On the use of evolutionary methods in metric theories of gravity XI.

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Plans and Aims:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

- **mass aspects in Einstein theory**
  - is there a mass aspect in the initial data?
  - Misner-Sharp mass
  - Geroch mass
  - monotonicity

    ... variation of the area
    ... variation of the Geroch mass
    ... construction of initial data with monotonous Geroch mass

Based on:

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, Class. Quantum Grav. 31 (2014) 155004
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. 32 (2015) 015006
- I. Rácz: *Constraints as evolutionary systems*, Class. Quantum Grav. **33** 015014 (2016)
- I. Rácz: *Construction of initial data with monotonous Geroch mass*, in preparation (2019)

Motivations:

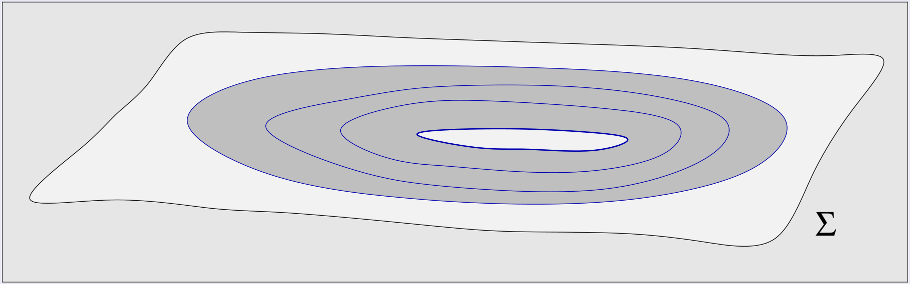
GR is a **metric theory of gravity**:

- it is highly non-trivial to talk about concepts of obvious physical interest such as the mass, energy, linear and angular momenta of bounded spatial regions
- "... it is almost certain that we have to understand conserved (or quasi conserved) quantities which can control the field in a more local manner. In other words, we expect some concept of quasi-local mass will be useful."
- efforts to prove the **positive mass theorem** and the **Penrose inequalities** using quasi-local techniques Geroch (1973), Wald, Jang (1977), Jang (1978), Kijowski (1986), Chruściel (1986), Jezierski, Kijowski (1987), Huisken, Ilmanen (1997, 2001), Frauendiener (2001), Bray (2001), Malec, Mars, Simon (2002), Bray, Lee (2009),. . .

The aim is to outline:

- a **simple construction** of a high variety of Riemannian three-spaces with prescribed, whence globally existing regular foliation and flow such that
    - the (quasi-local) **Geroch mass**—that can be evaluated on the leaves of the foliations—is **non-decreasing** with respect to the applied flow
    - the foliation gets to be **quasi-convex** w.r.t. the constructed three-metric

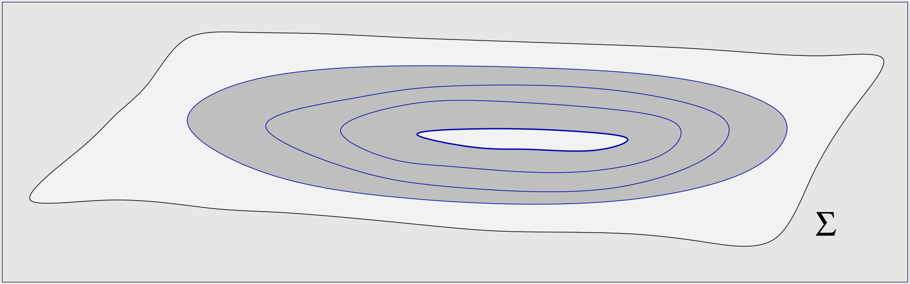
Foliations by topological two-spheres:



- consider a smooth 3-dimensional manifold Σ with a Riemannian metric h_{ij}
- assume $\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$ $origin(s)(!)$

  i.e. Σ is smoothly foliated by a one-parameter family of top. 2-spheres $\mathscr{S}_\rho$: $\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \rightarrow \mathbb{R}$ with $\partial_i \rho \neq 0$
- $\Longrightarrow$ $\boxed{\partial_i\rho \;\; \& \;\; h^{ij} \longrightarrow \widehat{n}_i \,, \widehat{n}^i = h^{ij}\widehat{n}_j \;\; \dots \;\; \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j}$
- $\cancel{\widehat{n}^i}$, '$\widehat{\ }$' to distinguish quantities that could also be viewed as fields on the leaves

