On the use of evolutionary methods in metric theories of gravity XII.

István Rácz
istvan.racz@fuw.edu.pl & racz.istvan@wigner.mta.hu
Faculty of Physics, University of Warsaw, Warsaw, Poland
Wigner Research Center for Physics, Budapest, Hungary

Supported by the POLONEZ programme of the National Science Centre of Poland which has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 665778.

National Science Centre Poland

European Commission | Horizon 2020 European Union funding for Research & Innovation

Institute of Theoretical Physics, University of Warsaw
Warsaw, 18 January 2019

Plans and Aims:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

- **horizons**
  - trapped codimenson-two surfaces, trapped regions
  - the generic local definition of MOTS
  - black holes

    ... event horizons

    ... dynamical horizons
- **evolutionary characterization of dynamical horizons**
  - recovering what was deduced by Bartnik and Isenberg and also get some new insight even in that case

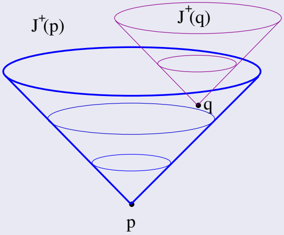
    .... in the spherically symmetric case

Based on:

- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. 32 (2015) 015006
- I. Rácz: *Constraints as evolutionary systems*, Class. Quantum Grav. **33** 015014 (2016)
- I. Rácz: *Evolutionary aspects of dynamical horizons*, to be published (2019)

What is a black hole?

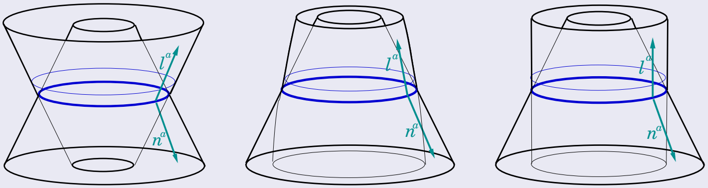
- **loosely speaking:**
  - synonym: region of no escape
  - no information can escape regardless of the physical processes we intend to use to forward messages



  - Be careful! one has to restrict attention to properly trapped surfaces as according to the previous definition any future set of Minkowski spacetime could be considered as a black hole ... better not to do so
  - **R. Penrose:** *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** 54-59 (1965) *introduced the concept of trapped surfaces*

Trapped surfaces

- consider the boundary of smooth spatially compact region
- flash some point-like light source at each point of this boundary
- consider the common wave-fronts emanating from the points of the boundary
- in normal situations there will be a front moving inward and another one moving outward with decreasing and increasing area, respectively



non-trapped $\theta^{(\ell)} > 0$ | *trapped* $\theta^{(\ell)} < 0$ | *marginally trapped* $\theta^{(\ell)} = 0$

- how to characterize this? introduce the concept of null expansion w.r.t. the outward pointing future directed null ℓ^a (the rate of the change of the volume form)

$$\mathscr{L}_\ell\, \boldsymbol{\epsilon}_q = \theta^{(\ell)}\, \boldsymbol{\epsilon}_q$$

Black hole:

- **Intiutively:** comprised by points which belong to some future trapped surfaces
- this is a completely quasi-local determination
- there exist a global definition too
  - this is even more popular or more widely used
  - the region that is missing from the observed or visible world of those observers who stay far away from the black hole region
  - as they can even exchange messages among each other the name of this region: the domain of outer communication: $\mathscr{D}$
- future event horizon $\mathscr{H}^+$: the boundary of the causal past of $\mathscr{D}$:

$$\boxed{\mathscr{H}^+ = \partial J^-[\mathscr{D}]}$$

- the black hole is the complement of the causal past of $\mathscr{D}$:

$$\boxed{\mathscr{B} = M \setminus J^-[\mathscr{D}]}$$

Schwarzschild spacetime

$$ds^2 = -\left(1 - \tfrac{2\,m}{r}\right) dt^2 + \left(1 - \tfrac{2\,m}{r}\right)^{-1} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\!\vartheta \, d\varphi^2\right)$$

