}^l{}_l - \widehat{\mathfrak{p}}_i\,, \\ \widehat{E}^{(\mathcal{EVOL})}_{ij} =&\ \widehat{R}_{ij} - \mathscr{L}_{\widehat{n}}\widehat{K}_{ij} - (\widehat{K}^l{}_l)\widehat{K}_{ij} + 2\,\widehat{K}_{il}\widehat{K}^l{}_j - \widehat{N}^{-1}\,\widehat{D}_i\widehat{D}_j\widehat{N} \\ &+ \widehat{\gamma}_{ij}\{\mathscr{L}_{\widehat{n}}\widehat{K}^l{}_l + \widehat{K}_{kl}\widehat{K}^{kl} + \widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\} - [\widehat{\mathfrak{S}}_{ij} - \widehat{\mathfrak{e}}\,\widehat{\gamma}_{ij}] \end{aligned} $$

where $\widehat{D}_i, \widehat{R}_{ij}$ and $\widehat{R}$ denote ....

$$ \widehat{\mathfrak{e}} = \widehat{n}^k\widehat{n}^l\,{}^{(n)}\mathscr{G}_{kl}\,,\quad \widehat{\mathfrak{p}}_i = \widehat{\gamma}^k{}_i\,\widehat{n}^l\,{}^{(n)}\mathscr{G}_{kl} \quad \text{and} \quad \widehat{\mathfrak{S}}_{ij} = \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,{}^{(n)}\mathscr{G}_{kl} $$

and

$$ \widehat{K}_{ij} = \widehat{\gamma}^l{}_i\,D_l\,\widehat{n}_j = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}}\widehat{\gamma}_{ij} $$

Relations between various parts of the basic equations I.:

Substituting the $[n-1]+1$ splitting of ${}^{(n)}E_{ij}$:

$$K^{ab}\,{}^{(n)}E_{ab} = 0$$
$$D^a[{}^{(n)}E_{ab}] - \epsilon\,\dot n^a\,{}^{(n)}E_{ab} = 0$$

as

$${}^{(n)}E_{ab} = h^e{}_a h^f{}_b\,E_{ef} = E^{(\mathcal{EVOL})}_{ab} + \cancel{h_{ab}\,E^{(\mathcal{H})}}$$

$$K^{ab\,(n)}E_{ab} = \boldsymbol{\kappa}\,\widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\,\widehat{E}^{(\mathcal{M})}_e + \mathbf{K}^{ef}\,\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ef}} + (\mathbf{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})}$$

$$\dot n^{a\,(n)}E_{ab} = [(\widehat{n}_a\dot n^a)\widehat{E}^{(\mathcal{H})} + (\dot n^a\,\widehat{E}^{(\mathcal{M})}_a)]\widehat{n}_b + (\widehat{n}_a\dot n^a)\widehat{E}^{(\mathcal{M})}_b + \dot n^a\,[\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ab}} + \widehat{\gamma}_{ab}\,\widehat{E}^{(\mathcal{H})}]$$

$$\widehat{n}^e D^a[{}^{(n)}E_{ae}] = \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{H})} + \widehat{D}^e\widehat{E}^{(\mathcal{M})}_e + (\widehat{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})} - [\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ef}} + \widehat{\gamma}_{ef}\,\widehat{E}^{(\mathcal{H})}]\,\widehat{K}^{ef} - 2\,\dot{\widehat{n}}^e\widehat{E}^{(\mathcal{M})}_e$$

$$\widehat{\gamma}^e{}_b D^a[{}^{(n)}E_{ae}] = \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{M})}_b + \widehat{D}^e[\cancel{\widehat{E}^{(\mathcal{EVOL})}_{eb}} + \widehat{\gamma}_{eb}\,\widehat{E}^{(\mathcal{H})}] + (\widehat{K}^e{}_e)\,\widehat{E}^{(\mathcal{M})}_b - \dot{\widehat{n}}^e\,\cancel{\widehat{E}^{(\mathcal{EVOL})}_{eb}}$$

