Problem Set 14 – Statistical Physics B

Problem 1: The Langevin equation and the Einstein relation

Consider the Langevin equation for one-dimensional Brownian motion

$$m\frac{dv}{dt} = -\zeta v(t) + \xi(t), \qquad \frac{dx}{dt} = v(t),$$

with m the mass of the Brownian particle and ζ the friction constant. Here, the noise $\xi(t)$ is a stochastic process with zero mean $\langle \xi(t) \rangle = 0$, and

$$\langle \xi(t)\xi(t') \rangle = \Gamma\delta(t - t'),$$

with Γ a constant. Here $\langle ... \rangle$ is defined as the average over a subensemble with the same initial velocity $v_0 = v(0)$ but a different realisation of the noise. Since $\xi(t)$ is a stochastic process, the Langevin equation makes $x(t)$ also a stochastic process whose stochastic properties follow from $\xi(t)$.

(a) We only specified the first two moments of the noise which suffices for this exercise. In general, we assume that $\xi(t)$ denotes Gaussian white noise. What does this condition imply for $\langle \xi(t_1)\xi(t_2)\xi(t_3)\xi(t_4) \rangle$?

(b) Determine explicitly $v(t)$ for given v_0. What is $\langle v(t) \rangle$? Conclude from your expression why the Brownian particle is out of equilibrium.

(c) Determine the correlation function $\langle v(t)v(t') \rangle$. What is the expression for the equal-time correlator?

(d) The limit $\lim_{t\to\infty}\langle v^2(t) \rangle$ is well defined. What is the corresponding value? From it, determine the constant Γ and relate your result to the fluctuation-dissipation theorem.

(e) Determine $x(t) - x(0)$ and from it compute the value of the mean-squared displacement $\langle [x(t) - x(0)]^2 \rangle$.

(f) Show that the particle for small enough times undergoes ballistic movement, whereas for long times the motion is diffusive. What is the relation between the diffusion constant D and ζ? This is called the Einstein relation. What would the result be in three spatial dimensions?

(g) Determine the correlators $\langle v(t)\xi(t) \rangle$ and $\langle x(t)\xi(t) \rangle$.

Problem 2: Driven damped harmonic oscillator

Consider the equation of motion of a damped harmonic oscillator in the presence of a driving force $f(t)$,

$$m\ddot{x}(t) + \gamma\dot{x}(t) + m\omega_0^2 x(t) = f(t),$$

with m the mass, γ the friction constant, and ω_0 the (undamped) angular frequency of the oscillator.

(a) The equilibrium position is given for $x = 0$, which allows us to define the response function as

$$x(t) = \int_{-\infty}^{\infty} dt'\, \chi(t - t')f(t').$$

Compute $\tilde{\chi}(\omega)$. How big is the error here within the linear-response approximation?

(b) Determine $\tilde{\chi}'(\omega)$ and $\tilde{\chi}''(\omega)$. What symmetry properties are satisfied by $\chi'(\omega)$ and $\chi''(\omega)$? Plot these functions in the underdamped and overdamped regime for representative values of the parameters. What happens for $\gamma \to 0$?

(c) Consider the analytical continuation $\tilde{\chi}(z)$. From its pole structure, show that $\chi(t)$ satisfies causality both in the underdamped and the overdamped regime.

(d) Determine $\chi(t)$ and sketch this function for representative parameter values.

(e) Consider harmonic driving, $f(t) = f_0 \cos(\Omega t)$. Computed $\bar{P}(t)$, the dissipated power averaged over a *full* cycle, and show that only $\tilde{\chi}''(\omega)$ contributes.

(f) Show that the explicit forms of $\tilde{\chi}'(\omega)$ and $\tilde{\chi}''(\omega)$ found in (b) satisfy the Kramers-Kronig relations.

Problem 3: Kramers-Kronig relations

In the lecture we derived the Kramers-Kronig relations by analysing the integral

$$\oint_{\mathcal{C}} dz \, \frac{\tilde{\chi}(z)}{z - \omega_0}, \quad \omega_0 \in \mathbb{R},$$

with $\tilde{\chi}(z)$ the analytical continuation of the Fourier transformed response function $\tilde{\chi}(\omega)$. The closed contour $\mathcal{C}$ does not enclose the pole of above integrand at $z = \omega_0$. Show that the Kramers-Kronig relations still follow if we choose a contour $\mathcal{C}'$ that encloses the pole on the real axis.