- $M = \mathbb{R}^2 \times \mathbb{S}^2$
  - t and r are such that $-\infty < t < \infty$, $0 \leq r < \infty$
  - whereas ϑ and φ are with their usual spherical ranges
- asymptotically flat: gets Minkowski while $r \to \infty$
- mass parameter: m
- singular at $r = 0$ and $r = R_S = 2\,m$
- the latter is called coordinate singularity as new coordinates can be used such that the latent singularity goes away
- **Kruskal-Szekeres:** (T, X) instead of (t, r)

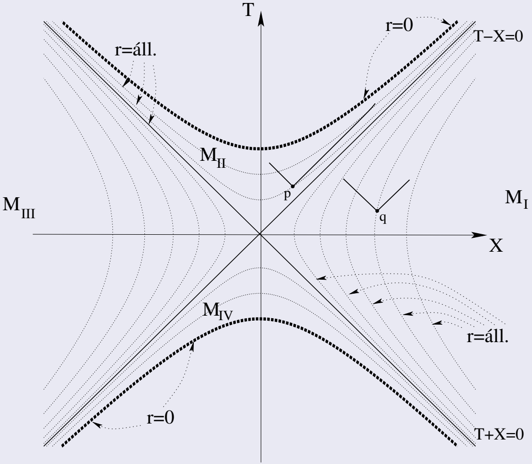
$$\left(\tfrac{r}{2\,m} - 1\right) e^{\frac{r}{2\,m}} = X^2 - T^2 \quad \& \quad \tanh\left(\tfrac{t}{4m}\right) = \tfrac{T}{X}$$

- in coordinates $(T, X, \vartheta, \varphi)$

$$ds^2 = \frac{32\, m^3 e^{-\frac{r}{2\,m}}}{r} \left(-dT^2 + dX^2\right) + r^2 \left(d\vartheta^2 + \sin^2\!\vartheta \, d\varphi^2\right)$$

$$ds^2 = \frac{32\, m^3 e^{-\frac{r}{2\,m}}}{r}\left(-dT^2 + dX^2\right) + r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right)$$

- conformally flat in the (T, X) section
- $r = 0 \quad \rightleftarrows \quad T^2 - X^2 = 1$ hyperbolas



Dynamical horizons:

It is a very special initial data surface:

- a spacelike hypersurface Σ that may tend to be null asymptotically
- it is smoothly foliated by MOTS: marginally outer trapped surfaces
- the geometric part of the initial data can be represented by a pair of smooth fields (h_{ij}, K_{ij}) on Σ
  - h_{ij} is a Riemannian metric while K_{ij} is a symmetric tensor field there
  - the restrictions coming from Einstein's equations $G_{ij} - \mathscr{G}_{ij} = 0$, with a source $\mathscr{G}_{kl}$ of vanishing divergence, on Σ are the constraints

$$
\begin{aligned}
{}^{(3)}\!R + \left(K^j{}_j\right)^2 - K_{ij}K^{ij} - 2\,\mathfrak{e} &= 0\,,\\
D_jK^j{}_i - D_iK^j{}_j + \mathfrak{p}_i &= 0
\end{aligned}
$$

  - ${}^{(3)}\!R$ and D_i denote the scalar curvature and the covariant derivative operator associated with h_{ij}, respectively
  - the source terms are as $\mathfrak{e} = n^k n^l\, \mathscr{G}_{kl}$ and $\mathfrak{p}_i = -h^k{}_i n^l\, \mathscr{G}_{kl}$

New variables by applying $2+1$ decompositions:

Splitting of the metric h_{ij}:

assume: $\Sigma \approx \mathbb{R} \times \mathscr{S}$

Σ is smoothly foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$:
$\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\Longrightarrow \quad \widehat{n}_i = \widehat{N}\,\partial_i\rho \ \ldots\ \&\ldots\ h^{ij} \longrightarrow \widehat{n}^i = h^{ij}\widehat{n}_j \longrightarrow \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j$$

