From quantum gravity to the Friedmann equation

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Much has happened in the last few years in QG. Using several of these developments (new vertex, coherent states, asymptotic analysis, 2-point function calculation, holomorphic representation, KKL extension, loop quantum cosmology, ACH construction ...), it is possible to outline a general QG theory. Here:

• 1. The theory without "derivations from classical GR".
→ Extremely natural definition; $SU(2) \subset SL(2, C) \rightarrow$ Einstein equations?

• 2. → A transition amplitude computation in cosmology. ("Spinfoam Cosmology")
LQG → Friedmann equation! **(Bianchi, CR, Vidotto)**.

Emanuele Alesci, Jonh Barrett, Marco Valerio Battisti, Eugenio Bianchi, Florian Conrady, You Ding, Richard Dowdall, Jonathan Engle, Winston Fairbairn, Laurent Freidel, Frank Hellmann, Wojciech Kaminski, Marcin Kisielowski, Kirill Krasnov, Etera Livine, Jurek Lewandowski, Elena Magliaro, Roberto Pereira, Claudio Perini, CR, Simone Speziale, Francesca Vidotto. And I am sure I forget somebody.

This presentation is strongly influenced by discussions with, and lectures by, **Eugenio Bianchi.**

Hilbert space

$$\tilde{\mathcal{H}} = \bigoplus_{\Gamma} \mathcal{H}_\Gamma.$$

A graph Γ is a set of L links l and N nodes n, together with two relations s (source) and t (target) assigning a source node $s(l)$ and a target node $t(l)$ to every link l. (**crf. Bahr)**

Graph Hilbert space: $\mathcal{H}_\Gamma = L_2[SU(2)^L/SU(2)^N]$ where: $\psi(U_l) \to \psi(V_{s(l)} U_l V_{t(l)}^{-1})$.

Peter Weyl: $L_2[SU(2)^L] = \oplus_{j_l} \otimes_l (\tilde{\mathcal{H}}^*_{j_l} \otimes \tilde{\mathcal{H}}_{j_l})$.

The two factors above are associated to the two nodes $s(l)$ and $t(l)$. Hence $\tilde{\mathcal{H}}_\Gamma = \oplus_{j_l} \otimes_n \tilde{\mathcal{H}}_n$

Node Hilbert space : $\tilde{\mathcal{H}}_n = \otimes_{l \in n} \mathcal{H}_l$.

Intertwiner space: $\mathcal{H}_n = \mathrm{Inv}_{SU(2)}[\tilde{\mathcal{H}}_n]$.

Then

$$|\Gamma, j_l, v_n\rangle \in \tilde{\mathcal{H}} = \bigoplus_{\Gamma} \bigoplus_{j_l} \bigotimes_{n} \mathcal{H}_n.$$

This is the Hilbert space on which loop quantum gravity is defined.

Divide $\tilde{\mathcal{H}}$ by appropriate identifications: $\tilde{\mathcal{H}}/\sim$. (i): if Γ is a subgraph of Γ' then we must identify $\mathcal{H}_\Gamma$ with a subspace of $\mathcal{H}_{\Gamma'}$. (ii) divide $\mathcal{H}_\Gamma$ by the action of the discrete group of the automorphisms of Γ.

Operators

L_l^i: flux of Ashtekar's electric field, or the flux of the inverse triad, across "an elementary surface cut by" the link l.

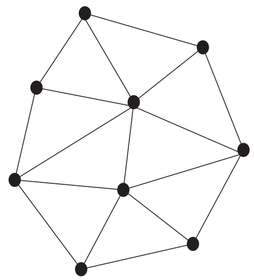
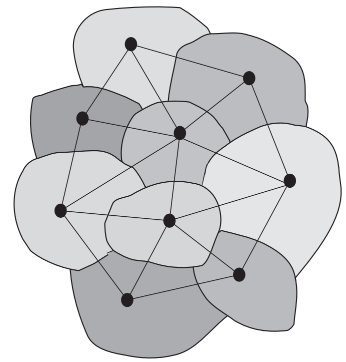
Area operator

$$A_\Sigma = 8\pi\gamma\hbar G \sum_{l \in \Sigma} \sqrt{L_l^i L_l^i}.$$

Volume operator

$$V_R = c_n (\gamma\hbar G)^{\frac{3}{2}} \sum_{n \in R} \sum_{l_a l_b l_c \in n} \sqrt{|\epsilon_{ijk} L_{l_a}^i L_{l_b}^j L_{l_c}^k|}.$$

Physical picture

$|\Gamma, j_l, v_n\rangle$

“Granular” space. A node n determines a “grain” or “chunk” of space.