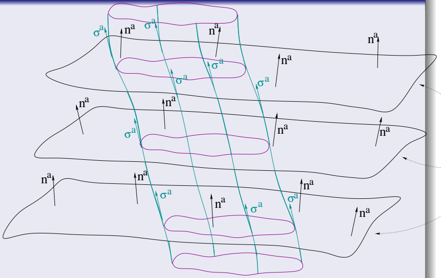
$$\mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{H})} + \widehat{\gamma}^{ef}\,\widehat{D}_e\widehat{E}^{(\mathcal{M})}_f = \widehat{\mathscr{E}}$$
$$\mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{M})}_b + \widehat{D}_b\widehat{E}^{(\mathcal{H})} = \widehat{\mathscr{E}}_b$$

$\Longrightarrow$ **IF** $\widehat{E}^{(\mathcal{EVOL})}_{ef} = 0$ holds: a **linear and homogeneous FOSH** for $(\widehat{E}^{(\mathcal{H})}, \widehat{E}^{(\mathcal{M})}_i)^T$

What do the above observations imply?

Theorem (I.)

- *Assume that the primary constraint expressions* $E^{(\mathcal{H})}$ *and* $E^{(\mathcal{M})}_a$ *vanish on the* $\sigma = const$ *level surfaces, also that*
- *the secondary constraint expressions* $\widehat{E}^{(\mathcal{H})}$ *and* $\widehat{E}^{(\mathcal{M})}_a$ *vanish along the hypersurface yielded by the Lie dragging,* $\mathscr{W}_{\rho_0} = \Phi_\sigma[\mathscr{S}_{\rho_0}]$*, of one of the level surfaces* $\mathscr{S}_{\rho_0}$ *foliating* Σ_0.
- $\Longrightarrow$ *Then, to get solutions to the full set of Einstein's equations* $G_{ab} - \mathscr{G}_{ab} = 0$ *it suffices*—**regardless whether the primary metric** g_{ab} **is Riemannian or Lorentzian**—*to solve, in addition, only the secondary reduced equations* $\widehat{E}^{(\mathcal{EVOL})}_{ij} = 0$.



Remark (i).: *the Lie dragging is done by using the one-parameter group of diffeomorphisms,* Φ_σ*, associated by the "time evolution vector field"* σ^a *— could be only a world-line*

Remark (ii): *if one wants to setup an initial-boundary value problem on either side of the hypersurface* $\mathscr{W}_{\rho_0}$ *the previous theorem provides a clear mean to identify the geometrical freedom we have on* $\mathscr{W}_{\rho_0}$

Relations between various parts of the basic equations II.:

$$K^{ab\,(n)}E_{ab} = 0$$

$$D^a[{}^{(n)}E_{ab}] - \epsilon\, \dot{n}^a\, {}^{(n)}E_{ab} = 0$$

$$K^{ab\,(n)}E_{ab} = \boldsymbol{\kappa}\, \widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\, \widehat{E}^{(\mathcal{M})}_e + \mathbf{K}^{ef}\, \widehat{E}^{(\mathcal{EVOL})}_{ef} + (\mathbf{K}^e{}_e)\, \widehat{E}^{(\mathcal{H})}$$

$$\dot{n}^{a\,(n)}E_{ab} = [(\widehat{n}_a\dot{n}^a)\widehat{E}^{(\mathcal{H})} + (\dot{n}^a\, \widehat{E}^{(\mathcal{M})}_a)]\widehat{n}_b + (\widehat{n}_a\dot{n}^a)\widehat{E}^{(\mathcal{M})}_b + \dot{n}^a\,[\widehat{E}^{(\mathcal{EVOL})}_{ab} + \widehat{\gamma}_{ab}\, \widehat{E}^{(\mathcal{H})}]$$

$$\widehat{n}^e D^a[{}^{(n)}E_{ae}] = \mathscr{L}_{\widehat{n}}\, \widehat{E}^{(\mathcal{H})} + \widehat{D}^e\widehat{E}^{(\mathcal{M})}_e + (\widehat{K}^e{}_e)\, \widehat{E}^{(\mathcal{H})} - [\widehat{E}^{(\mathcal{EVOL})}_{ef} + \widehat{\gamma}_{ef}\, \widehat{E}^{(\mathcal{H})}]\, \widehat{K}^{ef} - 2\,\dot{\widehat{n}}^e\widehat{E}^{(\mathcal{M})}_e$$

