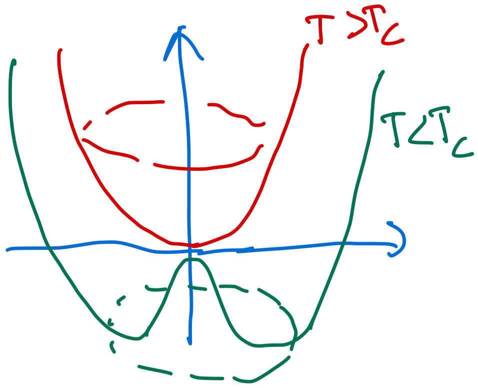
# Lecture 5: Correlations in Gaussian approximation.

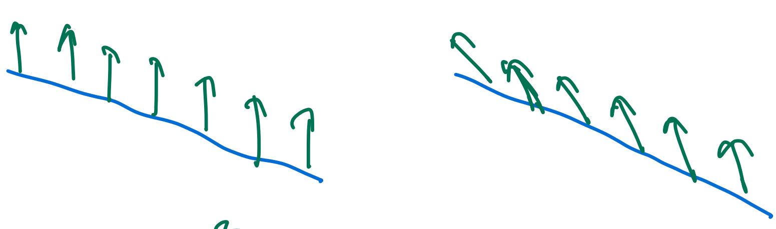
Partition function: $$\mathcal{Z} = \int \mathcal{D}\vec{\phi}\, e^{-\beta F_L[\vec{\phi}]} \quad (*)$$

functional integral

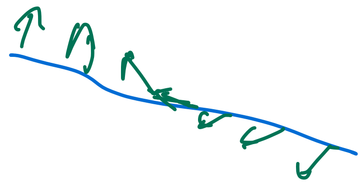
When $F_L[\vec{\phi}]$ has discrete symmetry $\to$ domain walls
continuous symmetry $\to$ Goldstone modes / bosons




Intuition for goldstone modes




But there exists excitations that look like



that cost little energy by stretching the winding over longer and longer distances.

Examples
- Phase fluctuations in superfluids.
- Phonons
- Magnons.
- Schlieren texture in nematic liquid crystals.

Take for example $O(3)$ model:

$$F_L[\vec{m}] = \int d^d\vec{r} \left[ \frac{K}{2} |\nabla \vec{m}|^2 + \frac{a}{2} |\vec{m}|^2 + \frac{b}{4} |\vec{m}|^4 \right]$$

and write $\vec{m} = m_0 \left( \cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta \right)$ $\quad \phi \in [0, 2\pi)$, $\theta \in [0, \pi)$

Then
$$F_L[\vec{m}] = \int d^d\vec{r} \left[ \frac{K}{2} |\nabla m_0|^2 + \frac{a}{2} m_0^2 + \frac{b}{4} m_0^4 + \frac{K}{2} m_0^2 \left[ (\nabla \theta)^2 + \sin^2\theta (\nabla \phi)^2 \right] + \ldots \right.$$

Two Goldstone modes. $\theta$ and $\phi$ and they interact.

Generally let's take effective Hamiltonian / Landau free energy that is invariant under symmetry group $G$. Suppose $G$ is spontaneously broken to $H$. Then the manifold of ground states is $G/H$ and $\#\text{goldstones} = \dim G - \dim H$.

Example $O(N)$ model. $G = O(N)$, $H = O(N-1)$

Ground state manifold: $\dfrac{O(N)}{O(N-1)} = S^{N-1}$

$\Rightarrow N-1$ goldstone modes.