Quasi-convex foliations:



- **the induced Riemannian metric** on the $\mathscr{S}_\rho$ level sets

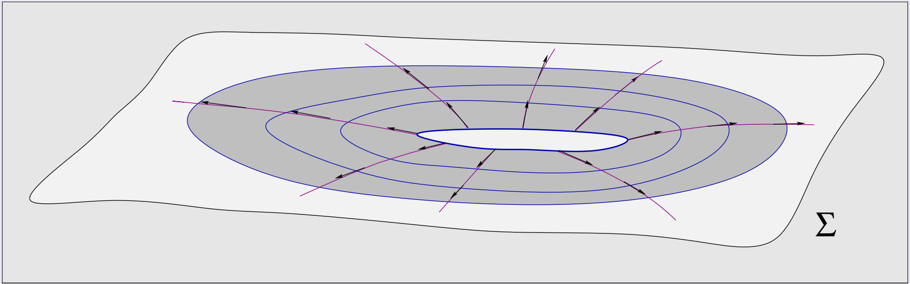
$$\boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j h_{kl}}$$

- **the extrinsic curvature** given by the symmetric tensor field

$$\boxed{\widehat{K}_{ij} = \widehat{\gamma}^l{}_i D_l \, \widehat{n}_j = \tfrac{1}{2} \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij},} \qquad D_i, \mathscr{L}_{\widehat{n}}$$

- a $\rho = const$ level surface is called to be **quasi-convex if its mean curvature**, $\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij} \widehat{K}_{ij} = \widehat{\gamma}^{ij} D_i \, \widehat{n}_j}$, is positive on $\mathscr{S}_\rho$

Flows:



- a smooth vector field ρ^i on Σ **is a flow** ("evolution vector field") w.r.t. $\mathscr{S}_\rho$
  - if the integral curves of ρ^i **intersect each leaves precisely once**, and
  - if ρ^i **is scaled such that** $\boxed{\rho^i\partial_i\rho = 1}$ holds throughout Σ
- any smooth flow can be decomposed in terms of its '**lapse**' and '**shift**' as

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i} \qquad \boxed{\widehat{N} = \rho^i\widehat{n}_i = (\widehat{n}^i\partial_i\rho)^{-1}} \quad \boxed{\widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j}$$

  - the lapse **measures the normal separation of the surfaces** $\mathscr{S}_\rho$

Variation of the area:

- to any quasi-convex foliation ∃ a (quasi-local) **orientation of the leaves** $\mathscr{S}_\rho$
- a flow ρ^i is called **outward pointing** if the area is increasing w.r.t. it
- variation of the area $\boxed{\mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \widehat{\boldsymbol{\epsilon}}}$ of the $\rho = const$ level surfaces, w.r.t. ρ^i

$$\boxed{\mathscr{L}_\rho \mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \mathscr{L}_\rho \widehat{\boldsymbol{\epsilon}} = \int_{\mathscr{S}_\rho} \left\{ \widehat{N}(\widehat{K}^l{}_l) + (\widehat{D}_i \widehat{N}^i) \right\} \widehat{\boldsymbol{\epsilon}} = \int_{\mathscr{S}_\rho} \widehat{N}(\widehat{K}^l{}_l)\, \widehat{\boldsymbol{\epsilon}},}$$

  the relations $\mathscr{L}_{\widehat{n}}\, \widehat{\boldsymbol{\epsilon}} = (\widehat{K}^l{}_l)\, \widehat{\boldsymbol{\epsilon}}$ and $\mathscr{L}_{\widehat{N}}\, \widehat{\boldsymbol{\epsilon}} = \frac{1}{2}\, \widehat{\gamma}^{ij} \mathscr{L}_{\widehat{N}} \widehat{\gamma}_{ij}\, \widehat{\boldsymbol{\epsilon}} = (\widehat{D}_i \widehat{N}^i)\, \widehat{\boldsymbol{\epsilon}}$, along with the vanishing of the integral of the total divergence $\widehat{D}_i \widehat{N}^i$, were applied.

- $\widehat{N}$ does not vanish on Σ unless the Riemannian three-metric

$$\boxed{h^{ij} = \widehat{\gamma}^{ij} + \widehat{N}^{-2}(\rho^i - \widehat{N}^i)(\rho^j - \widehat{N}^j)}$$

  gets to be singular

- for **quasi-convex foliations** $\boxed{\widehat{N}\widehat{K}^l{}_l > 0} \Longrightarrow$ the **area is increasing** w.r.t. ρ^i
- the orientations by $\widehat{n}^i$ and ρ^i coincide

Where is the mass in the initial data?