The volume of each grain n is v_n. Two grains n and n' are adjacent if there is a link l connecting the two. In this case the area of the elementary surface separating the two grains is $8\pi\gamma\hbar G\sqrt{j_l(j_l+1)}$.

Variants. Γ is the two-skeleton dual to a triangulation of a 3d space $\rightarrow$ flat tetrahedra. In some cases *to* 3d Regge geometry. “Twisted geometries”.

Important: Fixed time states $\rightarrow$ Boundary states.

Coherent states

Partially coherent: LS-states (**Livine-Speziale**);
Fully coherent: FS-states (**Freidel-Speziale**) or holomorphic states (**Thiemann-Bianchi-Magliaro-Perini**).

L^3 eigenstates: $L^3|j,m\rangle = m|j,m\rangle, \qquad m = -j, -j+1\ldots, j-1, j.$

Coherent states: $|j,\vec{n}\rangle = D^j(R_{\vec{n}})|j,j\rangle.$

LS-states: $$|j_l, i_{\vec{n}_l}\rangle = \int_{SU(2)} dg \ \bigotimes_{l\in n} D^{j_l}(g)|j_l,\vec{n}_l\rangle \in \mathcal{H}_n.$$

Heat kernel on $SU(2)$: $K_t(g) = \sum_j (2j+1)e^{-j(j+1)t}\chi(g)$

Holomorphic-states: $$\psi_{H_l}(U_l) = \int_{SU(2)^N} dg_n \bigotimes_{l\in\Gamma} K_t(g_{s(l)}H_l g_{t(l)}^{-1} U_l^{-1}).$$

cfr: Theimann's et al complexifier's states. Ashtekar's et al holomorphic representation.

Relation:

$$H = D^{\frac{1}{2}}(R_{\vec{n}})\ e^{-i(\xi+i\eta)\sigma_3/2}\ D^{\frac{1}{2}}(R^{-1}_{\vec{n}'})$$

such that one finds for large η_l

$$\langle j_l, i^n_{\vec{n}_l} | \psi_{H_l(\vec{n}_l, \vec{n}'_l, \eta_l, \xi_l)} \rangle \sim \prod_l e^{-\frac{(j_l - j^0_l)^2}{2\sigma_l}}\ e^{i\xi_l j_l}$$

identifying $\vec{n}$ and $\vec{\tilde{n}}$ with the $\vec{n}$ in $s(l)$ and $t(l)$ respectively and with $2j^0_l + 1 = \eta_l / t_l$ and $\sigma_l = 1/(2t_l)$.

The holomorphic states are superpositions of LS states forming wave packets on the spins!

"Geometrical" interpretation for the $(\vec{n}, \vec{n}', \xi, \eta)$ labels: ξ is the angle between the normals of the tetrahedra. (Freidel-Speziale).

$\mathcal{H}_\Gamma$ contains an (over-complete) basis of "wave packets" $\psi_{H_l} = \psi_{\vec{n}_l, \vec{n}'_l, \xi_l, \eta_l}$, with a nice interpretation as discrete classical geometries with intrinsic and extrinsic curvature.

Derivations

1.

Phase space of GR in the Ashtekar formulation $\longrightarrow$ Poisson algebra of observables $\longrightarrow$ represent in terms of operators on a Hilbert space $\longrightarrow$ factor away the relevant gauge invariances. 3d diffeomorphism "washes away" the location of the graph Γ in Σ.

2.

Discretize GR on a 4d lattice with a boundary, and study the resulting boundary Hilbert space of the lattice theory. Then

$$\mathcal{H}_\Gamma^{SL(2C)} = L_2[SL(2,C)^L/SL(2,C)^N].$$

States: $\psi(H_l), H_l \in SL(2C)$, where $H_l \sim P \exp \int_l A$.

Generators J of the Lorentz group are $\sim B = e \wedge e$. More precisely, since

$$S = \int (B + \frac{1}{\gamma} B^*) \wedge F[A]$$

$$J = (B + \frac{1}{\gamma} B^*).$$

Relation with $SL(2C)$: the map f

What is the relation between the QG Hilbert space $\mathcal{H}_\Gamma^{SU(2)}$ and the Lorentzian Hilbert space $\mathcal{H}_\Gamma^{SL(2C)}$?

