Problem Set 3 – Statistical Physics B

Problem 1: Tricritical point

Consider the Landau free energy density with scalar order parameter m, defined by

$$f_{\mathrm{L}}(m,T) = \frac{1}{2}A(T)m^2 + \frac{B(T)}{4}m^4 + \frac{D(T)}{6}m^6 - hm.$$

For $B > 0$ we can neglect the sixth order term, but otherwise it should be retained. Unless stated otherwise, we consider $h = 0$ (zero external magnetic field).

(a) What can we say about the sign of D when $B < 0$?

(b) Show that there is a first-order phase transition for $B < 0$. Complement your calculation by sketching $f_{\mathrm{L}}(m)$ for various values of A.

(c) Determine the value A and the value of m in the ordered phase at this first-order phase transition.

(d) Draw the phase diagram in the (A, B) plane. Clearly indicate the order of the phase transitions. Sketch the form of f_{L} in the various regions of your plot.

(e) The first and second order phase boundary meet at the so-called tricritical point, defined by $T = T_{\mathrm{c}3}$ and $B = 0$. Explain why it is called a tricritical point.

(f) Near this point we write $A = a(T - T_{\mathrm{c}3})$ and define $t = (T - T_{\mathrm{c}3})/T_{\mathrm{c}3}$. We define the critical exponents α, β, δ, and γ by the relations

$$C(t, h = 0) \sim t^{-\alpha}, \quad m(t, h = 0) \sim t^{\beta}, \quad \chi(t, h = 0) \sim t^{-\gamma}, \quad m(t = 0, h) = h^{1/\delta}.$$

Here, C is the heat capacity at constant external magnetic field, and χ is the magnetic susceptibility. Determine the above critical exponents in the mean-field approximation.

(g) Draw all stable, metastable, and unstable branches of m as a function of t. Mark in your plot all spinodal and binodal points and consider both $B > 0$ and $B < 0$. Can this system experience hysteresis? If so, explain this with your sketches.

Problem 2: Exact solution of the one-dimensional Ising model

In one spatial dimension the canonical partition function of the Ising model is

$$Z = \sum_{\{s_i\}} \exp\left[\sum_{i=1}^{N}(hs_i + Ks_is_{i+1})\right],$$

with $h = \beta\mu H$ and $K = \beta J$. Here H is an external magnetic field and J is a coupling constant between neighbouring spins. Consider periodic boundary conditions, $s_{N+1} = s_1$. This case can be solved analytically with the so-called transfer matrix method. This exercise will guide you through this procedure.

(a) Show that $\sum_{i=1}^{N} s_i = \sum_{i=1}^{N}(s_i + s_{i+1})/2$. Use this result to show that $Z = \operatorname{Tr}\mathbf{z}^N$, with the matrix $\mathbf{z}$ given by

$$\mathbf{z} = \begin{pmatrix} e^{-h+K} & e^{-K} \\ e^{-K} & e^{h+K} \end{pmatrix}.$$

(b) Show using the eigenrepresentation of $\mathbf{z}$ that $Z = \lambda_+^N + \lambda_-^N$, with $\lambda_\pm$ the eigenvalues of $\mathbf{z}$. We define $\lambda_+ > \lambda_-$.

(c) Determine $\lambda_\pm$ and show that the Helmholtz free energy for $N \to \infty$ becomes

$$\beta F/N = -K - \ln\left(\cosh h + \sqrt{\sinh^2 h + e^{-4K}}\right).$$

(d) Compute the average magnetization and show that it vanishes for $h \to 0^+$. Does the one-dimensional Ising model with nearest-neighbour interactions exhibit a finite-temperature phase transition? Compare your result with the mean-field result and comment on it.

(e) Determine the spin-spin correlation function $C_{ij} = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$ in the thermodynamic limit. Show the asymptotic relation $C_{ij} \sim e^{-|i-j|/\xi}$ and derive an expression for ξ. What is C_{ij} in the mean-field approximation?