This is consistent, since $\dim(O(N)) = \frac{1}{2}N(N-1)$

(Recall $O(N) = \{A \in \mathbb{R}^{N\times N} \mid A^T A = I\}$ $\longrightarrow$ $N(N+1)/2$ constraints. )

Mean-field theory does not describe Goldstone modes (cont. sym.) or domain walls (discrete symmetry). So let's take $O(N)$ model

$$F_L[\vec{\phi}] = \frac{1}{2}\int d^d\vec{r}\,\left[K|\nabla\vec{\phi}|^2 + a|\vec{\phi}|^2 + \frac{b}{2}|\vec{\phi}|^4\right]$$

with $\mathcal{Z} = \int \mathcal{D}\vec{\phi}\; e^{-\beta F_L[\vec{\phi}]}$ Write $\vec{\phi} = \langle\vec{\phi}\rangle + \delta\vec{\phi}$

Then: $\mathcal{Z} = e^{-\beta F_L[\langle\phi\rangle]}\int \mathcal{D}\delta\phi\, \exp\left[-\frac{\beta K}{2}\int d^d\vec{r}\,\{|\nabla\delta\phi|^2 + \xi(T)^{-2}|\delta\phi|^2\}\right]$

for $N=1$

(Note that this is de facto an application of the functional Taylor expansion)

$$\xi(T) = \begin{cases} \sqrt{\dfrac{K}{|a(T)|}} & T>T_c \\[2ex] \sqrt{\dfrac{K}{2|a(T)|}} & T<T_c \end{cases}$$

$\sim |T-T_c|^{-1/2}$

Recall from last lecture:

$$\mathcal{F}[u] = \mathcal{F}[u_0] + \int dx\, \left.\frac{\delta\mathcal{F}}{\delta u(x)}\right|_{u=u_0}[u(x)-u_0(x)]$$

$$+\frac{1}{2}\int dx\int dx'\, \left.\frac{\delta^2\mathcal{F}}{\delta u(x)\delta u(x')}\right|_{u=u_0}[u(x)-u_0(x)][u(x')-u_0(x')] + \ldots\ldots$$

The lowest-order correction to MFT is quadratic (why? ).

We are interested in computing correlators $\langle \delta\phi(\vec r)\,\delta\phi(\vec r\,')\rangle$ and to see what are the corrections to the mean-field result. So we need to know how to compute Gaussian functional integrals.

Let us consider a scalar order parameter first. So integrals are of the form:

$$\mathcal{Z}[j] = \int \mathcal{D}\phi \, \exp\Big[-\frac{1}{2}\int d^d\vec r \int d^d\vec r\,' \phi(\vec r)\, G^{-1}(\vec r,\vec r\,')\,\phi(\vec r\,') + \int d^d\vec r \, j(\vec r)\,\phi(\vec r)\Big].$$

(we add on purpose sources which will be clear in a moment).

Note that (see tutorials):

$$\mathcal{Z}(\vec j) = \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_n \, e^{-\frac{1}{2}\vec x\cdot \underline{A}\cdot \vec x + \vec j\cdot \vec x} = \mathcal{Z}(0)\exp\left(\frac{1}{2}\vec j\cdot \underline{A}^{-1}\cdot \vec j\right),$$

$$\text{where } \mathcal{Z}(0) = \frac{(2\pi)^{n/2}}{\sqrt{\det(\underline{A})}} = (2\pi)^{n/2}\exp\left[-\frac{1}{2}\mathrm{Tr}(\log \underline{A})\right].$$

Furthermore, $\langle x_{i_1} \dots x_{i_n}\rangle = \dfrac{1}{\mathcal{Z}(\vec j)} \dfrac{\partial^n}{\partial j_{i_1}\dots \partial j_{i_n}} \mathcal{Z}(\vec j)\Big|_{\vec j=0}$.

In particular:

$$\langle x_i x_j\rangle = \frac{1}{\mathcal{Z}(\vec j)}\frac{\partial^2}{\partial j_i \partial j_j}\mathcal{Z}(\vec j)\Big|_{\vec j=0} = A^{-1}_{ij}$$

Note that: $A_{ik}A^{-1}_{kj} = \delta_{ij}$

So now we can generalize to continuous case:

$$\mathcal{Z}[j] = \mathcal{Z}[0]\exp\left[\frac{1}{2}\int d^d\vec r\int d^d\vec r\,' j(\vec r)\,G(\vec r,\vec r\,')\,j(\vec r\,')\right]$$

$$\text{with } \mathcal{Z}[0] = \mathcal{N}\sqrt{\det G^{-1}} = \mathcal{N}\exp\left[-\frac{1}{2}\mathrm{Tr}(\log G^{-1})\right].$$

$$\text{where } \int d\vec r\,'' G^{-1}(\vec r,\vec r\,'')\,G(\vec r\,'',\vec r\,') = \delta(\vec r-\vec r\,')$$

and $\langle \phi(\vec{r})\phi(\vec{r}\,')\rangle = G(\vec{r},\vec{r}\,')$.