$$\widehat{\gamma}^e{}_b D^a[{}^{(n)}E_{ae}] = \mathscr{L}_{\widehat{n}}\, \widehat{E}^{(\mathcal{M})}_b + \widehat{D}^e[\widehat{E}^{(\mathcal{EVOL})}_{eb} + \widehat{\gamma}_{eb}\, \widehat{E}^{(\mathcal{H})}] + (\widehat{K}^e{}_e)\, \widehat{E}^{(\mathcal{M})}_b - \dot{\widehat{n}}^e\, \widehat{E}^{(\mathcal{EVOL})}_{eb}$$

$\Longrightarrow$ if the trace free part of $\widehat{E}^{(\mathcal{EVOL})}_{ef}$ vanishes:

$$\overset{\circ}{\widehat{E}}{}^{(\mathcal{EVOL})}_{ef} = \widehat{E}^{(\mathcal{EVOL})}_{ef} - \tfrac{1}{n-1}\, \widehat{\gamma}_{ef}\, \left[\widehat{\gamma}^{kl}\, \widehat{E}^{(\mathcal{EVOL})}_{kl}\right] = 0$$

$$K^{ab}E^{(\mathcal{EVOL})}_{ab} = (\boldsymbol{\kappa} + \mathbf{K}^e{}_e)\, \widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\, \widehat{E}^{(\mathcal{M})}_e + \tfrac{1}{n-1}\, (\mathbf{K}^e{}_e)\, (\widehat{\gamma}^{kl}\, \widehat{E}^{(\mathcal{EVOL})}_{kl}) = 0$$

$$\mathscr{L}_{\widehat{n}}\, \widehat{E}^{(\mathcal{H})} + \widehat{\gamma}^{ef}\, \widehat{D}_e\widehat{E}^{(\mathcal{M})}_f = \widehat{\mathscr{E}}$$

$$\mathscr{L}_{\widehat{n}}\, \widehat{E}^{(\mathcal{M})}_b - (\mathbf{K}^e{}_e)^{-1}\,[\boldsymbol{\kappa}\, \widehat{D}_b\widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\, \widehat{D}_b\widehat{E}^{(\mathcal{M})}_e] = \widehat{\mathscr{E}}_b$$

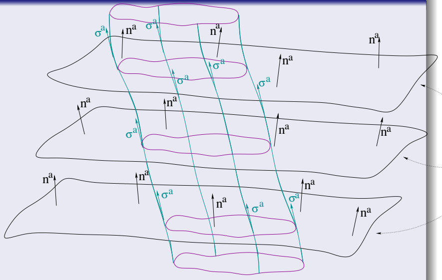
and $\boldsymbol{\kappa}\cdot \mathbf{K}^e{}_e < 0$!!! it is a **linear and homogeneous strongly hyperbolic system**

What is the meaning?

Theorem (II.)

- *Assume that the primary constraint expressions $E^{(\mathcal{H})}$ and $E_a^{(\mathcal{M})}$ vanish on the $\sigma = const$ level surfaces, also that*
- $\boldsymbol{\kappa} \cdot \mathbf{K}^e{}_e < 0$ *(on all $\mathscr{S}_{\sigma,\rho}$) and the secondary constraint expressions $\widehat{E}^{(\mathcal{H})}$ and $\widehat{E}_a^{(\mathcal{M})}$ vanish along the Lie dragging$^{(*)}$, $\mathscr{W}_{\rho_0} = \Phi_\sigma[\mathscr{S}_{\rho_0}]$, of one of the level surfaces $\mathscr{S}_{\rho_0}$ foliating Σ_0.*

  $^{(*)}$ *w.r.t. the one-parameter group of diffeomorphisms, Φ_σ, associated by the "time evolution vector field" σ^a*



Remark: *initial-boundary value problem on either side of the hypersurface $\mathscr{W}_{\rho_0}$..... — could be only a world-line*

- $\Longrightarrow$ *Then, to get solutions to the full set of the primary Einstein's equations $G_{ab} - \mathscr{G}_{ab} = 0$ it suffices*—**regardless whether the primary metric g_{ab} is Riemannian or Lorentzian**—*to solve, in addition, only the trace free part of the secondary reduced equations*

$$\mathring{\widehat{E}}{}_{ef}^{(\mathcal{EVOL})} = \widehat{E}_{ef}^{(\mathcal{EVOL})} - \frac{1}{n-1}\,\widehat{\gamma}_{ef}\,\left[\widehat{\gamma}^{kl}\,\widehat{E}_{kl}^{(\mathcal{EVOL})}\right] = 0$$

The complexity of the field equations has to be reduced I.

