Lecture 6: Symmetry breaking of continuous symmetries

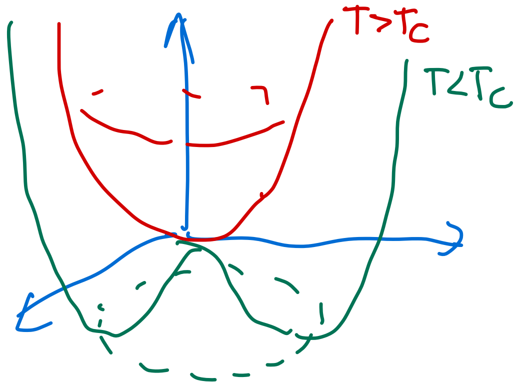
Spontaneous symmetry breaking in stat. physics. A Landau free energy F_L or effective Hamiltonian parametrised in terms of an order parameter ϕ. F_L (and therefore the disordered phase) is invariant under group G.

E.g. $O(N)$ model: $G = \{ g \in \mathbb{R}^{N\times N} \mid gg^T = \mathbb{1} \}$

then for $g \in G$ $\quad F_L[g\phi] = F_L[\phi]$.

For $O(2)$:

(*)



Below T_c, ground state spontaneously breaks symmetry of the underlying Hamiltonian to $H \subset G$.

Typically, we can write $F_L[\phi] = \int d^d\vec{r} \, \{ \text{"Gradient terms"} + \underbrace{V(\phi)}_{\text{"effective potential"}} \}$.

Then ground state $\langle \phi \rangle$ is a stationary point of V. We then can specify H further as $H = \{ h \in H \mid h\langle\phi\rangle = \langle\phi\rangle \}$.

Remarks: – If $H = G \rightarrow$ no spontaneous symmetry breaking

– For arbitrary state ϕ: $G_\phi = \{ g\phi \mid g \in G \}$ are set of states with the same $V(\phi)$. G_ϕ is called the orbit of ϕ.

– Note that H_ϕ and $H_{g\phi}$ are equivalent isotropy groups.

The number of Goldstone modes is precisely $\dim(G_{\langle\phi\rangle})$.

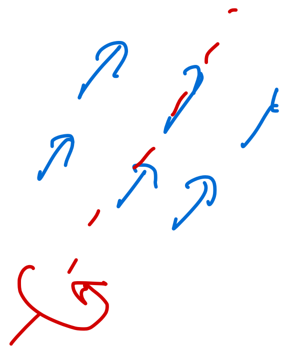
Furthermore: $G_{\langle\phi\rangle} = G/H$ and therefore $\dim(G_{\langle\phi\rangle}) = \dim G - \dim H$.

For example (i) $O(N) \longrightarrow O(N-1)$ $\quad$ $(\dim(O(N)) = \frac{1}{2}N(N-1))$.

\# Goldstones $= N-1$.

Let's create some intuition for it. E.g. $O(3)$. (three-dimensional magnetism)

The isotropy group is $O(2)$ since a specific ground state is invariant under rotations around the magnetization direction



The orbit $G\langle\phi\rangle$ is harder to visualize.

(ii). Take $O(2)$. Ground state is now invariant under $\mathbb{Z}_2$. i.e. identity and reflections in line parallel to magnetisation direction. (Not to be confused with $\hat{\mathbb{Z}}_2 = \{\pm 1\}$)

$O(2)/\mathbb{Z}_2 \sim S^1$. This is true, since in $(*)$ we can see that ground state manifold is a circle. (i.e $G\langle\phi\rangle$). Note $\mathbb{Z}_2 \cong O(1)$

(iii). Nematic liquid crystals. Disordered state is invariant under $O(3)$.

Order parameter is $\underline{\underline{Q}}$ and $V(\underline{\underline{Q}}) = \frac{a}{2}\operatorname{tr}\underline{\underline{Q}}^2 + \frac{b}{3}\operatorname{tr}\underline{\underline{Q}}^3 + \frac{c}{4}\operatorname{tr}\underline{\underline{Q}}^4$.

