Problem Set 4 – Statistical Physics B

Problem 1: Infinite-range Ising model

Consider a one-dimensional Ising model consisting of N spins. Each spin interacts with *every* other spin with an exchange coupling $-J/N$ with $J > 0$. The energy of a spin configuration $\{s_i\}$ is given by

$$E(\{s_i\}) = -\frac{J}{2N}\sum_{i,j} s_i s_j,$$

where each spin variable can take the values $s_i = \pm 1$ with $i = 1, ..., N$. The canonical partition function is

$$Z(N,T) = \sum_{\{s_i\}} \exp\left(\frac{J}{2Nk_\mathrm{B}T}\sum_{i,j} s_i s_j\right).$$

(a) Perform a Hubbard-Stratonovich transformation to the variable m, which satisfies $\langle m\rangle = \langle N^{-1}\sum_i s_i\rangle$. Show that the partition function is given by

$$Z(N,T) = \sqrt{\frac{NJ}{2\pi k_\mathrm{B}T}}\int_{-\infty}^{\infty} dm\,\exp\left[-\frac{Nf_\mathrm{L}(m)}{k_\mathrm{B}T}\right],$$

where

$$f_\mathrm{L}(m) = \frac{J}{2}m^2 - k_\mathrm{B}T\ln\left[2\cosh\left(\frac{mJ}{k_\mathrm{B}T}\right)\right].$$

(b) For $T < T_\mathrm{c}$ the system exhibits a second order phase transition with m as order parameter. What are the physical differences between the $T > T_\mathrm{c}$ and $T < T_\mathrm{c}$ case? Explain your answer in detail.

(c) Determine the critical temperature T_c.

(d) For $T \uparrow T_\mathrm{c}$, we find $\langle m\rangle \propto (T_\mathrm{c} - T)^\beta$. Determine the critical exponent β.

(e) We now add a magnetic field, such that

$$f_\mathrm{L}(m,B) = \frac{J}{2}m^2 - k_\mathrm{B}T\ln\left[2\cosh\left(\frac{mJ}{k_\mathrm{B}T}\right)\right] - mB.$$

We define the free energy per spin as

$$F(T,B) = -k_\mathrm{B}T\lim_{N\to\infty}\frac{1}{N}\ln\left\{\int_{-\infty}^{\infty} dm\,\exp\left[-\frac{Nf_\mathrm{L}(m)}{k_\mathrm{B}T}\right\}\right].$$

Show that $F(T,B)$ is non-analytic for $T < T_\mathrm{c}$. In particular, show that $F(T < T_\mathrm{c}, B) \propto |B|^\eta$ and determine η.

(f) The infinite-range Ising model predicts a phase transition in 1D, whereas there is no phase transition in the 1D Ising model with just nearest-neighbour interactions. Explain this.

Problem 2: Gymnastics with functional derivatives

(a) Consider the functional $F[u] = \int_{-\infty}^{\infty} dx\,a(x)u(x)$, for a given function a. Determine $\dfrac{\delta F[u]}{\delta u(x)}$.

(b) Let $t_1 > 0$, on which the functional $G[u;t_1]$ depends parametrically. In particular, we set $G[u;t_1] = \int_0^\infty dt\,K(t_1,t)u(t)$. Compute $\dfrac{\delta G[u;t_1]}{\delta u(t)}$.

(c) Take the functional $H[u;x'] = u(x')$. Compute $\dfrac{\delta H[u]}{\delta u(x)}$.

(d) Determine $\dfrac{\delta I[u]}{\delta u(x)}$ for the functional $I[u] = \int_{-\infty}^{\infty} dx\, \ln[1+u(x)]$.

(e) Let $K : \mathbb{R}^3 \to \mathbb{R}$ be a completely symmetric function. Determine $\dfrac{\delta J[u]}{\delta u(x_1,x_2)}$ for the functional

$$J[u] = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_3\, K(x_1,x_2,x_3) u(x_1,x_2) u(x_2,x_3) u(x_3,x_1).$$

(f) Take the functional $S[u] = \int_0^{\infty} dt\, f(u(t),\dot{u}(t))$, where $\dot{u}(t) = du/dt$. Determine $\dfrac{\delta S[u]}{\delta u(t)} = 0$. Where have you seen this equation before in physics?

Problem 3: Profile of a magnetic domain wall

Consider the Landau theory for an Ising-like magnet with scalar order parameter $m(\mathbf{r})$.

$$F_{\mathrm{L}}[m] = \int d^3\mathbf{r} \left[\frac{K}{2}|\nabla m(\mathbf{r})|^2 + \frac{a(T-T_{\mathrm{c}})}{2} m(\mathbf{r})^2 + \frac{b}{4} m(\mathbf{r})^4\right].$$

In this exercise, we will investigate the structure of a so-called domain wall, where a region of positive magnetisation is adjacent to a region of negative magnetisation.

(a) Explain why K cannot be negative.

(b) Determine from the saddle-point approximation the equation that governs the shape of the magnetisation profile $m(\mathbf{r})$.

(c) We now assume that the system is translational invariant in the xy plane and we impose the boundary conditions $m(z \to \pm\infty) = \pm\bar{m}$. Derive an expression for $\bar{m}$. What happens when $T > T_{\mathrm{c}}$?

(d) Solve the differential equation you found in (b) with the boundary conditions from (c). In particular, show that the profile of a so-called domain wall within this theory is given by

$$m_{\mathrm{dw}}(z) = \bar{m} \tanh\left(\frac{z-z_0}{\xi}\right)$$

and give an expression for z_0 and ξ. What is the interpretation of z_0 and ξ? Sketch $m_{\mathrm{dw}}(z)$ and indicate in your plot how the parameters z_0 and ξ affect the profile.

(e) Determine the free energy cost of creating a domain wall by evaluating $\Delta F = F_{\mathrm{L}}[m_{\mathrm{dw}}] - F_{\mathrm{L}}[\bar{m}]$. Show that close to the critical point ΔF vanishes as $(T_{\mathrm{c}} - T)^{3/2}$. What happens with the size of the domain wall?