Group field theory with non-commutative metric variables

Aristide Baratin

joint work with D. Oriti

Max Planck Institute for Gravitational Physics
(Albert-Einstein-Institute)

Zakopane 2010

arXiv:1002.4723

Introduction

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- Similar question in LQG: Relation LQG/simplicial geometry Dittrich-Ryan, Dittrich-Speziale.

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- Similar hint in LQG: metric data encoded in electric flux variables which do not commute.

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Overview

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- Dual description of LQG kin. Hilbert space in terms of cylindrical functions on Lie algebras A.B, B. Dittrich, D. Oriti J. Tambornino
  Dual variables x_j interpreted as elementary flux variables
  (see Johannes' talk)

Outline

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Group field theory in a nutshell

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  - GFT for 3D gravity Boulatov '92
  - Generalized to 4D lattice BF theories Ooguri '92.
  - GFT for the Barrett-Crane spin foam model: De Pietri, Freidel, Krasnov, Rovelli '00.

- Universal structure behind spin foam framework:

  Any local spin foam model can be viewed as Feynman graph of a GFT

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- For the spin foam perspective, GFTs tackle issue of triangulation dependence
- But they do much more:
  - Provides a framework to compute quantum gravity amplitudes including the sum over all topologies.
  - This is a field theory: lots of tools at our disposal!
    Symmetry, renormalization...

GFT for 3d Riemannian gravity: $D = 3$, $G = \mathrm{SO}(3)$.

- Field $\varphi_{123} := \varphi(g_1, g_2, g_3)$, with invariance $\varphi(hg_i) = \varphi(g_i)\ \forall h \in G$.

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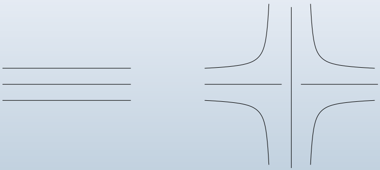
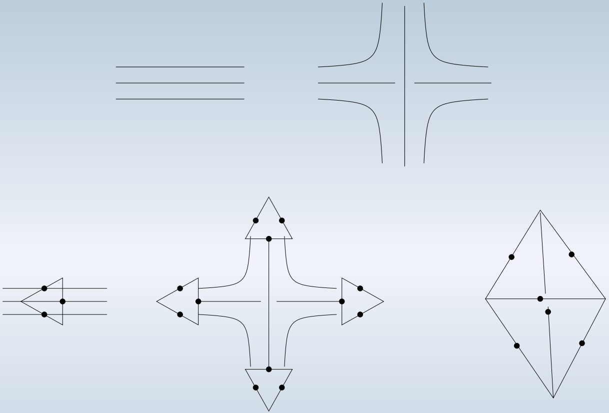
- Dynamics governed by the action: :

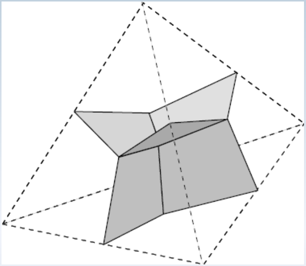
$$S = \frac{1}{2}\int [\mathrm{d}g]^3\, \varphi_{123}\varphi_{123} - \frac{\lambda}{4!}\int [\mathrm{d}g]^6\, \varphi_{123}\varphi_{345}\varphi_{526}\varphi_{641}$$



Group field theory in a nutshell
Building up space-time



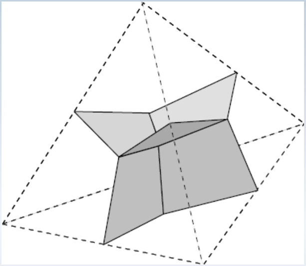
Feynman diagrams as 2-complexes dual to simplicial complexes (triangulated spaces)



- bulk variables $h_{t\tau}$: holonomy from the triangle t to the tetrahedron τ
- boundary variables g_{et}: holonomy from the edge e to the triangle t.

GFT Propagator and vertex:

$$\int \mathrm{d}h_t \prod_{e\subset t} \delta(g_{et} h_t \tilde{g}_{et}^{-1}), \quad \int \prod_{t} \mathrm{d}h_{t\tau} \prod_{\langle et \rangle} \delta(g_{et} h_{tt'} g_{et'}^{-1})$$



- Feynman amplitude: integral over discrete flat connections

$$I(\Gamma) = \int \prod_t \mathrm{d}h_t \prod_e \delta(\prod_{t \supset e} h_t)$$

- Spin representation of GFT using harmonic analysis on the gauge group

$$\varphi_{123} = \sum_{j_1,j_2,j_3} \phi^{j_1,j_2,j_3}_{m_1,m_2,m_3} D^{j_1}_{m_1 n_1}(g_1) D^{j_2}_{m_2 n_2}(g_2) D^{j_3}_{m_3 n_3}(g_3) C^{j_1,j_2,j_3}_{n_1,n_2,n_3}$$

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- Boundary observables described in terms of spin networks

- Lattice gauge theory picture:

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- Spin foam picture:

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$$I(\Gamma) = \int \prod_t \mathrm{d}h_t \prod_e \mathrm{d}X_e \prod_e e^{i \mathrm{Tr} X_e H_e(h)}, \quad H(e) = \delta(\prod_{t \supset e} h_t)$$

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Simplicial representation of GFT?