Assume: the shift of the "time evolution" vector field σ^a vanishes

- appears to be pretty strong as a requirement...
- !!! Müller and Sánches (2011): it is not as ill and hellish as it looks for the first glance
  - ... (in the Lorentzian case) to any globally hyperbolic spacetime (M, g_{ab}) there always exists a smooth time function $\sigma : M \to \mathbb{R}$ with timelike gradient such that the $\sigma = const$ level surfaces are Cauchy surfaces, and such that the metric can be given in the form

$$g_{ab} = \epsilon\, N^2\, (d\sigma)_a (d\sigma)_b + h_{ab}$$

    with a bounded lapse function $N : M \to \mathbb{R}$, and with a smooth Riemannian metric h_{ab} on the Σ_σ time level surfaces.
- In case of Riemannian spaces one cannot refer to the correspondent of the result of Müller and Sánches.
  - Nevertheless, based on the diffeomorphism invariance of the underlying theory we may simply require, without loss of generality, that the metric g_{ab} possesses the above "canonical form" [Christodoulou and Klainerman (1993)].

The complexity of the field equations can be reduced II.

Introduce the conformal structure by splitting the induced metric $\widehat{\gamma}_{ij}$:

- **Assume:** there exist a smooth function $\Omega : M \to \mathbb{R}$ which does not vanish—except at locations where the foliation $\mathscr{S}_{\sigma,\rho}$ smoothly reduces to a lover dimensional subset on the Σ_σ level surfaces—such that the induced metric $\widehat{\gamma}_{ij}$ can be decomposed as

  $$\widehat{\gamma}_{ij} = \Omega^2 \, \gamma_{ij}$$

  γ_{ij} is singled out by the condition:

  $$\gamma^{ij}(\mathscr{L}_\eta \gamma_{ij}) = 0$$

  that is expected to hold on each of the $\mathscr{S}_{\sigma,\rho}$ surfaces, where η^a stands for either of the "time evolution" vector fields $\sigma^a = (\partial_\sigma)^a$ or $\rho^a = (\partial_\rho)^a$

- $$\gamma^{ij}(\mathscr{L}_\eta \gamma_{ij}) = \mathscr{L}_\eta \, \ln[\det(\gamma_{ij})] = 0$$

  the determinant of γ_{ij} is independent of the coordinates σ and ρ
  !!! it may depend on directions tangential to the level surfaces $\mathscr{S}_{\sigma,\rho}$

Verifications

The conformal structure:

- Does the desired smooth function $\Omega : M \to \mathbb{R}$ and, in turn, the metric γ_{ij} exist?

$$\widehat{\gamma}^{ij}(\mathscr{L}_\eta \widehat{\gamma}_{ij}) = \cancel{\gamma^{ij}(\mathscr{L}_\eta \gamma_{ij})} + (n-1)\,\mathscr{L}_\eta(\ln \Omega^2)$$

  where η^a stands either for σ^a or for ρ^a

- **(i)** start with the smooth distribution of the induced metric $\widehat{\gamma}_{ij}$ on the $\mathscr{S}_{\sigma,\rho}$ surfaces. **(ii)** integrate the above relation first along the integral curves of ρ^a on Σ_0, starting at some $\mathscr{S}_0$, and then along the integral curves of σ^a, starting at the surface $\mathscr{S}_\rho$ on Σ_0, **(iii)** one gets $\Omega^2 = \Omega^2(\sigma, \rho, x^3, \dots, x^{n+1})$ as

$$\Omega^2 = \Omega_0^2 \cdot \exp\left[\tfrac{1}{n-1}\int_0^\rho \left(\widehat{\gamma}^{ij}(\mathscr{L}_\rho \widehat{\gamma}_{ij})\right)\, d\tilde{\rho}\right] \cdot \exp\left[\tfrac{1}{n-1}\int_0^\sigma \left(\widehat{\gamma}^{ij}(\mathscr{L}_\sigma \widehat{\gamma}_{ij})\right)\, d\tilde{\sigma}\right],$$

  where $\Omega_0 = \Omega_0(x^3, \dots, x^{n+1})$ denotes the conformal factor **chosen** at $\mathscr{S}_0$