So let us first consider the correlation function of our $\mathbb{Z}_2$ model.

$\Rightarrow \langle \delta\phi(\vec{r})\delta\phi(\vec{r}\,')\rangle = G(\vec{r},\vec{r}\,')$ where

$G^{-1}(\vec{r},\vec{r}\,') = \beta K\left(-\nabla^2 + \xi(T)^{-2}\right)\delta(\vec{r}-\vec{r}\,')$. So how to determine $G(\vec{r},\vec{r}\,')$?

$$\Rightarrow \int d\vec{r}\,'' G^{-1}(\vec{r},\vec{r}\,'')G(\vec{r}\,'',\vec{r}\,') = \int d\vec{r}\,'' \beta K\left(-\nabla^2 + \xi(T)^{-2}\right)\delta(\vec{r}-\vec{r}\,'')G(\vec{r}\,'',\vec{r}\,')$$

$$\Rightarrow \quad \beta K\left[-\nabla^2 + \xi(T)^{-2}\right]G(\vec{r}-\vec{r}\,') = \delta(\vec{r}-\vec{r}\,')$$

Solve with FT. For finite volume $V$, we have for our fields:

$$\phi_{\vec{k}} = \int d^d\vec{r}\; e^{-i\vec{k}\cdot\vec{r}}\phi(\vec{r}) \iff \phi(\vec{r}) = \frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}}\phi_{\vec{k}}.$$

(Recall since we did square gradient, we implicitly set $\phi_{\vec{k}} = 0$ for $|\vec{k}| > \Lambda$ with $\Lambda = \frac{\pi}{a}$).

Since it is finite spatial volume: $\vec{k} = \frac{2\pi}{L}\vec{n}$ $\quad \vec{n} \in \mathbb{Z}^d$ (periodic bcs)

with $V = L^d$. So in thermodynamic limit:

$$\phi(\vec{r}) = \int \frac{d^d\vec{k}}{(2\pi)^d}\, e^{i\vec{k}\cdot\vec{r}}\,\tilde{\phi}(\vec{k}). \iff \tilde{\phi}(\vec{k}) = \int d^d\vec{k}\; e^{-i\vec{k}\cdot\vec{r}}\phi(\vec{r})$$

So we find $\tilde{G}(\vec{k}) = \dfrac{k_B T}{K}\dfrac{1}{\vec{k}^2 + \xi(T)^{-2}}$ Let's check this result in $d=3$.

So $G(\vec{r}) = \dfrac{k_B T}{K}\displaystyle\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + \xi(T)^{-2}}$ (see Tutorials).

$$= \frac{k_B T}{4\pi K}\frac{e^{-r/\xi}}{r}. \quad \Rightarrow \langle \delta\phi(\vec{r})\delta\phi(\vec{r}\,')\rangle = \frac{k_B T}{4\pi K}\frac{e^{-|\vec{r}-\vec{r}\,'|/\xi}}{|\vec{r}-\vec{r}\,'|}.$$

$\xi$ is indeed correlation length!

At CP: $G(\vec{r}-\vec{r}\,') \sim \frac{1}{|\vec{r}-\vec{r}\,'|^{d-2+\eta}}$ At MF+Gauss $\Rightarrow$ $\eta = 0$ ! But $\eta = 1/4$ (2D) $\eta \simeq 0.036$ (3D)

Remarks: • $G(\vec{r})$ is a Green's function. In Field Theory language it is called a propagator or sometimes vertex function.

• We could have used $\int \mathcal{D}\phi = \lim_{N\to\infty} \prod_{i=1}^{N} \int d\phi_i$ and then make coordinate transf. with discrete FT.

