Problem Set 5 – Statistical Physics B

Problem 1: A useful integral

Consider the Fourier integral

$$I_d = \int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\alpha^2 + \gamma^2 \mathbf{k}^2}.$$

- (a) Compute I_1 using contour integration, i.e. by using the residue theorem.
- (b) Show that

$$I_3 = \frac{1}{4\pi\gamma} \frac{e^{-r/\xi}}{r},$$

and derive an expression for ξ . Hint: Go to spherical coordinates and choose a coordinate system where $\mathbf{k} \cdot \mathbf{r} = kr \cos \theta$ and perform first the integral over the angles.

The d=2 case will be considered in the hand-in exercise of this week.

Problem 2: Warming up with Gaussian integrals

- (a) Compute the Gaussian integral $G(\alpha) = \int_{-\infty}^{\infty} dx \, e^{-\frac{\alpha}{2}x^2}$ by first computing G^2 using polar coordinates. Is G convergent for all (complex) values of α ? What is the geometrical meaning of α ?
- (b) Consider now the Gaussian integral

$$G(\alpha, J) = \int_{-\infty}^{\infty} dx \, e^{-\frac{\alpha}{2}x^2 + Jx},$$

with J a real parameter. Evaluate $G(\alpha, J)$ by completing the square.

- (c) We can construct a probability distribution function P(x) by a suitable normalisation of the integrand of $G(\alpha)$. Determine P(x).
- (d) We define the expectation value as $\langle ... \rangle = \int_{-\infty}^{\infty} dx \, P(x)(...)$. Show by repeatedly acting with $\partial/\partial J$ on $G(\alpha, J)$ that

$$\langle x^{2n} \rangle = \frac{1}{\alpha^n} (2n-1)!!, \quad n = 1, 2, \dots$$

What is $\langle x^{2n+1} \rangle$?

(e) Sometimes $Z(J) := \langle e^{Jx} \rangle$ is called the moment-generating function of P(x). Explain this terminology.

Problem 3: Multivariable Gaussian integrals

We generalise Problem 2 to the multivariable N-dimensional case. We define real-valued vectors $\mathbf{x} = (x_1, ..., x_N)$ and $\mathbf{J} = (J_1, ..., J_N)$. Furthermore, \mathbf{A} is a positive definite, real, symmetric $n \times n$ matrix. Consider

$$Z(\mathbf{J}) = \int_{-\infty}^{\infty} dx_1 \dots \int_{-\infty}^{\infty} dx_N \, e^{-\frac{1}{2}\mathbf{x} \cdot \mathbf{A} \cdot \mathbf{x} + \mathbf{J} \cdot \mathbf{x}}.$$

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(a) Why can we always assume **A** to be symmetric?

(b) Consider the J=0 case. First assume that **A** is diagonal. Prove that

$$Z(0) = \frac{(2\pi)^{N/2}}{\sqrt{\det(\mathbf{A})}} = (2\pi)^{N/2} \exp\left[-\frac{1}{2} \text{Tr}(\log \mathbf{A})\right].$$

Use that any real symmetric matrix can be diagonalised by an orthogonal transformation to show that the above expression holds also for non-diagonal **A**.

(c) When $\mathbf{J} \neq 0$ it is convenient to find first the minimum of the quadratic form

$$\frac{\partial}{\partial x_k} \left(\frac{1}{2} \sum_{i,j=1}^N x_i A_{ij} x_j - \sum_{i=1}^N J_i x_i \right) = 0.$$

Explain why this is a good idea. Solve this equation for \mathbf{x} and perform a coordinate transformation to variables $(y_1, ..., y_N)$, where

$$x_i = \sum_{i=1}^{N} (\mathbf{A}^{-1})_{ij} J_j + y_i$$

to show that

$$Z(\mathbf{J}) = Z(0) \exp\left(\frac{1}{2}\mathbf{J} \cdot \mathbf{A}^{-1} \cdot \mathbf{J}\right).$$

(d) Compute using $Z(\mathbf{J})$ as a moment-generating function, the following expectation values:

$$\langle x_i \rangle, \quad \langle x_i x_j \rangle, \quad \langle x_i x_j x_k \rangle, \quad \langle x_i x_j x_k x_l \rangle.$$
 (1)

Can you find the general rule for computing expectation values of the form $\langle x_{i_1}...x_{i_m}\rangle$?

(e) Let $\mathbf{z} = (z_1, ..., z_N)$ be a complex vector and now we let \mathbf{J} to be complex as well. Show

$$Z(\mathbf{J}, \mathbf{J}^*) = \int \frac{d\mathbf{z}^* d\mathbf{z}}{(2\pi \mathbf{i})^n} \exp\left(-\mathbf{z}^* \cdot \mathbf{A} \cdot \mathbf{z} + \mathbf{z}^* \cdot \mathbf{J} + \mathbf{z} \cdot \mathbf{J}\right) = \exp\left[\mathbf{J}^* \cdot \mathbf{A}^{-1} \cdot \mathbf{J} - \text{Tr}(\log \mathbf{A})\right].$$

What conditions should be put on **A**?

(f) Determine $\langle z_i z_j^* \rangle$.