Check: $g \in O(3)$ then $V(\underline{\underline{Q}}) = V(g\underline{\underline{Q}}g^{-1})$ In nematic state

$\underline{\underline{Q}} = \frac{3}{2}S\left(\hat{n}\hat{n} - \frac{1}{3}\mathbb{1}\right)$ is invariant for $\{g \in O(3) \mid g\hat{n} = \pm\hat{n}\}$.

But this is $D_{\infty h}$. $O(3)/D_{\infty h} \cong \mathbb{R}P^2$ (S^2 with antipodal points identified)

$\dim(\mathbb{R}P^2) = 2 \Rightarrow 2$ Goldstones. (Note: there is some freedom in choosing G. See Mermin 1979)

In last lecture we considered a Landau-Ginzburg theory with a scalar order parameter, i.e:

$$F_L[\phi(\vec{r})] = \frac{1}{2}\int d^d\vec{r}\left[K|\nabla\phi(\vec{r})|^2 + a\,\phi(\vec{r})^2 + \frac{b}{2}\phi(\vec{r})^4\right]$$

and we found $\langle\delta\phi(\vec{r})\,\delta\phi(\vec{r}')\rangle = \dfrac{k_B T}{4\pi K}\,\dfrac{e^{-|\vec{r}-\vec{r}'|/\xi(T)}}{|\vec{r}-\vec{r}'|}$ within Gaussian approximation, for $d=3$

Here $\xi(T) = \begin{cases} \sqrt{\dfrac{K}{a(T)}} & T > T_c \\ \sqrt{\dfrac{K}{2a(T)}} & T < T_c \end{cases}$ is the correlation length and $\delta\phi(\vec{r}) = \phi(\vec{r}) - \langle\phi(\vec{r})\rangle$.

Note that $a(T) = a_0(T-T_c) \Rightarrow \xi(T) \sim |T-T_c|^{-\nu}$ with $\nu = \frac{1}{2}$ (MF + Gaussian)

Furthermore, close to T_c: $\langle\delta\phi(\vec{r})\delta\phi(\vec{r}\,')\rangle \sim \frac{1}{|\vec{r}-\vec{r}\,'|^{d-2+\eta}} \Rightarrow \eta = 0$. (MF + Gaussian)

We have also seen that including Gaussian fluctuations leads to diverging heat capacity for $d \leq 4$ (whereas MFT would just predict a discontinuity)

Q: When are fluctuations important?

$\Rightarrow$ We have only computed the correlation function for $d=3$. What about arbitrary dimension?

This amounts to computing the Fourier integral

$$\langle\delta\phi(\vec{r})\delta\phi(\vec{r}\,')\rangle = \frac{k_BT}{K}\int\frac{d^d\vec{k}}{(2\pi)^d}\frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2+\xi^{-2}} \quad (=G(\vec{r}-\vec{r}\,'))$$

In principle, one could compute this Fourier integral, but it is easier to come back to the original differential equation.

Recall that: $\beta K(-\nabla^2+\xi(T)^{-2})G(\vec{r}) = \delta(\vec{r})$

In spherical coordinates:

$$\left[-\frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}+\xi^{-2}\right]G(r) = \frac{k_BT}{K}\delta(\vec{r})$$

Define: $\vec{s} := \frac{1}{\xi}\vec{r}$

Then $\delta(\vec{r}) = \frac{1}{\xi^d}\delta(\vec{s})$ and the differential equation becomes

$$\left[-\frac{1}{s^{d-1}}\frac{\partial}{\partial s}s^{d-1}\frac{\partial}{\partial s}+\xi^{-2}\right]G(s) = g\,\delta(\vec{s}) \quad \text{with} \quad g := \frac{k_BT}{K}\xi^{2-d}$$