Then $\int \mathcal{D}\phi = \prod_{\vec{k}} C \int d\phi_{\vec{k}}\, d\phi^*_{\vec{k}}$ with $\phi^*_{\vec{k}} = \phi_{-\vec{k}}$.

Now (using shorthand notation $\delta\phi = \phi'$):

$$\int \mathcal{D}\phi' \exp\left\{-\frac{\beta K}{2}\int d^d\vec{r}\left[|\nabla\phi'|^2 + \xi^{-2}\phi'^2\right]\right\}$$

$$= \prod_{|\vec{k}|>0} C \int d\phi'_{\vec{k}}\, d\phi'^*_{\vec{k}} \exp\left\{-\frac{\beta K}{2}\sum_{k} \phi^{*\prime}_{\vec{k}}\left(\vec{k}^2 + \xi^{-2}\right)\phi'_{\vec{k}}\right\}.$$

$$\Rightarrow \langle \phi_{\vec{k}}\phi^*_{\vec{k}}\rangle = \frac{k_B T}{K}\frac{1}{\vec{k}^2 + \xi^{-2}}$$

But this is like using equipartition theorem !

$\Rightarrow H = \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m}$. $\left\langle \frac{p_{i\alpha}^2}{2m}\right\rangle = \frac{1}{2}k_B T$. $\alpha = x, y, z$. $i = 1, \dots, N$.

(Every quadratic degree of freedom in Hamiltonian receives $\frac{1}{2}k_B T$).

What about free energy?

$$Z = e^{-\beta F_L[\langle\phi\rangle]} \int \mathcal{D}\delta\phi \exp\left[-\frac{\beta K}{2}\int d^d\vec{r}\left\{|\nabla\delta\phi|^2 + \xi(T)^{-2}\delta\phi^2\right\}\right]$$

$$= e^{-\beta F_L[\langle\phi\rangle]}\, e^{-\frac{1}{2}\sum_{\vec{k}} \log\left[\beta K(\vec{k}^2 + \xi^{-2})\right]}$$

$$\beta F = \beta F_L[\langle\phi\rangle] + \frac{1}{2}\sum_{\vec{k}} \log\left[\beta K\left(\vec{k}^2 + \xi^{-2}\right)\right].$$ In continuum limit:

with $\beta\Delta F = \beta F - \underbrace{\beta F_{MF}}_{F[\langle\phi\rangle]}$ and $V = L^d$

$$\frac{\beta \Delta F}{V} = \frac{1}{2}\int \frac{d^d\vec{k}}{(2\pi)^d} \log\left[\beta K\left(\vec{k}^2 + \xi^{-2}\right)\right].$$

We find that in the Gaussian approximation, the same result as in mean-field theory (critical exponents, $T_c$), but the only correction is in the heat capacity. Define $c_V = \frac{C_V}{V}$, then $c_V = -T\,\frac{\partial^2 (F/V)}{\partial T^2}$.

Taking the derivative gives two contributions, which are proportional to

$$I_1 = \int_{|\vec{k}|<\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{(\vec{k}^2+\xi^{-2})^2} \quad \text{and} \quad I_2 = \int_{|\vec{k}|<\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\vec{k}^2+\xi^{-2}}.$$

Let us analyze both integrals.

First:

$$I_2 \propto \int_0^\Lambda dk \, \frac{k^{d-1}}{k^2+\xi^{-2}}$$

Let's do rescaling: $\vec{q} := \xi\vec{k}$

$$\propto \xi^{2-d}\int_0^{\xi\Lambda} dq \, \frac{q^{d-1}}{1+q^2}$$

Recall at $T_c$ $\xi \to \infty$

$\therefore$ inspect integrand for $q \to 0$ and $q \to \infty$

* For $q$ small and $d \geq 1$ there are no problems with divergence.
- For $q$ large integrand $\sim q^{d-3}$ so for convergence: $d-3 < -1$
  $d < 2$.

