Geometry of null hypersurfaces

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Abstract

We review some basic natural geometric objects on null hypersurfaces. Gauss-Codazzi constraints are given in terms of the analog of canonical ADM momentum which is a well defined tensor density on the null surface. Bondi cones are analyzed with the help of this object.

1 Introduction

In Synge's festshrift volume [10] Roger Penrose distinguished three basic structures which a null hypersurface N in four-dimensional spacetime M acquires from the ambient Lorentzian geometry:

- the degenerate metric $g|_N$ (see [9] for Cartan's classification of them and the solution of the local equivalence problem)
- the concept of an affine parameter along each of the null geodesics from the two-parameter family ruling N
- the concept of parallel transport for tangent vectors along each of the null geodesics

Using all three concepts on N one can define several natural geometric objects which we shall review in this article.

In Section 2 we remind the structures which are presented in [1]. In the next section we give solutions, which are mostly based on [2], to the following questions:

- What is the analog of canonical ADM momentum for the null surface?
- What are the "initial value constraints"?
- Are they intrinsic objects?

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More precisely, we remind the construction of external geometry in terms of tensor density which is a well defined intrinsic object on a null surface. We already developed some applications of these object to the following subjects:

- Dynamics of the light-like matter shell from matter Lagrangian which is an invariant scalar density on N [3]
- Dynamics of gravitational field in a finite volume with null boundary and its application to black holes thermodynamics [6] (see also in this volume)
- Geometry of crossing null shells [4].

In the last section we apply our construction to Bondi cones.

2 Natural geometric structures on TN/K

We remind some standard constructions on null hypersurfaces (see [1]):

- time-oriented Lorentzian manifold M with signature $(-,+,+,+)$.
- null hypersurface N – submanifold with codim=1 and degenerate induced metric $g|_N$ $(0,+,+)$, K – time-oriented non-vanishing null vector field such that $K_p^\perp = T_pN$ at each point $p \in N$
    1. K is null and tangent to N, $g(X,K)$=0 iff X is a vector field tangent to N
    2. integral curves of K are null geodesic generators of N
    3. K is determined by N up to scaling factor being any positive function.
- $T_pN/K := \left\{\overline{X} : X \in T_pN\right\}$ where $\overline{X} = [X]_{\mathrm{mod}\ K}$ is an equivalence class of the relation mod K defined as follows:

$$X \equiv Y(\mathrm{mod}\ K) \iff X - Y \text{ is parallel to } K.$$

- $TN/K := \cup_{p\in N} T_pN/K$ vector bundle over N with 2-dimensional fibers (equipped with Riemannian metric h), the structure does not depend on the choice of K (scaling factor)

$$h : T_pN/K \times T_pN/K \longrightarrow \mathbb{R}\,, \quad h(\overline{X},\overline{Y}) = g(X,Y)\ .$$

- null Weingarten map b_K (depending on the choice of scaling factor, in non-degenerate case one can always take unit normal to the hypersurface but in null case the vectorfield K is no longer transversal to N and has always scaling factor freedom because its length vanishes)

$$b_K : T_pN/K \longrightarrow T_pN/K\,, \quad b_K(\overline{X}) = \overline{\nabla_X K}\,;$$

$$b_{fK} = f b_K\,, \quad f \in C^\infty(N)\,, \quad f > 0\,.$$

- null second fundamental form B_K (bilinear form associated to b_K via h)
$$B_K : T_pN/K \times T_pN/K \longrightarrow \mathbb{R}$$
$$B_K(\overline{X}, \overline{Y}) = h(b_K(\overline{X}), \overline{Y}) = g(\nabla_X K, Y)$$
Moreover, b_K is self-adjoint with respect to h and B_K is symmetric.
- N is totally geodesic (i.e. restriction to N of the Levi-Civita connection of M is an affine connection on N, any geodesic in M starting tangent to N stays in N) $\iff$ $B = 0$ (non-expanding horizon is a typical example).
- null mean curvature of N with respect to K
$$\theta := \operatorname{tr} b = \sum_{i=1}^{2} B_K(\overline{e}_i, \overline{e}_i) = \sum_{i=1}^{2} g(\nabla_{e_i} K, e_i)$$
where $\overline{e}_i$ is an orthonormal basis for T_pN/K, e_i is an orthonormal basis for T_pS in the induced metric on S which is a two-dimensional submanifold of N transverse to K.

