Hand-in sheet 2 – Statistical Physics B

- *Please hand in your solution before Thursday 6 November 2025, 16:15. This can be done during the lectures or tutorials or in mailbox 50 (J. Everts) located close to the secretariat of the Institute of Theoretical Physics on the 5th floor.*
- *We only accept hand-written or printed solutions. Digital format, such as pdf, will not be accepted.*
- *Please do not forget to write your name and student number.*
- *Make sure to answer every question as completely as possible. When you do calculations, provide sufficient explanation for all steps. Unless stated otherwise, saying that you used Mathematica (or equivalent) does not count as an explanation!*
- *In total 100 points can be earned.*

Capillary waves from thermal fluctuations

Consider a gas and a liquid exactly at coexistence, with ρ_l the bulk liquid number density and ρ_g the bulk gas number density. For simplicity, we fix the interface between the two phases at $z = 0$, which we assume to be sharp. Although the interface will be located on average at $z = 0$, the instantaneous height can fluctuate and will be located at $z = h(x, y) \neq 0$, with h being a well-defined function (we neglect overhangs). Furthermore, note from mass conservation that $\int\int_\mathcal{D} dxdy\, h(x, y) = 0$. Here, $\int\int_\mathcal{D} dxdy = A$ the surface area of a flat gas-liquid interface. These height fluctuations due to thermal fluctuations are called capillary waves.

(a) (*10 points*) Explain in what sense capillary waves due to thermal fluctuations are Goldstone modes. Is this in contradiction with the fact that the gas-liquid phase transitions stems from breaking a discrete symmetry? *Hint: See Problem 4.3.*

(b) (*10 points*) Consider a single thermal excitation that distorts the otherwise flat interface at $z = 0$. Show that the surface area of the interface with instantaneous height $h(x, y)$ is

$$A + \delta A[h] = \int\int_\mathcal{D} dxdy \sqrt{1 + [\partial_x h(x, y)]^2 + [\partial_y h(x, y)]^2}.$$

(c) (*10 points*) Show that the gravitational energy $\Phi[h]$ of the system is given by

$$\Phi[h] = \frac{1}{2} mg(\rho_\text{l} - \rho_\text{g}) \int\int_\mathcal{D} dxdy\, h^2(x, y),$$

with g the gravitational acceleration and m the mass of the particles.

(d) (*5 points*) We define the capillary-wave Hamiltonian as $H_\text{cw}[h] = \gamma\delta A[h] + \Phi[h]$, with γ the surface tension. Show that for small height fluctuations, the above expression reduces to

$$H_\text{cw}[h] = \frac{\gamma}{2} \int\int_\mathcal{D} dxdy\, \left\{[\partial_x h(x, y)]^2 + [\partial_y h(x, y)]^2 + \ell_\text{c}^{-2} h^2(x, y)\right\}.$$

Here, ℓ_c is the so-called capillary length. Derive an expression for ℓ_c. What is its typical order of magnitude for atomic systems close to the triple point temperature?

(e) (*15 points*) The capillary-wave Hamiltonian is a quadratic functional of $h(x, y)$. Therefore, we can diagonalize $H_{\rm cw}[h]$ by a discrete Fourier transform

$$h(\mathbf{R}) = \sum_{\mathbf{q}} \tilde{h}(\mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{R}), \quad \tilde{h}(\mathbf{q}) = \frac{1}{A} \int d^2\mathbf{R}\, h(\mathbf{R}) \exp(-i\mathbf{q} \cdot \mathbf{R}),$$

with $\mathbf{R} = (x, y)$ and $\mathbf{q} = 2\pi(n_x, n_y)/L$. In the latter $n_{x,y} = 1, 2, 3....$ and L is the macroscopic size of the interface, such that $A = L^2$. Perform this diagonalisation and show that

$$\langle \tilde{h}(\mathbf{q})\tilde{h}(\mathbf{q}') \rangle = \frac{k_{\rm B}T}{\gamma A} \frac{\delta_{\mathbf{q},-\mathbf{q}'}}{q^2 + \ell_{\rm c}^{-2}},$$

where $\langle ... \rangle$ is an ensemble average using a Boltzmann distibution with the (effective) capillary-wave Hamiltonian. *Hint: Write down how this ensemble average looks like in terms of a functional integral over the instantaneous height profile.*

(f) (*15 points*) We define the average mean-squared height as $\overline{h^2} := A^{-1} \int d\mathbf{R}\, h(\mathbf{R})^2$. Show that

$$\langle \overline{h^2} \rangle = \frac{k_{\rm B}T}{4\pi\gamma} \ln\left(\frac{q_{\rm max}^2 + \ell_{\rm c}^{-2}}{q_{\rm min}^2 + \ell_{\rm c}^{-2}} \right),$$

with cut-off wavenumbers given by $q_{\rm max} = 2\pi/\xi$ (with ξ being the correlation length) and $q_{\rm min} = 2\pi/L$.

Here $q_{\rm min}$ is the smallest wavenumber dictated by the system size. In other words, capillary waves with wavelength larger than the system size cannot exist. Furthermore, $q_{\rm max}$ says that we do not take into account capillary waves with wavelength smaller than the correlation length, which far from the critical point is on the order of the size of the microscopic particles that constitute the fluid. These excitations are not taken into account because the continuum description that we use in this exercise would break down on such small length scales.

(g) (*10 points*) Consider the thermodynamic limit of $\langle \overline{h^2} \rangle$ in the limiting cases (i) $g = 0$ and (ii) $\xi \ll \ell_{\rm c} \ll L$. Give a physical interpretation of your results. In particular, what is the role of gravity?

(h) (*15 points*) Define the height-height correlation function as

$$g(\mathbf{R}) := \frac{1}{A} \int d^2\mathbf{R}' \langle h(\mathbf{R}')h(\mathbf{R}' + \mathbf{R}) \rangle.$$

Show that $\beta g(\mathbf{R}) = (2\pi\gamma)^{-1} K_0(R/\ell_{\rm c})$ with K_0 the zeroth order modified Bessel function of the second kind. The following integral might be useful

$$\int_0^\infty dx \frac{x J_0(\alpha x)}{x^2 + \beta^2} = K_0(\alpha\beta), \quad \alpha, \, \mathrm{Re}(\beta) > 0,$$

and J_0 the zeroth order Bessel function of the first kind. Discuss $g(\mathbf{R})$ for $r \ll \ell_{\rm c}$ and $r \gg \ell_{\rm c}$. *Hint: To obtain J_0 it might be beneficial to perform contour integration.*