The decomposition of the three-metric h_{ij}:

- in adopted (local) coordinates (σ, ρ, x^3, x^4) $\boxed{h_{ij}}$ read as

$$\boxed{h_{ij} = (\widehat{N}^2 + \widehat{N}_E \widehat{N}^E)\,(\mathrm{d}\rho)_i (\mathrm{d}\rho)_j + 2\,\widehat{N}_A\,(\mathrm{d}\rho)_{(i}(\mathrm{d}x^A)_{j)} + \widehat{\gamma}_{AB}\,(\mathrm{d}x^A)_i\,(\mathrm{d}x^B)_j}$$

$$\boxed{h_{ij} = \begin{pmatrix} \widehat{N}^2 + \widehat{N}_E \widehat{N}^E & \gamma_{AF}\widehat{N}^F \\ \gamma_{BG}\widehat{N}^G & \widehat{\gamma}_{AB} \end{pmatrix}}$$

$$\boxed{h^{ij} = \begin{pmatrix} \widehat{N}^{-2} & \widehat{N}^{-2}\widehat{N}^A \\ \widehat{N}^{-2}\widehat{N}^B & \widehat{\gamma}^{AB} - \widehat{N}^{-2}\widehat{N}^A\widehat{N}^B \end{pmatrix}}$$

- Schwarzschild spacetime

$$\boxed{ds^2 = -\left(1 - \tfrac{2\,M}{r}\right) dt^2 + \left(1 - \tfrac{2\,M}{r}\right)^{-1} dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)}$$

$$\boxed{\left(1 - \tfrac{2\,M}{r}\right) = \widehat{N}^{-2}} \quad \rightleftarrows \quad \boxed{M = \tfrac{r}{2}\left(1 - \widehat{N}^{-2}\right)}$$

What is the mass?

Various relations:

- the metric of as generic four-dimensional spherically symmetric spacetimes

$$\boxed{\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B + r^2(\mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2)}$$

g_{AB} Lorentzian two-metric; g_{AB} and r are smooth functions of x^A, excl.

- on the $x^1 = const$ surfaces the only non-trivial component is g^{22}
- Misner-Sharp mass (in spherically symmetric spacetimes)

$$\text{if} \quad \mathrm{x}^2 = \mathrm{r} \quad \boxed{M = \tfrac{r}{2}\,(1 - g^{rr})} \quad , \text{or} \quad \boxed{M = \tfrac{r}{2}\,\left(1 - g^{ab}\partial_a r \partial_b r\right)}$$

where r is assumed to be an area radial coordinate: i.e. for the Area

$$\boxed{\mathscr{A}_r \,(= \textstyle\int_{\mathscr{S}} \widehat{\boldsymbol{\epsilon}}) \; = 4\,\pi\, r^2}$$

$$\boxed{M = \sqrt{\tfrac{\mathscr{A}_r}{16\pi}}\,\left(1 - g^{ab}\partial_a r \partial_b r\right)}$$

Misner-Sharp mass:

- Misner-Sharp mass (in spherically symmetric spacetimes)

$$\boxed{M = \sqrt{\frac{\mathscr{A}_r}{16\pi}}\left(1 - \widehat{N}^{-2}\right) = \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}}\int_{\mathscr{S}}\left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2\right]\widehat{\boldsymbol{\epsilon}}}$$

- **Gauss-Bonnet theorem:** The integral of the Gauss curvature $\mathscr{K}_G = \frac{1}{2}\,\widehat{R}$ is a topological invariant

$$\boxed{\int_{\mathscr{S}}\mathscr{K}_G\,\widehat{\boldsymbol{\epsilon}} = 2\,\pi\,\chi_{\mathscr{S}}}$$

- a strange writing of '1': $\boxed{1 = (16\pi)^{-1}\int_{\mathscr{S}} 2\,\widehat{R}\,\widehat{\boldsymbol{\epsilon}}}$ (for topological spheres)

$$\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij}(\tfrac{1}{2}\,\mathscr{L}_{\widehat{n}}\widehat{\gamma}_{ij}) = N^{-1}(\tfrac{1}{2}\,\widehat{\gamma}^{ij}\,\mathscr{L}_\rho\widehat{\gamma}_{ij}) = N^{-1}\partial_r\ln(\sqrt{\det(\gamma^{ij})}) = \left(\tfrac{2}{r}\right)N^{-1}}$$