- choose ρ^i to be a vector field on Σ: the integral curves... & $\rho^i\partial_i\rho = 1$
- 'lapse' and 'shift' of ρ^i

$$\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i\,, \quad \text{where} \quad \widehat{N} = \rho^j\widehat{n}_j \quad \text{and} \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j$$

- induced metric, extrinsic curvature and acceleration of the $\mathscr{S}_\rho$ level surfaces:

$$\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i\,\widehat{\gamma}^l{}_j\,h_{kl} \qquad \widehat{K}_{ij} = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}}\widehat{\gamma}_{ij} \qquad \dot{\widehat{n}}_i := \widehat{n}^e\nabla_e\widehat{n}_i = -\widehat{D}_i \ln \widehat{N}$$

- the metric h_{ij} can then be given as

$$h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i\widehat{n}_j \quad \Longleftrightarrow \quad \{\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}\}$$

$2+1$ decompositions:

Splitting of the symmetric tensor field K_{ij}:

- $$K_{ij} = \boldsymbol{\kappa}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\,\mathbf{k}_j + \widehat{n}_j\,\mathbf{k}_i] + \mathbf{K}_{ij}$$

  where

  $$\boldsymbol{\kappa} = \widehat{n}^k\widehat{n}^l\,K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i\widehat{n}^l\,K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,K_{kl}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

  $$\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\,\mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\mathbf{K}^l{}_l$$

The new variables:

- $$(h_{ij}, K_{ij}) \quad \Longleftrightarrow \quad (\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

- these variables retain the physically distinguished nature of h_{ij} and K_{ij}

The new form of the constrains:

$$\mathscr{L}_{\widehat{n}}\mathbf{k}_i - \tfrac{1}{2}\,\widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i\boldsymbol{\kappa} + \widehat{D}^l\mathring{\mathbf{K}}_{li} + (\widehat{K}^l{}_l)\,\mathbf{k}_i + \boldsymbol{\kappa}\,\dot{\widehat{n}}_i - \dot{\widehat{n}}^l\,\mathbf{K}_{li} + \mathfrak{p}_l\,\widehat{\gamma}^l{}_i = 0$$

$$\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \boldsymbol{\kappa}\,(\widehat{K}^l{}_l) + \mathbf{K}_{kl}\widehat{K}^{kl} + 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l - \mathfrak{p}_l\,\widehat{n}^l = 0$$

$$\widehat{R} - \Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\Big\}$$
$$+2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0$$

where $\widehat{D}_i$ denotes the covariant derivative operator associated with $\widehat{\gamma}_{ij}$ and

$$\widehat{K}_{ij} = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}}\widehat{\gamma}_{ij} \qquad \dot{\widehat{n}}_i := \widehat{n}^e\nabla_e\widehat{n}_i = -\widehat{D}_i\ln\widehat{N}$$

- **(!)** Σ is assumed to be a **dynamical horizon** ... foliated by MOTS
- the null expansion $\theta^{(\ell)}$ with respect to the "outward pointing" null normal $\ell^a = n^a + \widehat{n}^a$ of the foliating two-surfaces $\mathscr{S}_\rho$ in Σ has to vanish

$$\theta^{(\ell)} = \tfrac{1}{2}\,(\mathscr{L}_\ell\widehat{\gamma}_{ij})\,\widehat{\gamma}^{ij} = \tfrac{1}{2}\,(\mathscr{L}_{n+\widehat{n}}\widehat{\gamma}_{ij})\,\widehat{\gamma}^{ij} = (\mathbf{K}_{ij} + \widehat{K}_{ij})\,\widehat{\gamma}^{ij} = \mathbf{K}^l{}_l + \widehat{K}^l{}_l$$

Implications:

- on a dynamical horizon $\mathbf{K}^l{}_l$ and $\widehat{K}^l{}_l$ cannot be treated independently

$$\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\,\widehat{K}_{ij} = \widehat{N}^{-1}[\,\tfrac{1}{2}\,\widehat{\gamma}^{ij}(\mathscr{L}_\rho\widehat{\gamma}_{ij}) - \widehat{D}_j\widehat{N}^j\,] = -\mathbf{K}^l{}_l$$

- using

$$\mathbf{K}^l{}_l + \widehat{K}^l{}_l = 0$$

- we get

$$\begin{aligned}
&\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l + \boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \mathring{\mathbf{K}}_{kl}\mathring{\widehat{K}}{}^{kl} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 + 2\,\dot{\widehat{n}}{}^l\,\mathbf{k}_l - \mathfrak{p}_l\,\widehat{n}^l = 0\\
&\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) + \boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) - \tfrac{1}{2}\mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}{}^{kl} - \tfrac{1}{2}\mathring{\mathbf{K}}_{kl}\mathring{\mathbf{K}}{}^{kl} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2\\
&\qquad\qquad - \widehat{N}^{-1}\,\widehat{D}^l\,\widehat{D}_l\widehat{N} - \mathbf{k}^l\mathbf{k}_l + \tfrac{1}{2}\,\widehat{R} - \mathfrak{e} = 0
\end{aligned}$$

- finally, by subtracting we get

$$\begin{aligned}
&\widehat{D}^l\mathbf{k}_l - \widehat{N}^{-1}\,\widehat{D}^l\,\widehat{D}_l\widehat{N} - \mathbf{k}^l\mathbf{k}_l - 2\,\dot{\widehat{n}}{}^l\,\mathbf{k}_l - \tfrac{1}{2}\big(\mathring{\widehat{K}}_{kl} + \mathring{\mathbf{K}}_{kl}\big)\big(\mathring{\widehat{K}}{}^{kl} + \mathring{\mathbf{K}}{}^{kl}\big)\\
&\qquad\qquad + \mathfrak{p}_l\,\widehat{n}^l + \tfrac{1}{2}\,\widehat{R} - \mathfrak{e} = 0
\end{aligned}$$

Recovering of Bartnik and Isenberg result:

In a spherically symmetric setup:

- **(!)**

$$\boxed{\widehat{\gamma}_{ij} = r^2\, \mathring{\gamma}_{ij}}$$

- both $\widehat{D}_i \widehat{N} \equiv 0$ and $\mathbf{k}_l \equiv 0$
- the trace free parts $\mathring{\mathbf{K}}_{kl}$ and $\mathring{\widehat{K}}_{kl}$ also vanish
- $$\boxed{\begin{aligned}\widehat{D}^l \mathbf{k}_l - \widehat{N}^{-1}\, \widehat{D}^l \widehat{D}_l \widehat{N} - \mathbf{k}^l \mathbf{k}_l - 2\, \dot{\widehat{n}}^l\, \mathbf{k}_l - \tfrac{1}{2}\big(\mathring{\widehat{K}}_{kl} + \mathring{\mathbf{K}}_{kl}\big)\big(\mathring{\widehat{K}}^{kl} + \mathring{\mathbf{K}}^{kl}\big)& \\ + \mathfrak{p}_l\, \widehat{n}^l + \tfrac{1}{2}\, \widehat{R} - \mathfrak{e} = 0&\end{aligned}}$$

  reduces to

$$\boxed{\mathfrak{p}_l\, \widehat{n}^l + \tfrac{1}{2}\, \widehat{R} - \mathfrak{e} = 0}$$

- there is no dynamical horizon in the pure spherically symmetric vacuum setup as then $\boxed{\widehat{R} = 0}$ would follow which is incompatible with the implication of the Gauss-Bonnet theorem for spherical MOTS

Strategy in the generic case:

- on a dynamical horizon

$$\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\,\widehat{K}_{ij} = \widehat{N}^{-1}[\,\tfrac{1}{2}\,\widehat{\gamma}^{ij}(\mathscr{L}_\rho\widehat{\gamma}_{ij}) - \widehat{D}_j\widehat{N}^j\,] = -\mathbf{K}^l{}_l}$$

or

$$\boxed{\tfrac{1}{2}\,\widehat{\gamma}^{ij}(\mathscr{L}_\rho\widehat{\gamma}_{ij}) = -\widehat{N}\,\mathbf{K}^l{}_l + \widehat{D}_j\widehat{N}^j}$$

- we cannot specify freely the entire content of the metric $\widehat{\gamma}_{ij}$ **(!)** BUT ...
- there exist a smooth function $\Omega : \Sigma \to \mathbb{R}$ such that the induced metric $\widehat{\gamma}_{ij}$ can be decomposed as

$$\boxed{\widehat{\gamma}_{ij} = \Omega^2\,\gamma_{ij}}$$

γ_{ij} is singled out by the condition on the $\mathscr{S}_\rho$ surfaces:

$$\boxed{\gamma^{ij}(\mathscr{L}_\rho\gamma_{ij}) = 0}$$

-

$$\boxed{\gamma^{ij}(\mathscr{L}_\rho\gamma_{ij}) = \mathscr{L}_\rho\,\ln[\det(\gamma_{ij})] = 0}$$

the determinant of γ_{ij} is independent of the coordinate ρ

How to get Ω?

- the desired smooth function $\Omega : M \to \mathbb{R}$ and, in turn, the metric γ_{ij} is constructed as

$$\boxed{\widehat{\gamma}^{ij}(\mathscr{L}_\rho \widehat{\gamma}_{ij}) = \cancel{\gamma^{ij}(\mathscr{L}_\rho \gamma_{ij})} + 2\,\mathscr{L}_\eta (\ln \Omega^2)}$$

  - start with the given smooth distribution of the induced metric $\widehat{\gamma}_{ij}$ on the $\mathscr{S}_\rho$ surfaces
  - integrate the above relation first along the integral curves of ρ^a on Σ_0, starting at some $\mathscr{S}_0$

$$\boxed{\Omega^2 = \Omega_0^2 \cdot \exp\left[\tfrac{1}{2}\int_0^\rho \left(\widehat{\gamma}^{ij}(\mathscr{L}_\rho \widehat{\gamma}_{ij})\right)\, d\tilde{\rho}\right]}$$

  where $\Omega_0 = \Omega_0(x^2, x^3)$ denotes the conformal factor **chosen** at $\mathscr{S}_0$

- we also have the tensor field

$$\boxed{C^k{}_{ij} = \delta^k{}_{(i}\mathbb{D}_{j)} \ln \Omega^2 - \tfrac{1}{2}\,\gamma_{ij}\,\mathbb{D}^k \ln \Omega^2}$$

  relating the covariant derivatives $\widehat{D}_i$ and $\mathbb{D}_i$ associated with $\hat{\gamma}_{ij}$ and γ_{ij}, resp

- whence e.g.

$$\boxed{\widehat{D}_j \widehat{N}^j = \mathbb{D}_j \widehat{N}^j + C^j{}_{kj}\widehat{N}^k = \mathbb{D}_j \widehat{N}^j + \widehat{N}^k\, \mathbb{D}_k(\ln \Omega^2)}$$

- using the above relation

$$\tfrac{1}{2}\,\widehat{\gamma}^{ij}(\mathscr{L}_\rho\widehat{\gamma}_{ij}) = -\widehat{N}\,\mathbf{K}^l{}_l + \widehat{D}_j\widehat{N}^j$$

