Problem Set 5 – Statistical Physics B

Problem 1: A useful integral

Consider the Fourier integral

$$I_d = \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\alpha^2 + \gamma \mathbf{k}^2}.$$

(a) Compute I_1 using contour integration, i.e. by using the residue theorem.

(b) Show that

$$I_3 = \frac{1}{4\pi\gamma} \frac{e^{-r/\xi}}{r},$$

and derive an expression for ξ. *Hint: Go to spherical coordinates and choose a coordinate system where* $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$ *and perform first the integral over the angles.*

The $d = 2$ case will be considered in the hand-in exercise of this week.

Problem 2: Warming up with Gaussian integrals

(a) Compute the Gaussian integral $G(\alpha) = \int_{-\infty}^{\infty} dx\, e^{-\frac{\alpha}{2}x^2}$ by first computing G^2 using polar coordinates. Is G convergent for all (complex) values of α? What is the geometrical meaning of α?

(b) Consider now the Gaussian integral

$$G(\alpha, J) = \int_{-\infty}^{\infty} dx\, e^{-\frac{\alpha}{2}x^2 + Jx},$$

with J a real parameter. Evaluate $G(\alpha, J)$ by completing the square.

(c) We can construct a probability distribution function $P(x)$ by a suitable normalisation of the integrand of $G(\alpha)$. Determine $P(x)$.

(d) We define the expectation value as $\langle ... \rangle = \int_{-\infty}^{\infty} dx\, P(x)(...)$. Show by repeatedly acting with $\partial/\partial J$ on $G(\alpha, J)$ that

$$\langle x^{2n} \rangle = \frac{1}{\alpha^n}(2n-1)!!, \quad n = 1, 2, \ldots .$$

What is $\langle x^{2n+1} \rangle$?

(e) Sometimes $Z(J) := \langle e^{Jx} \rangle$ is called the moment-generating function of $P(x)$. Explain this terminology.

Problem 3: Multivariable Gaussian integrals

We generalise Problem 2 to the multivariable N-dimensional case. We define real-valued vectors $\mathbf{x} = (x_1, ..., x_N)$ and $\mathbf{J} = (J_1, ..., J_N)$. Furthermore, $\mathbf{A}$ is a positive definite, real, symmetric $n \times n$ matrix. Consider

$$Z(\mathbf{J}) = \int_{-\infty}^{\infty} dx_1 ... \int_{-\infty}^{\infty} dx_N\, e^{-\frac{1}{2}\mathbf{x}\cdot\mathbf{A}\cdot\mathbf{x} + \mathbf{J}\cdot\mathbf{x}}.$$

(a) Why can we always assume $\mathbf{A}$ to be symmetric?

(b) Consider the $\mathbf{J} = 0$ case. First assume that $\mathbf{A}$ is diagonal. Prove that

$$Z(0) = \frac{(2\pi)^{N/2}}{\sqrt{\det(\mathbf{A})}} = (2\pi)^{N/2} \exp\left[-\frac{1}{2}\mathrm{Tr}(\log \mathbf{A})\right].$$

Use that any real symmetric matrix can be diagonalised by an orthogonal transformation to show that the above expression holds also for non-diagonal $\mathbf{A}$.

(c) When $\mathbf{J} \neq 0$ it is convenient to find first the minimum of the quadratic form

$$\frac{\partial}{\partial x_k}\left(\frac{1}{2}\sum_{i,j=1}^{N} x_i A_{ij} x_j - \sum_{i=1}^{N} J_i x_i\right) = 0.$$

Explain why this is a good idea. Solve this equation for $\mathbf{x}$ and perform a coordinate transformation to variables $(y_1, ..., y_N)$, where

$$x_i = \sum_{j=1}^{N} (\mathbf{A}^{-1})_{ij} J_j + y_i$$

to show that

$$Z(\mathbf{J}) = Z(0) \exp\left(\frac{1}{2}\mathbf{J} \cdot \mathbf{A}^{-1} \cdot \mathbf{J}\right).$$

(d) Compute using $Z(\mathbf{J})$ as a moment-generating function, the following expectation values:

$$\langle x_i \rangle, \quad \langle x_i x_j \rangle, \quad \langle x_i x_j x_k \rangle, \quad \langle x_i x_j x_k x_l \rangle. \tag{1}$$

Can you find the general rule for computing expectation values of the form $\langle x_{i_1} ... x_{i_m} \rangle$?

(e) Let $\mathbf{z} = (z_1, ..., z_N)$ be a complex vector and now we let $\mathbf{J}$ to be complex as well. Show that

$$Z(\mathbf{J}, \mathbf{J}^*) = \int \frac{d\mathbf{z}^* d\mathbf{z}}{(2\pi \mathrm{i})^n} \exp\left(-\mathbf{z}^* \cdot \mathbf{A} \cdot \mathbf{z} + \mathbf{z}^* \cdot \mathbf{J} + \mathbf{z} \cdot \mathbf{J}\right) = \exp\left[\mathbf{J}^* \cdot \mathbf{A}^{-1} \cdot \mathbf{J} - \mathrm{Tr}(\log \mathbf{A})\right].$$

What conditions should be put on $\mathbf{A}$?

(f) Determine $\langle z_i z_j^* \rangle$.