- a strange writing of $\widehat{N}^{-2}$:

$$\boxed{\widehat{N}^{-2} = \left(\frac{r^2}{4}\right)(\widehat{K}^l{}_l)^2 = \left(\frac{4\pi r^2}{16\pi}\right)(\widehat{K}^l{}_l)^2 = (16\pi)^{-1}\int_{\mathscr{S}}(\widehat{K}^l{}_l)^2\,\widehat{\boldsymbol{\epsilon}}}$$

The Geroch mass:

- the (quasi-local) Geroch mass (equal to the Hawking mass only if $K^{i}{}_{i} = 0$)

$$\boxed{m_{\mathcal{G}} = \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^{l}{}_{l})^2 \right] \widehat{\boldsymbol{\epsilon}}}$$

  where $\widehat{R}$ is the scalar curvature of the metric $\widehat{\gamma}_{ij}$ on the leaves
- for quasi-convex foliations the area $\mathscr{A}_\rho$ is monotonously increasing
- it suffices to investigate

$$\boxed{W(\rho) = \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^{l}{}_{l})^2 \right] \widehat{\boldsymbol{\epsilon}}}$$

- **if both $\mathscr{A}_\rho$ and $W(\rho)$ were non-decreasing**, and for some specific ρ_* value, $W(\rho_*)$ was zero or positive then $\boxed{m_{\mathcal{G}} \geq 0}$ would hold to the exterior of $\mathscr{S}_{\rho_*}$ in Σ

The variation of $W(\rho)$:

- the **key equation** we shall use **relates the scalar curvatures** of h_{ij} and $\widehat{\gamma}_{ij}$

$$\boxed{{}^{(3)}\!R = \widehat{R} - \Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\Big\} \qquad (*)}$$

$$\begin{aligned}
\mathscr{L}_\rho W &= -\int_{\mathscr{S}_\rho}\mathscr{L}_\rho\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol{\epsilon}}\Big] = -\int_{\mathscr{S}_\rho}\Big\{\widehat{N}\,\mathscr{L}_{\widehat{n}}\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol{\epsilon}}\Big] + \mathscr{L}_{\widehat{N}}\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol{\epsilon}}\Big]\Big\}\\
&= -\int_{\mathscr{S}_\rho}(\widehat{N}\widehat{K}^l{}_l)\,\Big[2\,\mathscr{L}_{\widehat{n}}\,(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2\Big]\,\widehat{\boldsymbol{\epsilon}} - \int_{\mathscr{S}_\rho}\widehat{D}_i\big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{N}^i\,\big]\widehat{\boldsymbol{\epsilon}}\\
&= -\int_{\mathscr{S}_\rho}(\widehat{N}\widehat{K}^l{}_l)\,\Big[\,(\,\widehat{R} - {}^{(3)}\!R\,) - \widehat{K}_{kl}\widehat{K}^{kl} - 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\,\Big]\,\widehat{\boldsymbol{\epsilon}}
\end{aligned}$$

- where on 1^{st} line $\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i$ and the Gauss-Bonnet theorem
- on 2^{nd} line the relations $\mathscr{L}_{\widehat{n}}\,\widehat{\boldsymbol{\epsilon}} = (\widehat{K}^l{}_l)\,\widehat{\boldsymbol{\epsilon}}$ and $\mathscr{L}_{\widehat{N}}\,\widehat{\boldsymbol{\epsilon}} = (\widehat{D}_i\widehat{N}^i)\,\widehat{\boldsymbol{\epsilon}}$
- on 3^{rd} line (*) and the vanishing of the integral of $\widehat{D}_i\big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{N}^i\,\big]$ were used

The variation of $W(\rho)$:

- by the Leibniz rule

$$\boxed{\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N} = \widehat{D}^l\big(\widehat{N}^{-1}\widehat{D}_l\widehat{N}\big) + \widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})}$$

- and by introducing the trace-free part of $\widehat{K}_{ij}$

$$\boxed{\overset{\circ}{\widehat{K}}_{ij} = \widehat{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\,(\widehat{K}^l{}_l), \qquad \widehat{K}_{kl}\widehat{K}^{kl} = \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + \tfrac{1}{2}\,(\widehat{K}^l{}_l)^2}$$

