A holonomy groupoid formulation of Loop Quantum Gravity

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Open problems in Loop Quantum Gravity

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March 1, 2010

Overview

Introduction

Quantisation of a Poisson system

classical Poisson system with a set of classical constraints	quantised algebra of canonical variables with a set of quantised constraints
$\downarrow$	$\downarrow$
reduced classical Poisson system	quantised algebra and Dirac states

Introduction

Implementing constraints on quantised level

A **quantum gravity system with constraints** is a pair $(\mathfrak{A}, \mathcal{C})$
$\mathfrak{A}$ ... unital concrete C^*-algebra $\mathfrak{A}$ of elementary variables
$\mathcal{C}$ ... set of constraints s.t. $\mathcal{C} \subset \mathfrak{A}$

The physical or Dirac state space is given by

$$\mathcal{S}_D := \{\omega \in \mathcal{S}(\mathfrak{A}) : \pi_\omega(C)\Omega_\omega = 0 \quad \forall C \in \mathcal{C}\}$$

But there are huge problems:

1. the holonomy-flux algebra does not contain the set of constraints in a satisfactory way
2. the set of constraints in LQG form a very complicated Poisson algebra

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The classical variables of LQG

Elementary variables

- (M, g) ... globally hyperbolic spacetime, $M \simeq \Sigma \times \mathbb{R}$
- $(\Sigma_t, q(t))$ 3-dimensional spatial submanifold
- $O^+(\Sigma_t, q)(\Sigma_t, SO(3), \pi)$ orthonormal frame bundle
- $A^i_a = \Gamma^i_a + \beta k^i_a$ ... represents Ashtekar connection form for Σ_t

  where

  Γ^i_a ... represents Levi-Cevita connection form for Σ_t

  k^i_a ... represents extrinsic curvature

  $\beta \in \mathbb{R}^*$ ... Immirzi Parameter
- R_A ... curvature associated to Ashtekar connection

The classical variables of LQG
Elementary variables

- E_a^i ... densitised triads, which are replaced by
  $\tilde{E}^i$ ... 2-form which can be integrated over a surface S
  $\tilde{E}_S(f_S) = \int_S f_S \tilde{E}_i$ where f_S smearing function
- Poisson bracket

$$\left\{ \int_\gamma A_a \,\mathrm{d}\, x^a, \tilde{E}_S(f_S) \right\} = \iota(\gamma, S) f_S(v) \tau_i$$

$f_S \in C_c^\infty(S)$

$\iota(\gamma, S) = \{0, \pm 1\}$, $v = S \cap \gamma$ and
$\iota(\gamma, S) = 0$ if $S \cap \gamma = \{\varnothing\}$ or $\gamma \subset S$

The classical variables of LQG

Duality of geometric objects

The **Barrett duality (1991)** of

infinitesimal connections

holonomies & parallel transports

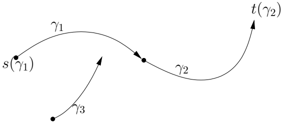
can be generalised to
the **Mackenzie duality (2005)** of

Lie algebroid $\mathcal{AG}$ connections and curvature

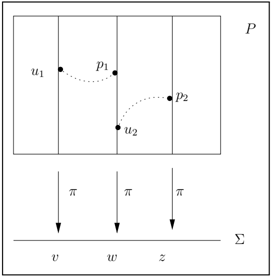
path connections in a Lie groupoid $\mathcal{G}$

The classical variables of LQG
Finite path groupoid and gauge groupoid

Γ... directed graph of independent edges



$\mathcal{P}_\Gamma\Sigma \underset{t}{\overset{s}{\rightrightarrows}} V_\Gamma$ ... **finite path groupoid**



$\frac{P\times P}{G} \underset{t_{\mathcal{G}}}{\overset{s_{\mathcal{G}}}{\rightrightarrows}} \Sigma$

... **gauge groupoid associated to** $P(\Sigma, G)$

The classical variables of LQG

Holonomy maps in LQG

$L\Sigma$ ... space of loops in Σ at base point v

Barrett's holonomy map $\mathfrak{h}_A : L\Sigma \to G$ is extended to

holonomy map for a path groupoid $\mathcal{P} \rightrightarrows \Sigma$
$=$ groupoid morphism

$$\mathfrak{h} : \mathcal{P} \to G, h : \Sigma \to \{e_G\}$$

$$\mathrm{Hol}(\Sigma) := \{\mathfrak{h}(\gamma) : \gamma \in \mathcal{P}\}$$

The classical variables of LQG

Holonomy maps in extended LQG

Barrett's holonomy map $\mathfrak{h}_A : L\Sigma \to G$ can be extended to

Mackenzie's holonomy groupoid morphism
for a Lie groupoid $\mathcal{G} \rightrightarrows \Sigma$ and associated to a path connection Λ:
$=$ groupoid morphism

$$\mathfrak{h}_\Lambda : \mathcal{P} \to \mathcal{G}, \quad h_\Lambda : \Sigma \to \Sigma$$

$$\mathrm{Hol}_\Lambda(\Sigma) := \{\mathfrak{h}_\Lambda(\gamma) : \gamma \in \mathcal{P}\}$$