- Is Ω consistently defined throughout M? ... the integrability condition for $\ln \Omega^2$ is

$$\mathscr{L}_\sigma\left[\widehat{\gamma}^{ij}(\mathscr{L}_\rho \widehat{\gamma}_{ij})\right] - \mathscr{L}_\rho\left[\widehat{\gamma}^{ij}(\mathscr{L}_\sigma \widehat{\gamma}_{ij})\right] = 0$$

  holds as the vector fields σ^a and ρ^a do commute by construction

The decomposition of h_{ij} and K_{ij}:

- in adopted (local) coordinates $(\sigma, \rho, x^3, \ldots, x^{n+1})$ $\boxed{h_{ij}}$ read as

$$\boxed{h_{ij} = (\widehat{N}^2 + \widehat{N}_E \widehat{N}^E)\,(\mathrm{d}\rho)_i (\mathrm{d}\rho)_j + 2\,\widehat{N}_A\,(\mathrm{d}\rho)_{(i}(\mathrm{d}x^A)_{j)} + \widehat{\gamma}_{AB}\,(\mathrm{d}x^A)_i\,(\mathrm{d}x^B)_j}$$

- whereas, as in the adopted coordinates $(\sigma, \rho, x^3, ..., x^{n+1})$, for any $\alpha = 2, 3, \ldots, n = 1 \quad (x^\alpha = \rho, x^A)$

$$\boxed{\mathscr{L}_n (\mathrm{d}x^\alpha)_i = N^{-1}\,\mathscr{L}_\sigma (\mathrm{d}x^\alpha)_i = 0}$$

$$\boxed{K_{ij} = \tfrac{1}{2}\,\mathscr{L}_n h_{ij}}$$

read as

$$\boxed{2\,K_{ij} = \mathscr{L}_n(\widehat{N}^2 + \widehat{N}_E \widehat{N}^E)\,(\mathrm{d}\rho)_i (\mathrm{d}\rho)_j + 2\,\mathscr{L}_n \widehat{N}_A\,(\mathrm{d}\rho)_{(i}(\mathrm{d}x^A)_{j)} + \mathscr{L}_n \widehat{\gamma}_{AB}\,(\mathrm{d}x^A)_i\,(\mathrm{d}x^B)_j}$$

The decomposition of h_{ij} and K_{ij}:

$$2\,K_{ij} = \mathscr{L}_n(\widehat{N}^2 + \widehat{N}_E\widehat{N}^E)\,(\mathrm{d}\rho)_i(\mathrm{d}\rho)_j + 2\,\mathscr{L}_n\widehat{N}_A\,(\mathrm{d}\rho)_{(i}(\mathrm{d}x^A)_{j)} + \mathscr{L}_n\widehat{\gamma}_{AB}\,(\mathrm{d}x^A)_i\,(\mathrm{d}x^B)_j$$

$$K_{ij} = \boldsymbol{\kappa}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\,\mathbf{k}_j + \widehat{n}_j\,\mathbf{k}_i] + \mathbf{K}_{ij}\,,$$

