Problem Set 6 – Statistical Physics B

Problem 1: Spin waves in the XY model

Consider a (hyper)cubic lattice with lattice constant a in d dimensions with "volume" L^d. On each lattice point resides a classical spin $\vec{s}_i$, with $|\vec{s}_i|^2 = 1$ for $i = 1, ..., N$. The energy of a spin configuration –where we only include nearest neighbour interactions– is given by

$$H(\{\vec{s}_i\}) = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j.$$

Here, $J > 0$ is the nearest-neighbour coupling constant. We assume that at sufficiently low temperatures the system breaks $O(2)$ symmetry, and the system chooses a ground state with a definite spin direction. In this exercise, we will investigate the angular fluctuations with respect to this ground state. For this reason, we introduce a coordinate system where the spin variables are parametrised as $\vec{s}_i = (\cos\theta_i, \sin\theta_i)$ and we use periodic boundary conditions.

(a) At low temperatures the set of angles $\{\theta_i\}$ varies slowly from site to site. In the continuum limit where the set of discrete angles becomes an angular field $\theta(\mathbf{r})$, show that we find up to an irrelevant constant

$$\beta H = \frac{K}{2} \int d^d\mathbf{r} \, [\nabla\theta(\mathbf{r})]^2 = \frac{K}{2L^d} \sum_{\mathbf{k}} \mathbf{k}^2 |\theta_{\mathbf{k}}|^2,$$

where $\{\theta_{\mathbf{k}}\}$ are the Fourier modes of $\theta(\mathbf{r})$. Is $\theta_{\mathbf{k}}$ independent from $\theta_{-\mathbf{k}}$? How is K related to J?

(b) Demonstrate that the same result follows from diagonalising the Hamiltonian in terms of discrete spins directly, and then taking the long-wavelength limit.

(c) Write down the partition function of the system and show that $\langle \cos\theta \rangle = e^{-W}$, where

$$W = \frac{S_d \Lambda^{d-2}}{2(2\pi)^d (d-2) K}, \qquad d > 2,$$

and that $W \to \infty$ for $d \to 2$. Here, S_d is the surface area of a d-dimensional unit sphere, and Λ is a microscopic cut-off wavelength imposed by a. Give a physical interpretation for W. What do you conclude about the existence of long-ranged order?

(d) Show that the spin-spin correlation function in $d = 2$ is given by

$$\langle \vec{s}(\mathbf{r}) \cdot \vec{s}(\mathbf{0}) \rangle = \mathrm{Re}\left[\left\langle e^{\mathrm{i}[\theta(\mathbf{r}) - \theta(\mathbf{0})]} \right\rangle\right] = \exp\left\{-\frac{1}{2\pi K}\left[\tilde{\gamma} + \mathcal{O}((\Lambda r)^{-3/2}) + \ln(\Lambda r)\right]\right\}.$$

To find this result, you may use

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \, \frac{1 - \cos(x \cos\phi)}{x} = \frac{1 - J_0(x)}{x},$$

with J_0 the zeroth-order Bessel function of the first kind. Furthermore, the constant $\tilde{\gamma}$ is given by

$$\tilde{\gamma} = \int_0^1 du \, \frac{1 - J_0(u)}{u} - \int_1^\infty du \, \frac{J_0}{u} \approx -0.116.$$

(e) Show that for $r \to \infty$

$$\langle \vec{s}(\mathbf{r}) \cdot \vec{s}(\mathbf{0}) \rangle \sim \begin{cases} \exp(-r/\xi), & d = 1, \\ \left(\frac{r}{a}\right)^{-1/(2\pi K)}, & d = 2, \\ e^{-2W}, & d = 3. \end{cases}$$

Interpret your answer.

Problem 2: Asymptotic behaviour of correlations

Consider the Green's function $G(\mathbf{r} - \mathbf{r}') = \langle \delta\phi(\mathbf{r})\delta\phi(\mathbf{r}') \rangle$ of a scalar ϕ^4 theory within the Gaussian approximation in d spatial dimensions. The Fourier representation is given by

$$G(\mathbf{r}) = \frac{k_{\mathrm{B}}T}{K} \int \frac{d^d\mathbf{k}}{(2\pi)^d} \frac{e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2 + \xi^{-2}},$$

with ξ the correlation length. In the lectures we have determined this expression in real space by relating it to the underlying differential equation. Here, we provide an alternative derivation and investigate its asymptotic behaviour.

(a) Show that

$$\frac{1}{k^2 + \xi^{-2}} = \int_0^\infty ds\, e^{-s(k^2+\xi^{-2})}.$$

(b) Use the result from (a) to show that

$$G(r) = \frac{k_{\mathrm{B}}T}{K(4\pi)^{d/2}} \int_0^\infty ds\, s^{-d/2} e^{-r^2/4s - s/\xi^2}.$$

(c) Compute the integral you found in (b) using the integral representation of the modified Bessel function of the second kind, i.e. $K_\nu(z) = \frac{1}{2}\int_0^\infty ds\, s^{\nu-1} e^{-z(s+1/s)/2}$.

In principle, we could now use the asymptotics of K_ν to determine the asymptotics of $G(r)$. However, here we use an alternative route by using the method of steepest descent (also known as the saddle-point approximation). This method amounts to approximating integrals of the form

$$\mathcal{I}(\lambda) = \int_{-\infty}^{\infty} dq\, e^{-f(q)/\lambda},$$

for $\lambda \downarrow 0$. The idea is that the maximum contribution to the integral comes from where f has its minimum, whereas other contribution are exponentially suppressed. If we assume that f has a single (global) minimum at $q = q^*$, then

$$\mathcal{I}(\lambda) = e^{-f(q^*)/\lambda} \sqrt{\frac{2\pi\lambda}{f''(q^*)}} e^{-\mathcal{O}(\lambda^{1/2})}. \tag{1}$$

Note that actually we have already used this method when we considered mean-field theory with Gaussian fluctuations.

(d) Prove Eq. (1). Why are the neglected terms in the exponent of order $\lambda^{1/2}$?

(e) Apply the steepest descent method to $G(r)$ and show that

$$G(r) \sim \begin{cases} \frac{1}{r^{d-2}}, & r \ll \xi, \\ \frac{e^{-r/\xi}}{r^{(d-1)/2}}, & r \gg \xi. \end{cases}$$

Strictly speaking one cannot use the steepest-descent method for $r \ll \xi$. Could you explain why somehow it gives the correct result?

(f) Explain why this result shows that at the critical point fluctuations occur at all scales.