This differential equation can be solved with modified Bessel functions of the second kind, i.e:

$$\frac{1}{g}G(s) = \begin{cases} e^{-s} & (d=1) \\ (2\pi)^{-d/2}s^{-(d-2)/2}K_{(d-2)/2}(s) & (d\geq 2)\end{cases}$$

We have the following asymptotic properties:

$$K_n(s) \sim \left(\frac{\pi}{2s}\right)^{1/2} e^{-s} \qquad s\to\infty \qquad (\text{for } n=\tfrac{1}{2}, \text{ it is valid for } \forall s)$$

$$K_n(s) \sim \frac{\Gamma(n)}{2}\left(\frac{s}{2}\right)^{-n} \qquad s\to 0,\ n\neq 0$$

$$K_0(s) \sim -\log s \qquad s\to 0. \qquad \left(\Gamma(z) := \int_0^\infty dt\, t^{z-1} e^{-t}\right)$$

We conclude that:

$$G(r) = \frac{\pi^{(1-d)/2}}{2^{(1+d)/2}} \frac{k_B T}{K} \frac{e^{-r/\xi}}{r^{(d-1)/2}} \frac{1}{\xi^{(d-3)/2}} \quad \text{for } r \gg \xi \text{ and } d \geq 2.$$

whereas

$$G(r) \sim \frac{k_B T}{K} \frac{\Gamma\left(\frac{d-2}{2}\right)}{4\pi^{d/2}} \frac{1}{r^{d-2}} \qquad r \ll \xi \text{ and } d > 2.$$

(logarithmic for $d=2$).

When can we trust mean-field theory?

In order to trust mean-field theory fluctuations in the order parameter should be smaller than the background around which they're fluctuating.

That is: $\langle \delta\phi^2 \rangle \ll \langle \phi \rangle^2$. Below T_c $\langle \phi \rangle = \pm \phi_0$

To estimate this, we use that fluctuations decay after a distance $r \gg \xi$.

$$R = \frac{\int_{r<\xi} d^d\vec{r}\, \langle \delta\phi(\vec{r})\delta\phi(0)\rangle}{\int_{r<\xi} d^d\vec{r}\, \langle \phi \rangle^2} \sim \frac{1}{\phi_0^2 \xi^d} \int_0^{\xi} dr \frac{r^{d-1}}{r^{d-2}} \sim \frac{\xi^{2-d}}{\phi_0^2}$$

$R \ll 1$ is called the Ginzburg criterion.

But close to CP according to mean-field theory: $\phi_0 \sim |T-T_c|^{1/2}$

and $\xi \sim |T-T_c|^{-1/2}$

So: $R \sim |T-T_c|^{(d-4)/2}$

So around the critical point we can trust MFT only if $d \geq d_u = 4$ (same result as last lecture)

(Note that MFT predicts its own demise for $d<4$:))

Sometimes fluctuations can destroy order (Peierls argument) for certain d. This defines the lower critical dimension d_ℓ. For ϕ^4 theory $d_\ell = 1$.

Up until now we have only considered breaking of a discrete symmetry, but what if underlying symmetry is continuous? In the Ising case the occurrence of domain walls destroy order, but what happens if there are Goldstone modes?

As an example, consider the O(N) model: $\vec{\phi} = (\phi_1, \dots, \phi_N)$

$$F_L[\vec{\phi}] = \frac{1}{2}\int d^d\vec{r}\left[K(\nabla\vec{\phi}(\vec{r}))^2 + a(T)|\vec{\phi}(\vec{r})|^2 + \frac{b}{2}|\vec{\phi}(\vec{r})|^4\right]. \quad (*)$$

Again we expand $\vec{\phi}(\vec{r}) = \langle\vec{\phi}\rangle + \delta\vec{\phi}(\vec{r})$. We are interested in the correlation function $\langle\delta\vec{\phi}(\vec{r})\,\delta\vec{\phi}(\vec{r}')\rangle$ which takes the form of a tensor. In components:

$$\langle\delta\phi_i(\vec{r})\,\delta\phi_j(\vec{r}')\rangle = G_{ij}(\vec{r}-\vec{r}').$$