$\widehat{n}_i = \widehat{N}(\mathrm{d}\rho)_i$ & $\widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j$ & $\Longrightarrow$ $\widehat{n}^i(\mathrm{d}\rho)_i = \widehat{N}^{-1}$, $\widehat{n}^i(\mathrm{d}x^A)_i = -\widehat{N}^{-1}\widehat{N}^A$ and $(\mathrm{d}\rho)_j\widehat{\gamma}^j{}_i = 0$

$$\begin{aligned}
\boldsymbol{\kappa} &= \widehat{n}^k\widehat{n}^l\,K_{kl} = \mathscr{L}_n \ln \widehat{N} \\
\mathbf{k}_i &= \widehat{\gamma}^k{}_i\widehat{n}^l\,K_{kl} = (2\widehat{N})^{-1}\,\widehat{\gamma}_{il}\,(\mathscr{L}_n\widehat{N}^l) \\
\mathbf{K}_{ij} &= \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,K_{kl} = \tfrac{1}{2}\,\widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,(\mathscr{L}_n\widehat{\gamma}_{kl})
\end{aligned}$$

$$\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\,\mathbf{K}_{kl} = \tfrac{1}{2}\,\widehat{\gamma}^{ij}(\mathscr{L}_n\widehat{\gamma}_{ij}) = \tfrac{n-1}{2}\,\mathcal{L}_n \ln \Omega^2$$

the (conformal invariant) projection operator reads as

$$\Pi^{kl}{}_{ij} = \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j - \tfrac{1}{n-1}\,\widehat{\gamma}_{ij}\widehat{\gamma}^{kl} = \gamma^k{}_i\gamma^l{}_j - \tfrac{1}{n-1}\,\gamma_{ij}\gamma^{kl}$$

$$\mathring{\mathbf{K}}_{ij} = \Pi^{ef}{}_{ij}\,\mathbf{K}_{ef} = \mathbf{K}_{ij} - \tfrac{1}{n-1}\,\gamma_{ij}(\gamma^{ef}\mathbf{K}_{ef}) = \tfrac{1}{2}\,\Omega^2\,\gamma^k{}_i\gamma^l{}_j\,\mathscr{L}_n\,\gamma_{kl}$$

The $n+1$ constraints

The momentum constraint:

$$E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\,(D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a) = 0$$

It is a first order **symmetric hyperbolic** system for the vector valued variable

$$(\mathbf{k}_B, \mathbf{K}^E{}_E)^T \quad \longrightarrow \quad (\mathcal{L}_n \widehat{N}^B, \mathcal{L}_n \ln \Omega^2)^T$$

where **the 'radial coordinate'** ρ **plays the role of 'time'**

regardless of the value of ϵ

The Hamiltonian constraint:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\,\{-\epsilon\,{}^{(n)}R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$

- it is a parabolic equation for $\widehat{N}$ $\quad [\widehat{K}^l{}_l = \widehat{N}^{-1}\,\overset{\star}{K}\ \&\ \widehat{N}^{-1}\,\widehat{D}^l \widehat{D}_l \widehat{N}]$
- algebraic equation for $\boldsymbol{\kappa} = \mathscr{L}_n \ln \widehat{N}$
- regardless of the value of ϵ

Solving the primary constraints:

The $n(n+1)$ independent components of (h_{ij}, K_{ij}) may be represented by

$$(\widehat{N}, \widehat{N}^i, \Omega, \gamma_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

- or by applying $\boldsymbol{\kappa} = \mathscr{L}_n \ln \widehat{N}$ and $\mathbf{k}_i = (2\widehat{N})^{-1}\, \widehat{\gamma}_{il}\, (\mathscr{L}_n \widehat{N}^l)$

  $\mathbf{K}^l{}_l = \frac{n-1}{2}\, \mathcal{L}_n \ln \Omega^2$ and $\mathring{\mathbf{K}}_{ij} = \frac{1}{2}\, \Omega^2\, \gamma^k{}_i \gamma^l{}_j\, (\mathscr{L}_n \gamma_{kl})$

$$(\widehat{N}, \widehat{N}^i, \Omega, \gamma_{ij}; \mathscr{L}_n \widehat{N}, \mathscr{L}_n \widehat{N}^i, \mathcal{L}_n \Omega, \mathscr{L}_n \gamma_{ij})$$