Peter-Weyl again

$$\mathcal{H}_\Gamma^{SL(2C)} = \sum_{(p_l,k_l)} \bigotimes_l (\mathcal{H}^*_{(p_l,k_l)} \otimes \mathcal{H}_{(p_l,k_l)}).$$

$(p \in R, k \in Z^+/2)$ labels of $SL(2C)$ unitary irreducible representations. Decompose each Lorentz irreducible into a sum of $SU(2)$ irreducibles.

$$\mathcal{H}_{(p,k)} = \bigoplus_{j'=k}^{\infty} \mathcal{H}_{j'}$$

The *first* term of this sum $\mathcal{H}_{j'=k} \subset \mathcal{H}_{(p,k)}$, namely the lowest spin irrep, is important.

Let

$$f : \mathcal{H}_\Gamma^{SU(2)} \rightarrow \mathcal{H}_\Gamma^{SL(2C)}$$

$$f : \mathcal{H}_j \longmapsto \mathcal{H}_j \subset \mathcal{H}_{(p=\gamma j, k=j)}.$$

Image of f is the subspace of $\mathcal{H}_\Gamma^{SL(2C)}$ where

$$p_l = \gamma j_l, \qquad k_l = j_l,$$

and $j'_l = k_l = j_l$.

$\vec{L}$ and $\vec{K}$: generators of the rotations and the boosts in $SL(2C)$ then:

$$\langle\psi|\vec{K} - \gamma\vec{L}|\phi\rangle = 0.$$

Thus, the image of f is a subspace of $L_2[SL(2,C)^L/SL(2,C)^N]$ where the constraints

$$\vec{K} = \gamma\vec{L}$$

are implemented weakly (You Ding, CR).

These implies $B = e \wedge e$.

Thus: the image of the natural map $f : \mathcal{H}_\Gamma^{SU(2)} \to \mathcal{H}_\Gamma^{SL(2C)}$ is the subspace where the boundary states of a covariant quantum GR live.

Transition amplitudes

Dynamics is given by a linear functional W on $\mathcal{H}$.

$$P(\psi) = |\langle W|\psi\rangle|^2$$

is the probability associated to the process described by the boundary state ψ.

Guidelines for constructing W (Eugenio Bianchi):

1. Locality. Elementary amplitude $W_v \leftrightarrow$ elementary process (QFT).
2. H is a density. Acts on nodes. This is the key results that started LQG at the end of the 80'. $W_v(j_l, v_n) = \langle W_v|j_l, v_n\rangle$.
3. Lorentz invariance. There should be a map from $SU(2)$ spin networks to a Lorentz covariant language that characterizes the vertex.

I am now ready to define the vertex. The vertex amplitude of LQG is : ....

.

$$\langle W_v | \psi \rangle = (f\psi)(\mathbb{1})$$

(**EPRL-FK-LS-KKL**)

$$\langle W_v | \psi \rangle = (f\psi)(\mathbb{1})$$

Astonishingly, this appears to give the Einstein equations in the large distance classical limit.

Cfr. QED: $$\langle W | \psi^A_{e_1} \otimes \psi^B_{e_2} \otimes \psi^\gamma_\mu \rangle = e\ \gamma^{AB}_\mu\ \delta(p_{e_1} + p_{e_2} + p_\gamma)$$

The amplitude takes a very manageable form when written in terms of coherent states

$$W_v(j_l, \vec{n}_l, \vec{n}'_l) = \int dg_n \bigotimes_l \langle \vec{n}_l | g_{s(l)} g^{-1}_{t(l)} | \vec{n}'_l \rangle_{(\gamma j, j)}$$

Even more in the Euclidean theory:

$$j^{\pm} = \frac{|1 \pm \gamma|}{2}\ j$$

$$W_v(j_l, \vec{n}_l, \vec{n}'_l) = \int dg dh \bigotimes_l \left(\langle \vec{n}_l | g_{s_l} g^{-1}_{t_l} | \vec{n}'_l \rangle^{2j^+} \langle \vec{n}_l | h_{s_l} h^{-1}_{t_l} | \vec{n}'_l \rangle^{2j^-} \right)$$

The full amplitude: spinfoams

A two-complex is a set of faces f meeting at edges e, in turn meeting at vertices v.

A spinfoam σ: is a two-complex colored with j_f and i_e.

If we "cut a spinfoam with a 2d-surface", we obtain a spin network:
edges $e \to$ nodes n ; faces $f \to$ links l.
In particular, an S_3 surface surrounding a vertex v of σ defines a spin network ψ_v.

The vertex amplitude of the vertex v of σ is defined to be

$$W_v(\sigma) := \langle W_v | \psi_v \rangle.$$

The amplitude of a spinfoam

$$\langle W_v | \psi \rangle = \sum_{\sigma} \prod_{f} d(j_f) \prod_{v} W_v(\sigma). \tag{1}$$

$d(j_f) = (2j_f + 1)$

The expression (**??**) fully defines a quantum theory of gravity. All that remains is to extract physics from this theory, and show that it gives GR in some limit.