- and using the vanishing of the integral of the total divergence $\widehat{D}^l(\widehat{N}^{-1}\widehat{D}_l\widehat{N})$

$$\boxed{\begin{aligned}\mathscr{L}_\rho W = {} & -\tfrac{1}{2}\int_{\mathscr{S}_\rho}(\widehat{N}\widehat{K}^l{}_l)\left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2\right]\widehat{\boldsymbol{\epsilon}} \\ & + \int_{\mathscr{S}_\rho}(\widehat{N}\,\widehat{K}^l{}_l)\left[{}^{(3)}\!R + \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\right]\widehat{\boldsymbol{\epsilon}}\end{aligned}}$$

Rigidity of the setup:

- **if** the product $\boxed{\widehat{N}\widehat{K}^{l}{}_{l}}$ **could be replaced** by its **mean value**
- $$\boxed{\overline{\widehat{N}\widehat{K}^{l}{}_{l}} = \frac{\int_{\mathscr{S}_\rho} \widehat{N}\widehat{K}^{l}{}_{l}\,\widehat{\boldsymbol{\epsilon}}}{\int_{\mathscr{S}_\rho}\widehat{\boldsymbol{\epsilon}}}} \qquad \boxed{\overline{\widehat{N}\widehat{K}^{l}{}_{l}} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

$$\boxed{[(64\,\pi^{3/2})/(\mathscr{A}_\rho)^{1/2}]\cdot\mathscr{L}_\rho\, m_{\mathcal{G}} = \mathscr{L}_\rho W + \tfrac{1}{2}\,(\mathscr{L}_\rho \log[\mathscr{A}_\rho])\,W \geq 0}$$

provided that $$\boxed{\int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}\!R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\Big]\,\widehat{\boldsymbol{\epsilon}}}$$

- once in addition to h_{ij} **a foliation and a flow are fixed** not only the **mean curvature** $\widehat{K}^{l}{}_{l}$ **BUT** the **lapse** $\widehat{N}$ and the **shift** $\widehat{N}^i$ get also to be fixed

$$\boxed{\widehat{K}^{l}{}_{l} = \widehat{\gamma}^{ij}\widehat{K}_{ij} = \widehat{\gamma}^{ij}D_i\,\widehat{n}_j} \qquad \boxed{\widehat{N} = \rho^i\widehat{n}_i = (\widehat{n}^i\partial_i\rho)^{-1}} \qquad \boxed{\widehat{N}^i = \widehat{\gamma}^{i}{}_{j}\,\rho^j}$$

- the only "freedom" is a relabeling of the leaves by using a function $\overline{\rho} = \overline{\rho}(\rho)$ but this cannot yield more than a rescaling $\widehat{N} \to \widehat{N}(\mathrm{d}\rho/\mathrm{d}\overline{\rho})$ of the lapse
- **(!)** at best $\boxed{\widehat{N}\widehat{K}^{l}{}_{l}}$ is a smooth positive function on the leaves of the foliation

How to get control on the monotonicity?

What we have by hands: $\{\widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}\,;\, \rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$

- a Riemannian metric h_{ij} defined on a three-surface Σ
- Σ is foliated by topological two-spheres: $\Sigma \approx \mathbb{R}\times\mathbb{S}^2$; $\quad \rho:\Sigma\to\mathbb{R}$ is chosen
- a flow ρ^i was also fixed on Σ such that $\rho^i\partial_i\rho = 1$
- the later two can be used to introduce coordinates (ρ, x^A) adapted to the flow:

  $$\boxed{\rho^i = (\partial_\rho)^i \ \leftrightarrow\ \delta^i{}_\rho}\,, \qquad \boxed{\widehat{N}^i = \delta^i{}_A\widehat{N}^A \quad \text{and} \quad \widehat{\gamma}_{ij} = \delta^A{}_i\,\delta^B{}_j\widehat{\gamma}_{AB}}$$

  $\widehat{N}^A$ and $\widehat{\gamma}_{AB}$ depend smoothly on ρ, x^A, where A takes the values $2, 3$
- line element of the Riemannian metric h_{ij}

  $$\boxed{\mathrm{d}s^2 = \widehat{N}^2\mathrm{d}\rho^2 + \widehat{\gamma}_{AB}\left(\mathrm{d}x^A + \widehat{N}^A\mathrm{d}\rho\right)\left(\mathrm{d}x^B + \widehat{N}^B\mathrm{d}\rho\right)}$$