We assume now that K is a geodesic vector field i.e. $\nabla_K K = 0$. Let us denote by prime covariant differentiation in the null direction:

$$\overline{Y}' := \overline{\nabla_K Y}\,, \quad b'(\overline{Y}) := b(\overline{Y})' - b(\overline{Y}')$$

From Riemann tensor we build the following curvature endomorphism

$$R : T_pN/K \longrightarrow T_pN/K\,, \quad R(\overline{X}) = \overline{Riemann(X,K)K}$$

and we get a Ricatti equation

$$b' + b^2 + R = 0\,. \tag{1}$$

Taking the trace of (1) we obtain well-known Raychaudhuri equation:

$$\theta' = -Ricci(K,K) - B^2\,, \quad B^2 = \sigma^2 + \frac{1}{2}\theta^2 \tag{2}$$

where σ is a shear scalar corresponding to the trace free part of B. A standard application of the Raychaudhuri equation gives the following

Proposition 1. *Let M be a spacetime which obeys the null energy condition, i.e. $Ricci(X,X) \geq 0$ for all null vectors X, and let N be a smooth null hypersurface in M. If the null generators of N are future geodesically complete then N has nonnegative null mean curvature i.e. $\theta \geq 0$.*

3 Canonical momentum on null surface

For non-degenerate hypersurface we define the canonical ADM momentum:

$$P^{kl} := \sqrt{\det g_{mn}}(g^{kl} g_{ij} \mathcal{K}^{ij} - \mathcal{K}^{kl})\,, \tag{3}$$

where $\mathcal{K}^{kl}$ is the second fundamental form (external curvature) of the imbedding of the hypersurface into the spacetime M.

Gauss-Codazzi equations for non-degenerate hypersurface are the following:

$$P_i{}^l{}_{|l} = \sqrt{\det g_{mn}}\, G_{i\mu} n^\mu \quad (= 8\pi \sqrt{\det g_{mn}}\, T_{i\mu} n^\mu)\,, \tag{4}$$

$$(\det g_{mn})\mathcal{R} - P^{kl}P_{kl} + \frac{1}{2}(P^{kl}g_{kl})^2 = 2(\det g_{mn}) G_{\mu\nu} n^\mu n^\nu \tag{5}$$
$$(= 16\pi (\det g_{mn}) T_{\mu\nu} n^\mu n^\nu)\,,$$

where $\mathcal{R}$ is the (three–dimensional) scalar curvature of g_{kl}, n^μ is a four–vector normal to the hypersurface, $T_{\mu\nu}$ is an energy–momentum tensor of the matter field, and the calculations have been made with respect to the non-degenerate induced three–metric g_{kl} (”$|$” denotes covariant derivative, indices are raised and lowered with respect to that metric etc.)

A null hypersurface in a Lorentzian spacetime M is a three-dimensional submanifold $N \subset M$ such that the restriction g_{ab} of the spacetime metric $g_{\mu\nu}$ to N is degenerate.

We shall often use adapted coordinates, where coordinate x^3 is constant on N. Space coordinates will be labeled by $k, l = 1, 2, 3$; coordinates on N will be labeled by $a, b = 0, 1, 2$; finally, coordinates on S will be labeled by $A, B = 1, 2$. Spacetime coordinates will be labeled by Greek characters α, β, μ, ν.

We will show in the sequel that null-like counterpart of initial data $(g_{kl}, P^k{}_l)$ consists of the metric g_{ab} and tensor density $Q^a{}_b$ which is a mixed (contravariant-covariant) tensor density.

The non-degeneracy of the spacetime metric implies that the metric g_{ab} induced on N from the spacetime metric $g_{\mu\nu}$ has signature $(0, +, +)$. This means that there is a non-vanishing null-like vector field K^a on N, such that its four-dimensional embedding K^μ to M (in adapted coordinates $K^3 = 0$) is orthogonal to N. Hence, the covector $K_\nu = K^\mu g_{\mu\nu} = K^a g_{a\nu}$ vanishes on vectors tangent to N and, therefore, the following identity holds:

$$K^a g_{ab} \equiv 0\ . \tag{6}$$

It is easy to prove that integral curves of K^a, after a suitable reparameterization, are geodesic curves of the spacetime metric $g_{\mu\nu}$. Moreover, any null hypersurface N may always be embedded in a one-parameter congruence of null hypersurfaces.