- can be rephased as

$$\mathscr{L}_\rho(\ln\Omega^2) - \widehat{N}^k\,\mathbb{D}_k(\ln\Omega^2) = -\widehat{N}\,\mathbf{K}^l{}_l + \mathbb{D}_j\widehat{N}^j$$

or as

$$\mathscr{L}_\rho\Omega - \widehat{N}^k\,\mathbb{D}_k\Omega = \tfrac{1}{2}\,\Omega\,[\,\mathbb{D}_j\widehat{N}^j - \widehat{N}\,\mathbf{K}^l{}_l\,]$$

- the equations have to be solved simultaneously but we have to eliminate the implicit use of $\widehat{\gamma}_{ij}$
- some relations to be checked
  - $\widehat{D}^l\,\mathbf{k}_l = \Omega^{-2}\,\big[\,\mathbb{D}^l\,\mathbf{k}_l - \gamma^{ij}C^k{}_{ij}\,\mathbf{k}_k\,\big] = \Omega^{-2}\,\mathbb{D}^l\,\mathbf{k}_l$, where the vanishing of the contraction $\gamma^{ij}C^k{}_{ij}$ was used
  - as $\gamma^{ij}C^k{}_{il}\,\mathring{\mathbf{K}}_{jk} = 0$ we also get

$$\begin{aligned}\widehat{D}^j\,\mathring{\mathbf{K}}_{jl} &= \Omega^{-2}\left[\mathbb{D}^j\,\mathring{\mathbf{K}}_{jl} - \gamma^{ij}C^k{}_{ij}\,\mathring{\mathbf{K}}_{jl} - \gamma^{ij}C^k{}_{il}\,\mathring{\mathbf{K}}_{jk}\right]\\ &= \Omega^{-2}\left[\mathbb{D}^j\,\mathring{\mathbf{K}}_{jl} - \gamma^{ij}C^k{}_{il}\,\mathring{\mathbf{K}}_{jk}\right] = \Omega^{-2}\,\mathbb{D}^j\,\mathring{\mathbf{K}}_{jl}\end{aligned}$$

The evolutionary equations:

$$\mathscr{L}_{\hat{n}}\mathbf{k}_i - \tfrac{1}{2}\,\mathbb{D}_i(\mathbf{K}^l{}_l) - \mathbb{D}_i\boldsymbol{\kappa} + \Omega^{-2}\,\mathbb{D}^l\mathring{\mathbf{K}}_{li} - (\mathbf{K}^l{}_l)\,[\,\mathbf{k}_i - \tfrac{1}{2}\,\dot{\hat{n}}_i\,] + \boldsymbol{\kappa}\,\dot{\hat{n}}_i - \Omega^2\,\dot{\hat{n}}^l\,\mathring{\mathbf{K}}_{li} + \mathfrak{p}_l\,\gamma^l{}_i = 0$$

$$\mathscr{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \Omega^{-2}\,\mathbb{D}^l\mathbf{k}_l + \boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) - \Omega^{-4}\,\mathring{\mathbf{K}}_{kl}\mathring{\mathbf{K}}^{kl} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 + 2\,\Omega^{-2}\,\dot{\hat{n}}^l\,\mathbf{k}_l - \mathfrak{p}_l\,\hat{n}^l = 0$$

$$\mathscr{L}_\rho\Omega - \widehat{N}^k\,\mathbb{D}_k\Omega - \tfrac{1}{2}\,\Omega\,[\,\mathbb{D}_j\widehat{N}^j - \widehat{N}\,\mathbf{K}^l{}_l\,] = 0$$

all the indices are raised and lowered by γ^{ij} and γ_{ij}, resp., and $\dot{\hat{n}}_k = -\mathbb{D}_k(\ln\widehat{N})$