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(The last sentence is a provocative over-statment.)

Physical transition amplitudes

$\mathcal{R}$: ball in spacetime; Σ its boundary. For any state ψ on Σ, interpret

$$P(\psi) = |\langle W|\psi_{H_l}\rangle|^2$$

as a probability associated to the process defined by the boundary state ψ.

In particular: if $\mathcal{R}$ is a ball in Einstein spacetime; Σ its boundary. Σ has intrinsic and and extrinsic geometry (g, k). Choose holomorphic-state ψ_{H_l} peaked on (g, k). Then I expect

$$P(\psi_{H_l}) = |\langle W|\psi_{H_l}\rangle|^2 \sim 1.$$

And

$$W_{H_l}(E_1, ..., E_n) = \langle W|E_1...E_n|\psi_{H_l}\rangle :$$

scattering amplitude between the n "particles" (quanta).

But: **there is no physics without approximation!**

Approximations:

1. Graph expansion Γ with a small number L of links. $\sim$ multipole expansion.

2. Large-distance expansion $\mathcal{R}$ large. Boundary state peaked on boundary geometry large compared with the Planck length. Holomorphic boundary states ψ_{H_l} where $\eta_l >> 1$ in each H_l.

   Vertex in this limit (Barrett, Dowdall, Fairbairn, Hellmann, Pereira):

$$W_v \sim e^{iS_{Regge}} + e^{-iS_{Regge}} + K$$

   where S_{Regge} is the functional of the boundary variables, under the identifications of these with variables describing a Regge geometry.

   Using the holomorphic representation:

$$W_v \sim e^{iS_{Regge}}$$

   The sum disappears in the holomorphic representation! (Bianchi Magliaro Perini) $\rightarrow$ strong hint that the theory can yield GR.

3. Vertex expansion In the number N of vertices of σ. In which regime is this expansion valid? Hint: Regge interpretation of the vertex amplitude $\rightarrow$ flatness!

Can we extract physics?

$(N{=}1,\ L{=}10,\ EEEE{=}g^{ab}g^{cd}) \rightarrow$ **The graviton propagator**

Choose $N = 1$, $L = 10$ and first order in $1/\eta_l$. $\Gamma = \Gamma_5$. ψ_L given by the (intrinsic and extrinsic) geometry of the boundary of a *regular* 4-simplex of size L.

$$
\begin{aligned}
W^{abcd}_{mn} &= \langle W|\vec{L}_{na}\cdot\vec{L}_{nb}\ \vec{L}_{mc}\cdot\vec{L}_{md}|\psi_L\rangle \\
&\quad -\langle W|\vec{L}_{na}\cdot\vec{L}_{nb}|\psi_L\rangle\langle W\vec{L}_{mc}\cdot\vec{L}^{j}_{md}|\psi_L\rangle.
\end{aligned} \tag{2}
$$

$m, n, a, b... = 1, ..., 5$: nodes of Γ_5.

Compare with

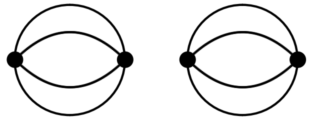
$$
\begin{aligned}
W^{abcd}(x_m, x_n) &= \langle 0|g^{ab}(x_n)g^{cd}(x_m)|0\rangle \\
&\quad - \langle 0|g^{ab}(x_n)|0\rangle\ \langle 0|g^{cd}(x_m)|0\rangle.
\end{aligned}
$$

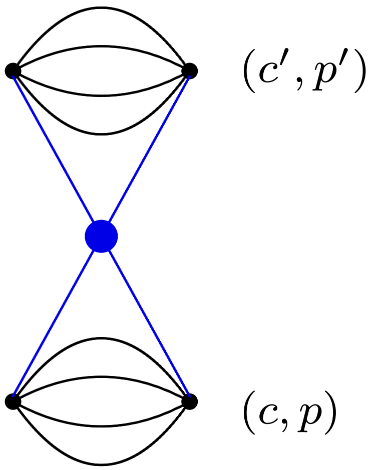
in conventional QFT, where $g^{ab}(x)$ is the gravitational field operator.

(**??**) has been computed recently in the Euclidean theory and shown to converge to the free progatator of GR in the large L limit (Bianchi, Magliaro, Perini).

It is clear that n point functions in gravity can be computed order by order.