We assume that topologically we have $N = \mathbb{R}^1 \times S^2$. Since our considerations are purely local, we fix the orientation of the $\mathbb{R}^1$ component and assume that null-like vectors K describing degeneracy of the metric g_{ab} of N will be always compatible with this orientation. Moreover, we shall always use coordinates such that the coordinate x^0 increases in the direction of K, i.e. inequality $K(x^0) = K^0 > 0$ holds. In these coordinates degeneracy fields are of the form $K = f(\partial_0 - n^A\partial_A)$, where $f > 0$, $n_A = g_{0A}$ and we rise indices with the help of the two-dimensional matrix $\tilde{\tilde{g}}^{AB}$, inverse to g_{AB}.

If by λ we denote the two-dimensional volume form on each surface $\{x^0 = \text{const.}\}$:

$$\lambda := \sqrt{\det g_{AB}}\ , \tag{7}$$

then for any degeneracy field K of g_{ab} the following object

$$v_K := \frac{\lambda}{K(x^0)}$$

is a well defined scalar density on N. This means that

$$\mathbf{v}_K := v_K \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2$$

is a coordinate-independent differential three-form on N. However, v_K depends upon the choice of the field K.

It follows immediately from the above definition that the following object:

$$\Lambda = v_K \; K$$

is a well defined (i.e. coordinate-independent) vector density on N.

Obviously, it *does not depend* upon any choice of the field K:

$$\Lambda = \lambda(\partial_0 - n^A \partial_A) \tag{8}$$

and it is an intrinsic property of the internal geometry g_{ab} of N. The same is true for the divergence $\partial_a \Lambda^a$ which is, therefore, an invariant, K-independent, scalar density on N. Mathematically (in terms of differential forms) the quantity Λ represents the two-form:

$$\mathbf{L} := \Lambda^a \left(\partial_a \rfloor \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \right) \,,$$

whereas the divergence represents its exterior derivative (a three-from): $\mathrm{d}\mathbf{L} := (\partial_a \Lambda^a) \, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2$.

In particular, a null surface with vanishing $\mathrm{d}\mathbf{L}$ is the *non-expanding horizon.*

Both objects $\mathbf{L}$ and $\mathbf{v}_K$ may be defined geometrically, without any use of coordinates. For this purpose we note that at each point $p \in N$ the tangent space T_pN may be quotiented with respect to the degeneracy subspace spanned by K. The quotient space T_pN/K carries a non-degenerate Riemannian metric h and, therefore, is equipped with a volume form ω (its coordinate expression would be: $\omega = \lambda \; \mathrm{d}x^1 \wedge \mathrm{d}x^2$).

The two-form $\mathbf{L}$ is equal to the pull-back of ω from the quotient space T_pN/K to T_pN

$$\pi : T_pN \longrightarrow T_pN/K \,, \quad \mathbf{L} := \pi^* \omega \,.$$

The three-form $\mathbf{v}_K$ may be defined as a product:

$$\mathbf{v}_K = \alpha \wedge \mathbf{L} \,,$$

where α is *any* one-form on N, such that $< K, \alpha > \equiv 1$.

We have

$$\mathrm{d}\mathbf{L} = \theta \mathbf{v}_K$$

where θ is a null mean curvature of N.

The degenerate metric g_{ab} on N does not allow to define *via* the compatibility condition $\nabla g = 0$, any natural connection, which could be applied to generic tensor fields on N. Nevertheless, there is one exception: the degenerate metric defines *uniquely* a certain covariant, first order differential operator. The operator may be applied only to mixed (contravariant-covariant) tensor density fields $\mathbf{H}^a{}_b$, satisfying the following algebraic identities:

$$\mathbf{H}^a{}_b K^b = 0 \ , \quad \mathbf{H}_{ab} = \mathbf{H}_{ba} \ , \tag{9}$$

where $\mathbf{H}_{ab} := g_{ac}\mathbf{H}^c{}_b$. Its definition cannot be extended to other tensorial fields on N. Fortunately, the extrinsic curvature of a null-like surface and the energy-momentum tensor of a null-like shell are described by tensor densities of this type.