- $2\,\widehat{N}\,\gamma^{ij}$ times of '(1)', $\Omega^2\,\widehat{N}$ times of '(1)' in (local) coordinates (ρ, x^2, x^3), adopted to the foliation $\mathscr{S}_\rho$ and the vector field ρ^i

$$\left\{\begin{pmatrix} 2\,\gamma^{AB} & 0 & 0 \\ 0 & \Omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\partial_\rho + \begin{pmatrix} -2\,\widehat{N}^K\,\gamma^{AB} & -\widehat{N}\,\gamma^{AK} & 0 \\ -\widehat{N}\,\widehat{\gamma}^{BK} & -\Omega^2\widehat{N}^K & 0 \\ 0 & 0 & -\widehat{N}^K \end{pmatrix}\partial_K\right\}\begin{pmatrix} \mathbf{k}_B \\ \mathbf{K}^E{}_E \\ \Omega \end{pmatrix} + \begin{pmatrix} \mathscr{B}^A_{(\mathbf{k})} \\ \mathscr{B}_{(\mathbf{K})} \\ \mathscr{B}_{(\Omega)} \end{pmatrix} = 0$$

- it is a FOSH provided that $\Omega \neq 0$

$$\boxed{\mathcal{A}^{(\rho)}\,\partial_\rho \mathbf{u} + \mathcal{A}^{(K)}\,\partial_K \mathbf{u} + \mathcal{B} = 0} \quad \text{for} \quad \mathbf{u} = (\mathbf{k}_B, \mathbf{K}^E{}_E, \Omega)^T$$

- the 'radial coordinate' ρ plays the role of 'time'
- we have to solve

$$\widehat{D}^l \mathbf{k}_l - \widehat{N}^{-1}\,\widehat{D}^l \widehat{D}_l \widehat{N} - \mathbf{k}^l \mathbf{k}_l - 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l - \tfrac{1}{2}\big(\mathring{\widehat{K}}_{kl} + \mathring{\mathbf{K}}_{kl}\big)\big(\mathring{\widehat{K}}^{kl} + \mathring{\mathbf{K}}^{kl}\big) + \mathfrak{p}_l\,\widehat{n}^l + \tfrac{1}{2}\,\widehat{R} - \mathfrak{e} = 0$$

- as an elliptic equation for $\widehat{N}$:
- allows to have an appropriate evolutionary description of dynamical horizons
- using Ω and γ_{ij} the scalar curvature $\widehat{R}$ on the $\mathscr{S}_\rho$ level surfaces can be given as

$$\boxed{\widehat{R} = \Omega^{-2}\left[{}^{(\gamma)}R - \mathbb{D}^l \mathbb{D}_l \ln \Omega^2\right]}$$

  where ${}^{(\gamma)}R$ and $\mathbb{D}_i$ are associated with γ_{ij}
- if the terms in the square brackets do not compensate each other $\widehat{R}$ blows up while one approaches the 'origin' (Ω vanishes there)

The main result:

Theorem

Suppose that Σ represents a dynamical horizon and that a smooth choice had been made for the freely specifiable variables

$$\widehat{N}^i, \gamma_{ij}\,; \boldsymbol{\kappa}, \mathring{\mathbf{K}}_{ij}\,; \mathfrak{e}, \mathfrak{p}_i \tag{1}$$

throughout Σ. Assume that smooth initial data $({}_0\mathbf{k}_i, {}_0\mathbf{K}^l{}_l, {}_0\Omega)$ had also been chosen to the first order symmetric hyperbolic system on one of the level surfaces $\mathscr{S}_{\rho_0}$ in Σ. Assume that the elliptic equation is solved also for ${}_0\widehat{N}$. Then there exists a unique smooth (local) solution $(\widehat{N}, \mathbf{k}_i, \mathbf{K}^l{}_l, \Omega)$ to the coupled FOSH-elliptic system such that $\widehat{N}|_{\mathscr{S}_{\rho_0}} = {}_0\widehat{N}, \mathbf{k}_i|_{\mathscr{S}_{\rho_0}} = {}_0\mathbf{k}_i, \mathbf{K}^l{}_l|_{\mathscr{S}_{\rho_0}} = {}_0\mathbf{K}^l{}_l, \Omega|_{\mathscr{S}_{\rho_0}} = {}_0\Omega$.