$(N{=}1,\ L{=}8,\ \text{no } EE) \rightarrow$ Spinfoam Cosmology

Choose $N = 1$, $L = 8$ where $\Gamma =$  and first order in $1/\eta_l$.



Choose coherent states $|H_l\rangle$ describing a homogeneous and isotropic geometry. This geometry is determined by (c, p) in the past and two (c', p') in the future. Let $z = c + ip$.

Then $$\langle W|\psi_{H_{l(z,z')}}\rangle = W(z, z').$$

This can be computed explicitly! (Bianchi, Vidotto, CR)

$$W(z, z') = C\ zz' e^{-\frac{z^2+(z')^2}{\hbar}}$$

This resulting amplitude happens to satisfy an equation

$$\hat{H}(z, \frac{d}{dz})W(z', z) = -\left(-\hbar^2 \frac{d^2}{dz^2} + z^2 + \frac{3}{2}\hbar\right)^2 W(z', z) = 0.$$

Take the classical limit of this operator (that is: $\hbar\frac{d}{dz} \to \bar{z}, \hbar \to 0$). This gives

$$H \to -(-\bar{z}^2 + z^2)^2 = p^2 c^2 = (Vol)\sqrt{p}c^2 = (Vol)\dot{a}^2 a$$

This is precisely the hamiltonian constraint of a homogeneous and isotropic cosmology.

$\to$ LQG yields the Friedmann equation.

Homeworks

1. Find other computable amplitudes.
2. Compute the propagator (**??**) in the Lorentzian theory (easier in the holomorphic basis?).
3. Compute the three point function and compare it with the vertex amplitude of conventional perturbative quantum gravity on Minkowski space.
4. Compute the next order of the two-point function, for $N = 2$.
5. Compute the next order of the two-point function, for $L > 10$.
6. Understand the normalization factors in these terms, and their relative weight. Find out under which conditions the expansion is viable.
7. Study the possible (infrared) divergences in (**??**). The sum can be split into a sum over two-complexes and a sum over labelings (spin and intertwiners) for a given two complex. The potential divergences of the second are associated to "bubbles" (nontrivial elements of the second homotopy class) in the two complex. Classify them and study how do deal with these.
8. Do the infrared radiative corrections renormalize the vertex amplitude?
9. Do radiative corrections generate new vertices? Or is the vertex protected by the Lorentz symmetry?
10. Use the analysis of the these divergences to study the scaling of the theory.

11. Does G scale?

12. Study the quantum corrections that this theory adds to the tree-level n-point functions of classical general relativity. Can any of these be connected to potentially observable phenomena?

13. Is there any reason for a breaking of local Lorentz invariance, that could lead to observable phenomena such as γ ray bursts energy-dependent time of arrival delays, in this theory?

14. Find a simple group field theory whose expansion gives (**??**).

15. Compute more in spinfoam cosmology, and compare with standard LQC.

16. Compare with the Ashtekar-Campiglia-Henderson way of building spinfoams from canonical LQC.

17. Find the relation between this formalism and the way dynamics can be treated in the canonical theory. Formally, if H is the Hamiltonian constraint, we expect something like the main equation

$$HP = 0$$

where the operator P satisfy $\langle W|\overline{\psi} \otimes \phi\rangle = \langle \psi|P|\phi\rangle$, since P is formally a projector on the solutions of the Wheeler de Witt equation

$$H\psi = 0.$$

Can we construct the Hamiltonian operator in canonical LQG such that this is realized?

18. What fixes the correct face amplitude?
19. How to couple fermions and YM fields to this formulation?
20. To couple matter, can we use the super-simple group theoretical argument that selects the gravitational vertex?
21. Where is the cosmological constant in the theory?
22. ...
23. Prove that this theory is well-defined, free of uncontrollable divergences, gives GR in the low energy limit, and compute a predictions that will be later confirmed by experiments.
24. Or alternatively: show that this theory is wrong and therefore:
25. correct it.

There is much to do !

Summary (incomplete)

1. I have sketched a tentative general framework for defining a full quantum theory of gravity and computing transitions out of it.

2. There is a specific theory defined by a simple and rather natural vertex.

3. To first order, this theory leads to the correct form of the 2-point function.

4. To first order, it leads to the Friedmann equation.

Summary

1. I have sketched a tentative general framework for defining a full quantum theory of gravity and computing transitions from of it.

2. There is a specific theory defined by a simple and rather natural vertex.

3. To first order, this theory leads to the correct form of the 2-point function.

4. To first order, it leads to the Friedmann equation.

5. **Many** aspects of this theory are not fixed.

6. We do not know if the theory is consistent.