The challenge is:

- choose a maximal subset of the fields $\{h_{ij}\,;\,\rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]} \qquad \boxed{{}^{(3)}R + \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N}) \geq 0}$$

Solution 1°: using the inverse mean curvature flow (IMCF)

- choose a maximal subset of the fields $\{h_{ij}\,;\,\rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l=\overline{\widehat{N}\widehat{K}^l{}_l}=\mathscr{L}_\rho\log[\mathscr{A}_\rho]}\qquad\boxed{{}^{(3)}R+\mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl}+2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\geq 0}$$

- what is if we keep (Σ,h_{ij}) but drop $\rho:\Sigma\to\mathbb{R}$ and the shift from $\rho^i=(\partial_\rho)^i$

The foliation and part of the flow is to be determined dynamically

- the **inverse mean curvature flow**

$$\boxed{\rho^i_{\{IMCF\}}=\big(\widehat{K}^l{}_l\big)^{-1}\,\widehat{n}^i+\widehat{N}^i_{\{IMCF\}}}$$

- as for the corresponding foliation $\widehat{N}\widehat{K}^l{}_l\equiv 1$ hold: if this flow existed globally the Geroch mass would be non-decreasing w.r.t it
- one can relax these condition by using a generalized IMCF

$$\boxed{\rho^i=\mathscr{L}_\rho(\log[\mathscr{A}_\rho])\,\rho^i_{\{IMCF\}}}$$

- **(!)** global existence and regularity remains a serious issue

Solution 2°: using a prescribed, globally existing foliation

- choose a maximal subset of the fields $\{h_{ij}\,;\,\rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l=\overline{\widehat{N}\widehat{K}^l{}_l}=\mathscr{L}_\rho\log[\mathscr{A}_\rho]}\qquad \boxed{{}^{(3)}R+\mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl}+2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\geq 0}$$

- what is if we drop the three-metric h_{ij} BUT choose a globally well-defined foliation $\rho:\Sigma\to\mathbb{R}$, a flow ρ^i and the induced metric $\widehat{\gamma}_{ij}$ on the leaves: in coordinates (ρ,x^A) adapted to the flow $\rho^i=(\partial_\rho)^i$ the induced metric: $\widehat{\gamma}_{AB}$

Using prescribed foliation, flow, induced metric: $h_{ij}\leftrightarrow\widehat{N},\ \widehat{N}^A,\ \widehat{\gamma}_{AB}$

- $\boxed{\rho^i=\widehat{N}\,\widehat{n}^i+\widehat{N}^i}$ however counterintuitive it is: we may always construct shift $\widehat{N}^i$ with desirable properties:

$$\boxed{\widehat{N}\widehat{K}^l{}_l=\tfrac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij}-\widehat{D}_i\widehat{N}^i}$$

- as $\widehat{N}\widehat{K}^l{}_l=\overline{\widehat{N}\widehat{K}^l{}_l}=\mathscr{L}_\rho\log[\mathscr{A}_\rho]$ wished to be guaranteed,

$$\boxed{\widehat{D}_A\widehat{N}^A=\mathscr{L}_\rho\log\big[\sqrt{\det(\widehat{\gamma}_{AB})}\,\big]-\mathscr{L}_\rho\log[\mathscr{A}_\rho]\qquad(**)}$$

Solution 2°: using prescribed foliation, flow and $\widehat{\gamma}_{AB}$

Solving $\widehat{D}_A\widehat{N}^A = \mathscr{L}_\rho \log[\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ (**) on $\mathscr{S}_\rho$

- on topological two-spheres using then the Hodge decomposition of the shift $\boxed{\widehat{N}^A = \widehat{D}^A\chi + \widehat{\epsilon}^{AB}\widehat{D}_B\eta}$, χ and η are some smooth functions on $\mathscr{S}$, (**)

$$\boxed{\widehat{D}^A\widehat{D}_A\chi = \mathscr{L}_\rho \log[\,\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

- solubility of this parabolic equation level surface by level surface is guaranteed
- standard Schauder estimates guarantee that if the coefficients and the source terms are are smooth function of the parameter ρ then the solutions will also depend smoothly on ρ.