The operator, which we denote by $\overline{\nabla}_a$, is defined by means of the four-dimensional metric connection in the ambient spacetime M in the following way:
Given $\mathbf{H}^a_b$, take any its extension $\mathbf{H}^{\mu\nu}$ to a four-dimensional, symmetric tensor density, "orthogonal" to N, i.e. satisfying $\mathbf{H}^{\perp\nu} = 0$ ("$\perp$" denotes the component transversal to N). Define $\overline{\nabla}_a\mathbf{H}^a_b$ as the restriction to N of the four-dimensional covariant divergence $\nabla_\mu\mathbf{H}^\mu_\nu$. The ambiguities, which arise when extending three-dimensional object $\mathbf{H}^a_b$ living on N to the four-dimensional one, cancel finally and the result is unambiguously defined as a covector density on N. It turns out, however, that this result does not depend upon the spacetime geometry and may be defined intrinsically on N as follows:

$$\nabla_a\mathbf{H}^a_b = \partial_a\mathbf{H}^a_b - \frac{1}{2}\mathbf{H}^{ac}g_{ac,b} \ , \tag{10}$$

where $g_{ac,b} := \partial_b g_{ac}$, a tensor density $\mathbf{H}^a_b$ satisfies identities (9), and moreover, $\mathbf{H}^{ac}$ is *any* symmetric tensor density, which reproduces $\mathbf{H}^a_b$ when lowering an index:

$$\mathbf{H}^a_b = \mathbf{H}^{ac}g_{cb} \, . \tag{11}$$

It is easily seen, that such a tensor density always exists due to identities (9), but the reconstruction of $\mathbf{H}^{ac}$ from $\mathbf{H}^a_b$ is not unique because $\mathbf{H}^{ac} + CK^aK^c$ also satisfies (11) if $\mathbf{H}^{ac}$ does. Conversely, two such symmetric tensors $\mathbf{H}^{ac}$ satisfying (11) may differ only by CK^aK^c. Fortunately, this non-uniqueness does not influence the value of (10).

Hence, the following definition makes sense:

$$\overline{\nabla}_a\mathbf{H}^a{}_b := \partial_a\mathbf{H}^a_b - \frac{1}{2}\mathbf{H}^{ac}g_{ac,b} \ . \tag{12}$$

The right-hand-side does not depend upon any choice of coordinates (i.e. it transforms like a genuine covector density under change of coordinates).

To express directly the result in terms of the original tensor density $\mathbf{H}^a{}_b$, we observe that it has five independent components and may be uniquely reconstructed from $\mathbf{H}^0{}_A$ (2 independent components) and the symmetric two-dimensional matrix $\mathbf{H}_{AB}$ (3 independent components). Indeed, identities (9) may be rewritten as follows:

$$\mathbf{H}^A_B = \tilde{\tilde{g}}^{AC}\mathbf{H}_{CB} - n^A\mathbf{H}^0{}_B \ , \tag{13}$$

$$\mathbf{H}^0_0 = \mathbf{H}^0{}_A n^A \ , \tag{14}$$

$$\mathbf{H}^B_0 = \left(\tilde{\tilde{g}}^{BC}\mathbf{H}_{CA} - n^B\mathbf{H}^0{}_A\right) n^A \ . \tag{15}$$

The correspondence between $\mathbf{H}^a{}_b$ and $(\mathbf{H}^0{}_A, \mathbf{H}_{AB})$ is one-to-one.

To reconstruct $\mathbf{H}^{ab}$ from $\mathbf{H}^a{}_b$ up to an arbitrary additive term CK^aK^b, take the following (coordinate dependent) symmetric quantity:

$$\mathbf{F}^{AB} := \tilde{\tilde{g}}^{AC}\mathbf{H}_{CD}\tilde{\tilde{g}}^{DB} - n^A\mathbf{H}^0{}_C\tilde{\tilde{g}}^{CB} - n^B\mathbf{H}^0{}_C\tilde{\tilde{g}}^{CA} \ , \tag{16}$$

$$\mathbf{F}^{0A} := \mathbf{H}^0{}_C\tilde{\tilde{g}}^{CA} =: \mathbf{F}^{A0} \ , \tag{17}$$

$$\mathbf{F}^{00} := 0 \ . \tag{18}$$

It is easy to observe that any $\mathbf{H}^{ab}$ satisfying (11) must be of the form:

$$\mathbf{H}^{ab} = \mathbf{F}^{ab} + \mathbf{H}^{00}K^aK^b \ . \tag{19}$$

The non-uniqueness in the reconstruction of $\mathbf{H}^{ab}$ is, therefore, completely described by the arbitrariness in the choice of the value of $\mathbf{H}^{00}$. Using these results, we finally obtain:

$$\begin{aligned} \overline{\nabla}_a\mathbf{H}^a{}_b \quad &:= \quad \partial_a\mathbf{H}^a{}_b - \frac{1}{2}\mathbf{H}^{ac}g_{ac,b} = \partial_a\mathbf{H}^a{}_b - \frac{1}{2}\mathbf{F}^{ac}g_{ac,b} \\ &= \quad \partial_a\mathbf{H}^a{}_b - \frac{1}{2}\left(2\mathbf{H}^0{}_A\ n^A{}_{,b} - \mathbf{H}_{AC}\tilde{\tilde{g}}^{AC}{}_{,b}\right) \ . \end{aligned} \tag{20}$$

The operator on the right-hand-side of (20) is called the (three-dimensional) covariant derivative of $\mathbf{H}^a{}_b$ on N with respect to its degenerate metric g_{ab}. It is well defined (i.e. coordinate-independent) for a tensor density $\mathbf{H}^a{}_b$ fulfilling conditions (9). One can also show that the above definition coincides with the one given in terms of the four-dimensional metric connection and, due to (10), it equals:

$$\nabla_\mu\mathbf{H}^\mu{}_b = \partial_\mu\mathbf{H}^\mu{}_b - \frac{1}{2}\mathbf{H}^{\mu\lambda}g_{\mu\lambda,b} = \partial_a\mathbf{H}^a{}_b - \frac{1}{2}\mathbf{H}^{ac}g_{ac,b} \ , \tag{21}$$

hence, it coincides with $\overline{\nabla}_a\mathbf{H}^a{}_b$ defined intrinsically on N.

To describe exterior geometry of N we begin with covariant derivatives *along* N of the "orthogonal vector K". Consider the tensor ∇_aK^μ. Unlike in the non-degenerate case, there is no unique "normalization" of K and, therefore, such an object does depend upon a choice of the field K. The length of K vanishes. Hence, the tensor is again orthogonal to N, i.e. the components corresponding to $\mu = 3$ vanish identically in adapted coordinates. This means that ∇_aK^b is a purely three-dimensional tensor living on N. For our purposes it is useful to use the "ADM-momentum" version of this object, defined in the following way:

$$Q^a{}_b(K) := -s\left\{v_K\left(\nabla_bK^a - \delta^a_b\nabla_cK^c\right) + \delta^a_b\partial_c\Lambda^c\right\} \ , \tag{22}$$

where $s := \operatorname{sgn} g^{03} = \pm 1$. Due to the above convention, the object $Q^a{}_b(K)$ feels only *external orientation* of N and does not feel any internal orientation of the field K.

Remark: If N is a *non-expanding horizon*, the last term in the above definition vanishes.

The last term in (22) is K-independent. It has been introduced in order to correct algebraic properties of the quantity

$$v_K\left(\nabla_bK^a - \delta^a_b\nabla_cK^c\right) \ .$$

One can show that ${Q^a}_b$ satisfies identities (9) and, therefore, its covariant divergence with respect to the degenerate metric g_{ab} on N is uniquely defined. This divergence enters into the Gauss–Codazzi equations, which relate the divergence of Q with the transversal component ${\mathcal{G}^\perp}_b$ of the Einstein tensor density ${\mathcal{G}^\mu}_\nu = \sqrt{|\det g|}\left({R^\mu}_\nu - \delta^\mu_\nu \frac{1}{2} R\right)$. The transversal component of such a tensor density is a well defined three-dimensional object living on N. In coordinate system adapted to N, i.e. such that the coordinate x^3 is constant on N, we have ${\mathcal{G}^\perp}_b = {\mathcal{G}^3}_b$. Due to the fact that $\mathcal{G}$ is a tensor density, components ${\mathcal{G}^3}_b$ *do not change* with changes of the coordinate x^3, provided it remains constant on N. These components describe, therefore, an intrinsic covector density living on N.

Proposition 2. *The following null-like-surface version of the Gauss–Codazzi equation is true:*

$$\overline{\nabla}_a {Q^a}_b(K) + s v_K \partial_b \left(\frac{\partial_c \Lambda^c}{v_K}\right) \equiv -{\mathcal{G}^\perp}_b \ . \tag{23}$$

The proof is given in [3]. We remind the reader that the ratio between two scalar densities: $\partial_c \Lambda^c$ and v_K, is a scalar function θ. Its gradient is a covector field. Finally, multiplied by the density v_K, it produces an intrinsic covector density on N. This proves that also the left-hand-side is a well defined geometric object living on N.