- The constraints can be solved either
  - (i) as a parabolic–hyperbolic system for

    $$\widehat{N}, \mathscr{L}_n \widehat{N}^i, \mathscr{L}_n \Omega$$

    with freely specifiable variables on Σ_0: $(\widehat{N}^i, \Omega, \boldsymbol{\gamma_{ij}}; \mathscr{L}_n \widehat{N}, \boldsymbol{\mathscr{L}_n \gamma_{ij}})$
  - (ii) or as a symmetrizable hyperbolic system & algebraic equation for

    $$(\mathscr{L}_n \widehat{N}, \mathscr{L}_n \widehat{N}^i, \mathcal{L}_n \Omega)$$

    with freely specifiable variables on Σ_0: $(\widehat{N}, \widehat{N}^i, \Omega, \boldsymbol{\gamma_{ij}}; \boldsymbol{\mathscr{L}_n \gamma_{ij}})$

Principal parts of the secondary constraints:

$\widehat{E}^{(\mathcal{H})} = 0$

$$-2\,\mathscr{L}_n^2\,\Omega + \widehat{D}^l\widehat{D}_l\,\Omega + \Omega\,\widehat{D}^l\widehat{D}_l(\ln N) - \tfrac{1}{n-1}\,{}^{(\gamma)}\!R\,\Omega^{-1} + \{\text{lower order terms}\} = 0$$

$\Longrightarrow$ $\mathscr{L}_n^2\,\Omega = \ldots.$ $\Longrightarrow$ $\mathscr{L}_n\,\Omega = \ldots.$ on $\mathscr{W}_{\rho_0} = \Phi_\sigma[\mathscr{S}_{\rho_0}]$

$\widehat{E}_i^{(\mathcal{M})} = 0$

$$(2\widehat{N})^{-1}\Omega^2\,\gamma_{il}(\mathscr{L}_n^2\widehat{N}^l) + \widehat{N}^{-1}\,\widehat{D}_i(\widehat{D}_l\widehat{N}^l) + \tfrac{1}{n-1}\Omega^2\,\widehat{D}^l[\mathscr{L}_{\widehat{n}}\gamma_{li}] - \widehat{D}_i[\mathscr{L}_{\widehat{n}}(\ln N)]$$
$$+\ \Omega^{-1}\,\widehat{D}_i[\mathscr{L}_{\widehat{n}}\,\Omega] - 2\,(\widehat{N}\Omega)^{-1}\widehat{D}_i[\mathscr{L}_\rho\,\Omega] + \{\text{lower order terms}\} = 0$$

$\Longrightarrow$ $\mathscr{L}_n^2\,\widehat{N}^l = \ldots.$ $\Longrightarrow$ $\mathscr{L}_n\,\widehat{N}^l = \ldots.$ on $\mathscr{W}_{\rho_0} = \Phi_\sigma[\mathscr{S}_{\rho_0}]$

Solubility of the mixed system:

The principal parts of $\widehat{E}^{(\mathcal{EVOL})}_{ij} = 0$:

$$\widehat{E}^{(\mathcal{EVOL})}_{ij} = \Pi^{kl}{}_{ij}\, \widehat{E}^{(\mathcal{EVOL})}_{kl} + \tfrac{1}{n-1}\, \widehat{\gamma}_{ij} \left\{ \widehat{\gamma}^{kl}\, \widehat{E}^{(\mathcal{EVOL})}_{kl} \right\} = 0\,,$$

$$\tfrac{1}{2}\, \gamma^{k}{}_{i}\gamma^{l}{}_{j}(\epsilon\, \mathscr{L}^2_n \gamma_{kl} + \mathscr{L}^2_{\widehat{n}}\gamma_{kl}) + \tfrac{1}{n-1}\, \widehat{N}^{-1}\gamma_{ij}\, \mathscr{L}_{\widehat{n}}(\mathbb{D}_l \widehat{N}^l) + \Omega^{-2}\, \Pi^{kl}{}_{ij}[\mathbb{D}_k\mathbb{D}_l(\ln N + \ln \widehat{N})] + \{\text{lower order terms}\} = 0$$

$$\epsilon\, \mathscr{L}^2_n(\ln \widehat{N}) + \mathscr{L}^2_{\widehat{n}}(\ln \widehat{N}) + \widehat{D}^l\, \widehat{D}_l(\ln \widehat{N}) + \tfrac{n-2}{2(n-1)}\, \Omega^{-2}\, [\gamma^{pq}\gamma^{st}(\partial_t\partial_q\gamma_{ps} - \partial_t\partial_s\gamma_{pq})] + \{\text{lower order terms}\} = 0$$

