Hand-in sheet 3 – Statistical Physics B

- *Please hand in your solution before Thursday 27 November 2025, 16:15. This can be done during the lectures or tutorials or in mailbox 50 (J. Everts) located close to the secretariat of the Institute of Theoretical Physics on the 5th floor.*
- *We only accept hand-written or printed solutions. Digital format, such as pdf, will not be accepted.*
- *Please do not forget to write your name and student number.*
- *Make sure to answer every question as completely as possible. When you do calculations, provide sufficient explanation for all steps. Unless stated otherwise, saying that you used Mathematica (or equivalent) does not count as an explanation!*
- *In total 100 points can be earned.*

A one-dimensional gas

In some cases the partition function can be computed analytically, even for an interacting system. Consider a line of length L with N particles denoted with positions $x_1, ..., x_N$ and linear momenta $p_1, ..., p_N$, where $x_i \in [0, L]$ and $p_i \in (-\infty, \infty)$. There are fixed particles at $x_0 = 0$ and $x_{N+1} = L$, which effectively act as external potentials confining the particles to the line. The Hamiltonian is

$$H(p^N, x^N) = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \Phi(x^N), \quad \Phi(x^N) = \sum_{i=1}^{N} \left[\sum_{j>i} v(|x_i - x_j|) + v(x_i) + v(L - x_i) \right]. \tag{1}$$

Observe that the second and third term in $\Phi(x^N)$ denote external potentials. Furthermore, we assume that $v(r)$ is of the form

$$v(x) = \begin{cases} \infty, & |x| < \sigma, \\ \varphi(x), & \sigma < |x| < 2\sigma, \\ 0, & |x| > 2\sigma. \end{cases} \tag{2}$$

This means all particles have a hard "core" and only interact with their immediate neighbours, i.e. we only include nearest-neighbour interactions (not the same as pair-wise additive).

(a) (*15 points*) Show that the canonical partition function can be written as

$$Z(N, L, T) = \frac{1}{\Lambda^N} \int_0^L dx_N \int_0^{x_N} dx_{N-1} ... \int_0^{x_2} dx_1 \, \exp\left[-\beta \sum_{i=1}^{N+1} v(x_i - x_{i-1}) \right]. \tag{3}$$

Give an expression for Λ and give a physical interpretation for this quantity.

(b) (*10 points*) Use the result of a. to derive the recursion relation

$$Z(N, L, T) = \frac{1}{\Lambda} \int_0^L dx_N \exp[-\beta v(L - x_N)] Z(N-1, x_N, T). \tag{4}$$

(c) (*15 points*) Define the Laplace transform $\mathcal{L}$ of a function f as $\mathcal{L}[f](s) = \hat{f}(s) = \int_0^\infty dx \exp(-sx) f(x)$. Specifically introduce $J(x) = \exp[-\beta v(x)]$ and show that

$$\hat{Z}(N, s, T) = \frac{1}{\Lambda^N} [\hat{J}(s)]^{N+1}. \tag{5}$$

(d) (*10 points*) Instead of performing the inverse Laplace transform of the result in (c), we perform an easier approach. Perform a Legendre transformation to go to the (N, p, T) ensemble (isobaric-isothermal ensemble). What is the relevant thermodynamic potential? Give an expression in terms of the chemical potential. Give an expression for the total differential of the thermodynamic potential.

(e) (*10 points*) The partition function in the (N, p, T) ensemble is

$$\Delta(N, p, T) = \frac{1}{\Lambda} \int_0^\infty dL \, \exp(-\beta p L) Z(N, L, T) \tag{6}$$

How does it relate to the thermodynamic potential in (d)? Show that for $N \to \infty$ that the fugacity is $z(p, T) = 1/\hat{J}(\beta p)$, with $z(p, T) = \exp(\beta\mu)/\Lambda$.

(f) (*10 points*) The average line density is $\rho = N/L$. Show that

$$\rho = -\frac{\hat{J}(\beta p)}{\hat{J}'(\beta p)}, \tag{7}$$

with a prime denoting differentiation to the argument.

(g) (*15 points*) Find the equation of state $p(\rho, T)$ for a hard-line system (the so-called Tonks gas), with $v(x) = \infty$ for $|x| < \sigma$ and zero otherwise.

(h) (*15 points*) Compute the isothermal compressibility κ_T and show that

$$\kappa_T = -\frac{1}{L}\left(\frac{\partial L}{\partial p}\right)_{N,T} = \frac{\beta N}{L}[\langle y^2 \rangle - \langle y \rangle^2]. \tag{8}$$

with $y = x_{i+1} - x_i$. Is it possible to have a long-range ordered lattice in one dimension?