The component $K^b {\mathcal{G}^\perp}_b$ of the equation (23) is nothing but a densitized form of Raychaudhuri equation (2) for the congruence of null geodesics generated by the vector field K.

4 Initial data on asymptotic Bondi cones

Recall (see [7]) that in Bondi-Sachs coordinates (u, x, x^A) the space-time metric takes the form:

$${}^4g = -xV\mathrm{e}^{2\beta}\mathrm{d}u^2 + 2\mathrm{e}^{2\beta}x^{-2}\mathrm{d}u\mathrm{d}x + x^{-2}h_{AB}\left(\mathrm{d}x^A - U^A\mathrm{d}u\right)\left(\mathrm{d}x^B - U^B\mathrm{d}u\right) \ . \tag{24}$$

Let us derive explicitly canonical data $(g_{ab}, {Q^a}_b)$ on null surfaces $N := \{u = \text{const.}\}$ which we call *Bondi cones.* The intrinsic coordinates on null surface N are $x^a = (x, x^A)$. We choose null field

$$K := \mathrm{e}^{-2\beta} x^2 \partial_x \, . \tag{25}$$

The components of the degenerate metric g_{ab} are as follows:

$$g_{AB} = x^{-2} h_{AB} \ , \quad g_{xA} = 0 = g_{xx} \ .$$

From (24), (25) and (22) we obtain the following formulae:

$$s{Q^a}_x(K) = 0 \tag{26}$$

$$s{Q^A}_B(K) = -\frac{1}{2}\sin\theta\, h^{AC}\left(x^{-2}h_{CB}\right)_{,x} \tag{27}$$

$$s{Q^x}_A(K) = x^{-2}\sin\theta\left(\beta_{,A} + \frac{1}{2}\mathrm{e}^{-2\beta}h_{AB}U^B_{,x}\right) \tag{28}$$

If we assume that Bondi cone data is polyhomogeneous and conformally $C^1 \times C^0$-compactifiable, it follows that (cf. [8])

$$h_{AB} = \breve{h}_{AB}(1 + \frac{x^2}{4}\chi^{CD}\chi_{CD}) + x\chi_{AB} + x^2\zeta_{AB} + x^3\xi_{AB} + O_{\ln^* x}(x^4) \ ,$$

where ζ_{AB} and ξ_{AB} are polynomials in $\ln x$ with coefficients which smoothly depend upon the x^A's. By definition of the Bondi coordinates we have $\det h = \det \breve{h} = \sin\theta$, which implies $\breve{h}^{AB}\chi_{AB} = \breve{h}^{AB}\zeta_{AB} = 0$. Further,

$$\beta = -\frac{1}{32}\chi^{CD}\chi_{CD}x^2 + Bx^3 + O_{\ln^* x}(x^4) \ , \tag{29}$$

$$h_{AB}U^B = -\frac{1}{2}\chi_A{}^B{}_{||B}x^2 + W_A x^3 + O_{\ln^* x}(x^4) \ , \tag{30}$$

where B and W_A are again polynomials in $\ln x$ with smooth coefficients depending upon the x^A's, while $\underset{\smile}{||}$ denotes covariant differentiation with respect to the unit sphere metric $\breve{h}$. This leads to the following approximate formulae:

$$sQ^A{}_B(K) \quad = \quad x^{-2}\sin\theta\left(x^{-1}\delta^A{}_B - \chi^A{}_B + O(x^2)\right) \tag{31}$$

$$sQ^x{}_A(K) \quad = \quad x^{-2}\sin\theta\left(-\frac{1}{2}x\chi_A{}^B{}_{||B} + O(x^2)\right) \tag{32}$$

$$g_{AB} \quad = \quad x^{-2}\left(\breve{h}_{AB} + x\chi_{AB} + O(x^2)\right) \tag{33}$$

It is easy to verify that the asymptotic behaviour of canonical data $(g_{ab}, Q^a{}_b)$ is determined by "free data" χ_{AB} which agrees with standard Bondi-Sachs approach to the null initial value formulation.

We hope that the variational formula on a truncated cone, which is space-like inside and light-like near Scri, (proposed in [5]) can be formulated with the help of the object $Q^a{}_b$ for arbitrary hypersurfaces, i.e. without assumption that the null part of the initial surface is a Bondi cone.

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