Solubility of the mixed system:

$$\tfrac{1}{2}\,\gamma^{k}{}_{i}\gamma^{l}{}_{j}(\epsilon\,\mathscr{L}_n^2\gamma_{kl} + \mathscr{L}_{\widehat{n}}^2\gamma_{kl}) + \tfrac{1}{n-1}\,\widehat{N}^{-1}\gamma_{ij}\,\mathscr{L}_{\widehat{n}}(\mathbb{D}_l\widehat{N}^l) + \Omega^{-2}\,\Pi^{kl}{}_{ij}\,[\mathbb{D}_k\mathbb{D}_l(\ln N + \ln\widehat{N})] + \{\text{lower order terms}\} = 0$$

$$\epsilon\,\mathscr{L}_n^2(\ln\widehat{N}) + \mathscr{L}_{\widehat{n}}^2(\ln\widehat{N}) + \widehat{D}^l\widehat{D}_l(\ln\widehat{N}) + \tfrac{n-2}{2(n-1)}\,\Omega^{-2}\,[\gamma^{pq}\gamma^{st}(\partial_t\partial_q\gamma_{ps} - \partial_t\partial_s\gamma_{pq})] + \{\text{lower order terms}\} = 0$$

1. Provided that suitable fields $\Omega, \gamma_{AB}, \widehat{N}^A, \mathscr{L}_n\,\gamma_{AB}, \mathscr{L}_n\,\widehat{N}$ are chosen on Σ_0 **and** $\widehat{N}, \mathscr{L}_n\widehat{N}^A, \mathscr{L}_n\Omega$ on some $\mathscr{S}_{\rho_0}$ $\Longrightarrow$ the parabolic-hyperbolic system $E_b^{(\mathcal{M})} = 0$ & $E^{(\mathcal{H})} = 0$ can be solved for $\widehat{N}, \mathscr{L}_n\widehat{N}^A, \mathscr{L}_n\Omega$
2. Notice that once the fields $\widehat{N}, \mathscr{L}_n\widehat{N}, \gamma_{AB}, \mathscr{L}_n\,\gamma_{AB}$ are known on the initial data surface Σ_0 we have initial data for the above two evolutionary equations.
3. The fields $\Omega, \widehat{N}^A$ can also be determined on the succeeding Σ_σ level surfaces by integrating $\mathscr{L}_n\Omega, \mathscr{L}_n\widehat{N}^A$ along the $\sigma^a = N\,n^a = (\partial/\partial\sigma)^a$ 'time lines'.
4. This way the corresponding inductive process may be closed by evaluating all the fields $\Omega, \mathscr{L}_n\Omega, \widehat{N}, \mathscr{L}_n\widehat{N}, \widehat{N}^A, \mathscr{L}_n\widehat{N}^A, \gamma_{AB}, \mathscr{L}_n\gamma_{AB}$ on the succeeding Σ_σ level surfaces.

Summary:

1. **Euclidean and Lorentzian signature** Einsteinian spaces of $\mathbf{n+1}$-dimension $(\mathbf{n} \geq \mathbf{3})$ were considered. **The topology of** M was restricted by assuming:
   - smoothly foliated by a one-parameter family of homologous hypersurfaces
   - one of these level surfaces is smoothly foliated by a one-parameter family of homologous codimension-two surfaces
2. the **Bianchi identity** and **a pair of nested decompositions** were used **to explore interrelations** of various projections of the field equations
3. a proposal to setup the initial-boundary problem in GR is given by applying some **geometrically distinguished variables** and using the interrelation of various parts of Einstein's equations
4. the **conformal structure** $\boldsymbol{\gamma_{ij}}$, defined on the foliating codimension-two surfaces $\mathscr{S}_{\sigma,\rho}$, appears to provide a **convenient embodiment** of the $\frac{(n-1)\,n}{2} - 1$ degrees of freedom in Einstein's theory of gravity
5. **!!!** all these results apply **regardless** whether the primary space is Riemannian or Lorentzian

That is all...