So $$I_2 \propto \begin{cases} \xi^{2-d} & \text{for } d=1 \\ \sim \ln \Lambda\xi & \text{for } d=2 \\ \Lambda^{d-2} & \text{for } d>2 \end{cases}$$

Now let us analyse $I_1$:

$$I_1 \propto \xi^{4-d} \int_0^{\xi\Lambda} dq \, \frac{q^{d-1}}{(1+q^2)^2} .$$

- At $q \to 0$ again convergent for $d \geq 1$
- At $q \to \infty$ integrand $\sim q^{d-5}$ so integrable at infinity iff $d < 4$

$$I_1 \propto \begin{cases} \xi^{4-d} & 1 \leq d < 4 \\ \ln \Lambda\xi & d = 4 \\ \Lambda^{d-4} & d > 4 \end{cases}$$

(Recall $\Lambda \sim \frac{1}{a}$)

Since $\Lambda$ is a finite scale imposed by a microscopic length scale; leading order divergence at $T_c$ goes like $\xi^{4-d}$ $1 \leq d \leq 4$.

and $\xi \sim |T-T_c|^{-\nu}$ $\nu = 1/2$

So $\Delta c_V \sim |T-T_c|^{\nu(4-d)} \underset{\substack{\Uparrow \\ \nu=\frac{1}{2}\,(MF)}}{=} |T-T_c|^{-(2-d/2)}$ for $1 \leq d < 4$.

$\alpha$ ← first correction to MF.

For $d \geq 4$ we find that $\Delta c_V$ is a constant.

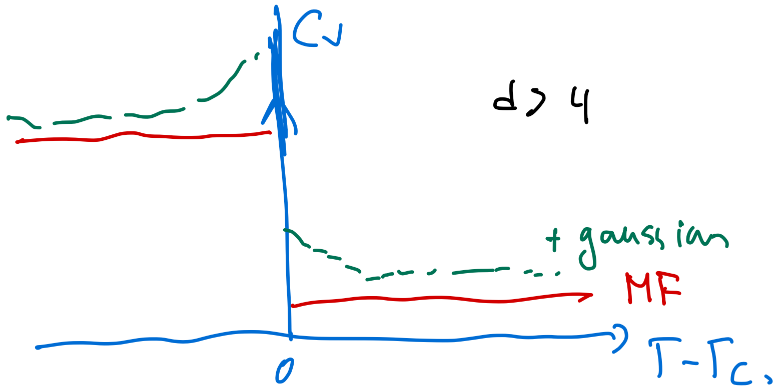
So recall $C_V \sim |T-T_c|^{-\alpha}$

In MF theory $\alpha = 0$ (discontinuity)

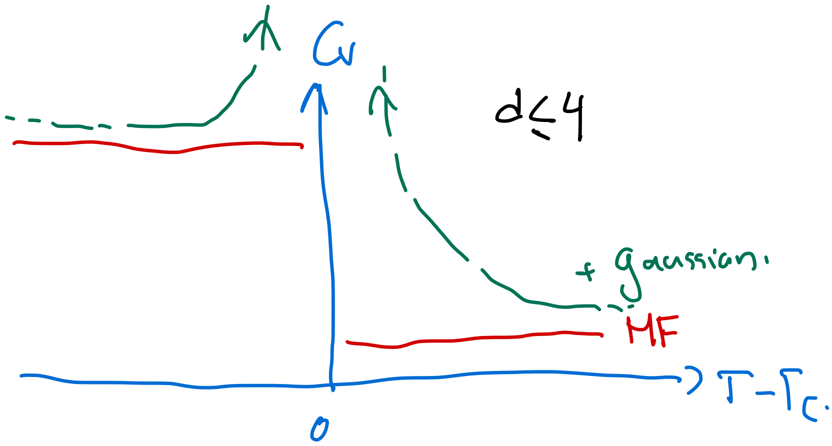
with Gaussian correction $\alpha = 2 - \frac{d}{2}$ $d < 4$.

$\alpha = 0$ $d \geq 4$.

Critical exponents are unaffected for $d \geq 4$. This is called the upper critical dimension ($d_u = 4$)




Still discontinuous, critical exponent is unaffected.




Heat capacity diverges at critical point!