For $T > T_c$, we have: $F_L[\delta\vec{\phi}] = \sum_{i=1}^{N} F_L[\delta\phi_i]$ and therefore

$$G_{ij}(\vec{r}-\vec{r}') = \delta_{ij}\frac{k_B T}{4\pi K}\frac{e^{-|\vec{r}-\vec{r}'|/\xi}}{|\vec{r}-\vec{r}'|} \quad \text{with } \xi(T>T_c) = \sqrt{\frac{K}{a(T)}}$$

For $T < T_c$ the fourth-order term in $(*)$ gives the contributions:

$$\frac{b}{4}\sum_{i,j}\int d\vec{r}\left[2\langle\phi_i\rangle^2\,\delta\phi_j(\vec{r})^2 + 4\langle\phi_i\rangle\langle\phi_j\rangle\,\delta\phi_i(\vec{r})\,\delta\phi_j(\vec{r})\right]$$

As in discrete case, we write

$$Z = e^{-\beta F_L[\langle\vec{\phi}\rangle]}\int\mathcal{D}\delta\vec{\phi}\,\exp\left[-\frac{1}{2}\int d\vec{r}\int d\vec{r}'\,\delta\vec{\phi}(\vec{r})\cdot\underline{\underline{G}}^{-1}(\vec{r}-\vec{r}')\,\delta\vec{\phi}(\vec{r}')\right]$$

with $G^{-1}_{ij}(\vec{r},\vec{r}\,') = \frac{1}{k_BT}\delta_{ij}\left[-K\nabla^2 + a(T) + b|\langle\vec{\phi}\rangle|^2\right] - \frac{2b}{k_BT}\langle\phi_i\rangle\langle\phi_j\rangle$

Now we will use that $|\langle\vec{\phi}\rangle|^2 = \frac{-a(T)}{b}$ and define $\hat{\nu} = \frac{\langle\vec{\phi}\rangle}{|\langle\vec{\phi}\rangle|}$

Then in Fourier space:

$$\tilde{G}^{-1}_{ij}(\vec{k}) = \underbrace{\frac{1}{k_BT}\left[K\vec{k}^2 - 2a(T)\right]\hat{\nu}_i\hat{\nu}_j}_{\substack{\text{fluctuating modes parallel to } \hat{\nu} \\ \text{(longitudinal modes)}}} + \underbrace{\frac{K\vec{k}^2}{k_BT}(\delta_{ij} - \hat{\nu}_i\hat{\nu}_j)}_{\substack{\text{fluctuating modes} \\ \text{orthogonal to } \nu. \\ \text{(transverse modes)}}}.$$

Note that $\hat{\nu}\hat{\nu}$ and $\underline{\underline{I}} - \hat{\nu}\hat{\nu}$ are projection operators.

$\hat{\nu}\hat{\nu}$: projects onto space parallel to "magnetization" direction

$\underline{\underline{I}} - \hat{\nu}\hat{\nu}$: projects onto space perpendicular to "magnetization" direction.

Furthermore, * $(\hat{\nu}\hat{\nu})\cdot(\hat{\nu}\hat{\nu}) = \hat{\nu}\hat{\nu}$

* $(\underline{\underline{I}} - \hat{\nu}\hat{\nu})\cdot(\underline{\underline{I}} - \hat{\nu}\hat{\nu}) = \underline{\underline{I}} - \hat{\nu}\hat{\nu}$