We have not done yet(!) ${}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N}) \geq 0$

- in clearing up the picture have a glance again of the key equation

$$\boxed{{}^{(3)}R = \widehat{R} - \Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\Big\} \qquad (*)}$$

A parabolic equation for $\widehat{N}$:

- as noticed first by Bartnik (1993), while applying quasi-spherical foliations (*) can be viewed as a parabolic equation for $\widehat{N}$
- remarkably, (*) **can always be seen to be a parabolic eqn** for $\widehat{N}$ **IF** ${}^{(3)}R$, $\widehat{\gamma}_{AB}$ and $\widehat{N}^A$ can be treated as prescribed fields
- introducing $\boxed{\overset{\star}{K}_{AB} = \widehat{N}\widehat{K}_{AB}}$ and $\boxed{\overset{\star}{K} = \frac{1}{2}\,\widehat{\gamma}^{AB}\mathscr{L}_\rho\widehat{\gamma}_{AB} - \widehat{D}_A\widehat{N}^A}$ to **eliminate hidden occurrence** of the lapse in (*) we get

$$\boxed{\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - \widehat{N}^A(\hat{D}_A\widehat{N})\,] = \widehat{N}^2(\widehat{D}^A\widehat{N}_A\widehat{N}) + \mathcal{A}\,\widehat{N} - \tfrac{1}{2}\,(\,\widehat{R} - {}^{(3)}R\,)\,\widehat{N}^3}$$

where $\boxed{\mathcal{A} = \partial_\rho\overset{\star}{K} + \frac{1}{2}\,[\overset{\star}{K}{}^2 + \overset{\star}{K}_{AB}\overset{\star}{K}{}^{AB}]}$ with $\boxed{\overset{\star}{K} = \overline{\widehat{N}\widehat{K}^A}{}_A = \mathscr{L}_\rho\log[\mathscr{A}_\rho] > 0}$

- it is standard to obtain **existence of unique solutions to this (Bernoulli type) uniformly parabolic PDE** in a sufficiently small one-sided neighborhood of $\mathscr{S}$ in Σ

Global existence of unique solutions:

- our main concern is **global existence (!)**
- it should not come as a surprise that **an analogous parabolic equation** came up **in deriving the evolutionary form** of the Hamiltonian constraints in [Rácz I: *Constrains as evolutionary systems*, Class. Quant. Grav. **33** 015014 (2016)]
- if, e.g., $\boxed{{}^{(3)}R + \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} = 0}$ **global unique solutions exist** to

$$\boxed{\overset{\star}{K}[(\partial_\rho \widehat{N}) - \widehat{N}^A(\hat{D}_A\widehat{N})] = \widehat{N}^2(\widehat{D}^A\widehat{N}_A\widehat{N}) + (\partial_\rho \overset{\star}{K} + \tfrac{3}{4}\,\overset{\star}{K}{}^2)\widehat{N} - \tfrac{1}{2}\,\widehat{R}\widehat{N}^3}$$

- for **any smooth positive initial data** ${}_0\widehat{N}$ on some $\mathscr{S}_{\rho_0}$ a unique **positive bounded** solution $\widehat{N}$ **exists for all** $\rho \geq \rho_0$
- if $\Sigma \approx \mathbb{R}^3$ and the freely specifiable data $\widehat{\gamma}_{AB}$ is chosen such that suitable integral terms approximate their "asymptotically flat forms" then in the $\rho \to \infty$ limit $\widehat{N} \to 1$ can also be guaranteed

Summary:

a **simple construction** of a high variety of Riemannian three-spaces with prescribed, whence globally existing regular foliation and flow:

1. the (quasi-local) **Geroch mass**—that can be evaluated on the leaves of the foliations—is **non-decreasing** with respect to the applied flow
2. the foliation gets to be **quasi-convex** w.r.t. the constructed h_{ij} : $\widehat{N}\widehat{K}^l{}_l = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ & the flow gets to be a (generalized) IMCF
3. **ultimate aim** is to construct initial data sets with these properties
4. the topology of Σ **could be**: $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{R}\times\mathbb{S}^2$, $\mathbb{S}^1\times\mathbb{S}^2$, $(1,2,0,0)$
5. the construction **applies to wide range of geometrized theories of gravity**
    - concerning the metric (on M or on Σ): no use of Einstein's equations or any other field equation had been applied anywhere in our construction
    - as only the Riemannian character of the metric on Σ was used the signature of the metric on the ambient space could be either Lor. or Euc.