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Simplicial representation of 3d GFT

Going to metric variables: Fourier transform

- Define plane waves $\mathsf{e}_g(x) = e^{i\vec{p}_g \cdot \vec{x}}$ as functions on $\mathfrak{g} \sim \mathbb{R}^n$
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- Algebra structure on Im$\,\widehat{}\,$: $\mathsf{e}_{g_1} \star \mathsf{e}_{g_2} = \mathsf{e}_{g_1 g_2}$
  inherited from the convolution product on the group

Simplicial representation of 3d GFT

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- Fourier transform invertible on functions $f(g) = f(\text{-}g)$ of $\mathrm{SO}(3) \sim \mathrm{SU}(2)/\mathbb{Z}_2$:

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- With more work full Fourier transform on $\mathrm{SU}(2)$
[Freidel Majid 05; Joung, Mourad, Noui 08]

Simplicial representation of 3d GFT

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  $\star$-product $\leftrightarrow$ deformed addition of momenta preserving the bound.
- $\mathbb{R}^3$ seen with a finite resolution: Functions that can be sampled by discrete values f^j_{mn} without loss of information:

$$\widehat{f}(x) = \sum_{j,m,n} f^j_{mn} \widehat{D}^j_{mn}(x)$$

Simplicial representation of 3d GFT

Dual field, action, Feynman rules

- Fourier transform of the Boulatov field φ_{123}:

$$\widehat{\varphi}_{123} := \hat{\varphi}(x_1, x_2, x_3) = \int [\mathrm{d}g]^3 \, \varphi_{123} \, \mathsf{e}_{g_1}(x_1) \mathsf{e}_{g_2}(x_2) \mathsf{e}_{g_3}(x_3)$$

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- Gauge invariance dual to a closure constraint:

$$\widehat{P\varphi} = \widehat{C} \star \widehat{\varphi}, \quad \widehat{C}(x_1, x_2, x_3) = \delta_0(x_1 + x_2 + x_3)$$

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  - Dual field as a (non-commutative) triangle
  - Field variables as metric variables associated to the edges.

Action

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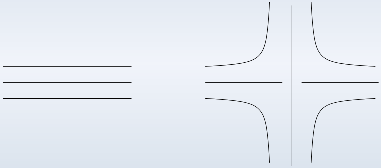

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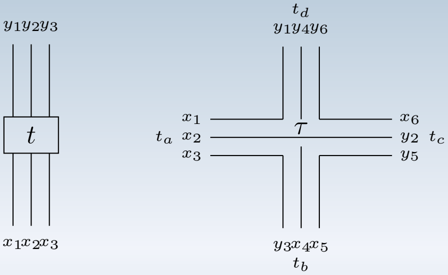

$$\int \mathrm{d}h_t \prod_{i=1}^{3} (\delta_{-x_i} \star \mathsf{e}_{h_t})(y_i), \quad \int \prod_t \mathrm{d}h_{t\tau} \prod_{i=1}^{6} (\delta_{-x_i} \star \mathsf{e}_{h_{tt'}})(y_i)$$

- h_t: parallel transport through the triangle t.
- $h_{t\tau}$: parallel transport from the tetrahedron τ to triangle t.
- $h_{tt'} := h_{t\tau} h_{\tau t'}$

- Join strands using the $\star$-product, keeping track of ordering.
- Each loop of strands bound a face of the 2-complex, dual to an edge of the triangulation.
- Under integration over holonomies h, product of face amplitudes $A_f[h]$.

Simplicial representation of 3d GFT
Feynman amplitudes

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- After integration over $x_j, \quad j \neq 0$:

$$I(\Gamma) = \int \prod_t \mathrm{d}h_t \prod_e \mathrm{d}x_e \, e^{i \sum_e \mathrm{Tr}\, x_e H_e}$$

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  - x_e metric associated to e in the frame of a reference tetrahedron around e

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  $A_f[h]$ is the cyclic $\star$-product:

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- identifies edge metrics x_j seen from the different tetrahedra j up to holonomies
- After integration over $x_j, \quad j \neq 0$:

$$I(\Gamma) = \int \prod_t \mathrm{d}h_t \prod_e \mathrm{d}x_e \, e^{i\sum_e \mathrm{Tr}\, x_e H_e}$$

  - x_e metric associated to e in the frame of a reference tetrahedron around e
  - H_e holonomy of e starting from the reference frame.

Simplicial representation of 3d GFT

Feynman amplitudes

- Boundary observables inserted using $\star$-product

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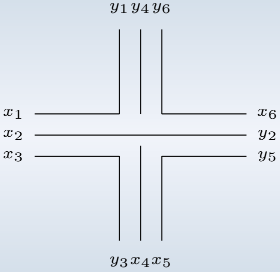
- One-vertex contribution to the 4-point function:



$$\int \prod_t \mathrm{d}h_{t\tau} \prod_{i=1}^{6} (\delta_{-x_i} \star \mathsf{e}_{h_{tt'}})(y_i)$$

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  - Intertwiner C^{j_i} implement closure of each triangle.

Outline

Towards 4d gravity models

Going up dimensions: Ooguri model

- Fourier transform extend to fields on $\mathrm{SO}(4) \sim \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$

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Imposing simplicity: non-commutative tetrahedron

- $S_{\text{grav}}[e, A] = \int \text{tr}\, B \wedge F(A), \qquad B = \star e \wedge e$
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Projector onto fields on $\mathrm{SO}(4)/\mathrm{SO}(3)_k$. $k = 1$: Barrett-Crane projector

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imposes simplicity of bivectors x_t in each of the frames associated to the simplices j around t.

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- Study of the model in progress...

Outline

Conclusion

Summary

- New representation of GFT as non-local and non-commutative fields on Lie algebras:

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Explicit GFT duality spin foams/ simplicial path integrals

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- New proposal for GFT model: study under way.

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  - deepen links between GFT formalism and non-commutative geometry