* $\hat{\nu}\hat{\nu}\cdot(\underline{\underline{I}} - \hat{\nu}\hat{\nu}) = 0$

With the decomposition in transverse and longitudinal modes, we thus find:

$$\underline{\underline{\tilde{G}}}(\vec{k}) = \frac{k_BT}{K}\left[\frac{1}{\vec{k}^2 + \xi(T)^{-2}}\hat{\nu}\hat{\nu} + \frac{1}{\vec{k}^2}(\underline{\underline{I}} - \hat{\nu}\hat{\nu})\right].$$

$$=: \tilde{G}_L(\vec{k})\,\hat{\nu}\hat{\nu} + \tilde{G}_T(\vec{k})\,(\underline{\underline{I}} - \hat{\nu}\hat{\nu}).$$

Let us check first the case where $d=3$:

Then $G_L(\vec{r}-\vec{r}\,') = \frac{k_BT}{4\pi K}\,\frac{e^{-|\vec{r}-\vec{r}\,'|/\xi(T)}}{|\vec{r}-\vec{r}\,'|}$

whereas $G_T(\vec{r}-\vec{r}\,') = \dfrac{k_B T}{4\pi K|\vec{r}-\vec{r}\,'|}$

$\Rightarrow$ Longitudinal fluctuations decay exponentially (like in the discrete case) whereas transverse fluctuations decay algebraically!

This is typical for symmetry breaking of a continuous symmetry!

These transverse fluctuations are precisely the Goldstone modes, and from the decay of their correlations, we learn that they are massless

The nomenclature for mass comes from high-energy physics where the Greens-function (propagator) takes on the form:

$$\tilde{G}(\vec{k}) \propto \frac{1}{(\hbar c\vec{k})^2 + m^2c^4} \qquad (c: \text{speed of light})$$

m: rest mass.

Goldstones behave like elementary particles with zero mass.

Summarise above results for $T<T_c$:

$$\langle\vec{\phi}(\vec{r})\vec{\phi}(\vec{r}\,')\rangle = \langle\vec{\phi}\rangle^2\hat{\nu}\hat{\nu} + \frac{k_BT}{4\pi K}\frac{e^{-|\vec{r}-\vec{r}\,'|/\xi(T)}}{|\vec{r}-\vec{r}\,'|}\hat{\nu}\hat{\nu} + \frac{k_BT}{4\pi K|\vec{r}-\vec{r}\,'|}(\underline{\underline{I}}-\hat{\nu}\hat{\nu})$$

$$= \langle\vec{\phi}\rangle^2\left\{\hat{\nu}\hat{\nu} + \frac{k_BTb\,\xi(T)^2}{2\pi K^2}\frac{e^{-|\vec{r}-\vec{r}\,'|/\xi(T)}}{|\vec{r}-\vec{r}\,'|}\hat{\nu}\hat{\nu} + \frac{k_BTb\,\xi(T)^2}{2\pi K^2|\vec{r}-\vec{r}\,'|}(\underline{\underline{I}}-\hat{\nu}\hat{\nu})\right\},$$

So for $|\vec{r}-\vec{r}\,'|\to\infty$, we find $\langle\vec{\phi}(\vec{r})\vec{\phi}(\vec{r}\,')\rangle = \langle\vec{\phi}\rangle^2\hat{\nu}\hat{\nu}$

This is called long-range order. Here we have been a bit sloppy with cutoff

$$G_T(\vec{r}) = \frac{k_B T}{K} \int_{|\vec{k}|<\Lambda} \frac{d^d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2} \sim \begin{cases} \Lambda^{d-2} - r^{2-d} & d>2, \\ \log(\Lambda r) & d=2 \\ r - \Lambda^{-1} & d=1 \end{cases}$$

Transverse fluctuations are finite for $d>2$. For $d \leq 2$ fluctuations of Goldstones grow indefinitely $\Rightarrow$ destroys ordered phase for $d=2$.

$\Rightarrow$ In Tutorials, we will demonstrate this for the XY model.

This is called the Mermin-Wagner theorem:

A continuous symmetry cannot be spontaneously broken for $d \leq 2$.

There are no Goldstone modes in $d=2$ dimensions. (So for cont. symmetry lower critical dimension is $d_\ell = 2$!)

What is the fate of the "would-be" Goldstone modes?