The classical variables of LQG
Path connection on a Lie groupoid

$\mathcal{G} \rightrightarrows \Sigma$ ... Lie groupoid

Path connection is a map

$$\Lambda : P\Sigma \to \mathcal{P}_{\Sigma}^{s_{\mathcal{G}}}(\mathcal{G}),$$
$$\gamma(t) \mapsto \Lambda(\gamma(t), s) =: \tilde{\gamma}(s) \text{ for } s \in I$$

$+$ more conditions

Holonomy groupoid is a collection of endpoints of "lifted" paths in a Lie groupoid $\mathcal{G}$

$$\mathrm{Hol}_{\Lambda} := \{\mathfrak{h}_{\Lambda}(\gamma) = \Lambda(\gamma, 1) : \gamma \in P\Sigma\}$$

The classical variables of LQG
Duality of geometric objects

infinitesimal connections and curvature

holonomies and parallel transports

Lie algebroid $\mathcal{AG} = \frac{TP}{G}$ connections and curvature

path connections in a Lie groupoid $\mathcal{G} = \frac{P \times P}{G} \rightrightarrows \Sigma$

The classical variables of LQG

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Lie algebroid of $\mathrm{Hol}_\Lambda(\mathcal{G})$ $=$ least subalgebroid of $\mathcal{AG}$ containing Lie algebroid connections and curvature[a]

[a] Mackenzie 2005

holonomy groupoid $\mathrm{Hol}_\Lambda(\mathcal{G})$ $:= \{\mathfrak{h}_\Lambda(\gamma) : \gamma \in \mathcal{P}\Sigma\}$ for a path connection Λ $=$ subgroupoid of $\mathcal{G}$ ($\mathcal{P}\Sigma$ path space on Σ)

(Generalised Ambrose-Singer theorem)

The classical variables of LQG

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(Generalised Ambrose-Singer theorem)

The classical variables of LQG

Duality of path and infinitesimal connections on a Lie groupoid

In particular, direct correspondence between

- path connection $\Lambda(\gamma, s) : P\Sigma \longrightarrow \mathcal{P}^{s_{\mathcal{G}}}_{\Sigma}(\mathcal{G})$ and Lie algebroid connection $\gamma_A : T\Sigma \longrightarrow \frac{TP}{G}$
- connection forms $\Omega^1_{\text{basic}}(P, \mathfrak{g})^G$ & Lie algebroid connections γ_A
- curvature two forms $\Omega^2_{\text{basic}}(P, \mathfrak{g})^G$ & γ_A

The classical variables of LQG

Duality of path and infinitesimal connections on a Lie groupoid

- exponential map $\exp : \Gamma A\mathcal{G} \longrightarrow \Gamma\mathcal{G}$

$$X \mapsto \exp(tX(v)), \quad t \in \mathbb{R}, v \in \Sigma$$

- smooth family of local bisections

$$(t, v) \mapsto \exp(tX(v))$$

The classical variables of LQG

The classical configuration space

path groupoid holonomy maps are holonomy group homomorphisms associated to the set of path connections $\hat{\Lambda}$

$$\mathcal{A}^G_{\hat{\Lambda}} := \mathrm{Hom}_{\hat{\Lambda}}(\mathcal{P}\Sigma, G)$$

path groupoid holonomies are holonomy groupoid morphisms associated to the set of path connections $\hat{\Lambda}$

$$\mathcal{A}_{\hat{\Lambda}} := \mathrm{Hom}_{\hat{\Lambda}}(\mathcal{P}\Sigma, \mathcal{G})$$

The quantum algebra of LQG

Holonomy C^*-algebra

$\mathcal{P} \rightrightarrows \Sigma$ ... (semi-)analytic path groupoid

The algebra of cylindrical functions $f \in \mathrm{Cyl}^0(\mathcal{A}^G_{\hat{\Lambda}})$ is given by all elements of the form

$$f(\mathfrak{h}_\Lambda) = (f_{\Gamma_i} \circ \pi_{\Gamma_i})(\mathfrak{h}_\Lambda) \text{ for } \mathfrak{h}_\Lambda \in \mathrm{Hom}_{\hat{\Lambda}}(\mathcal{P}, G) \\ \text{where } \pi_{\Gamma_i} : \mathcal{A}^G_{\hat{\Lambda}} \longrightarrow G^{N_i}, f_{\Gamma_i} \in C(G^{N_i}) \tag{1}$$

for all directed graphs Γ_i and $N_i := |E^I_{\Gamma_i}|$ number of independent paths. $\mathrm{Cyl}^0(\mathcal{A}^G_{\hat{\Lambda}})$ completed in sup-norm is called the unital **analytic holonomy C^*-algebra** $\mathrm{Cyl}(\mathcal{A}^G_{\hat{\Lambda}})$.

$\mathrm{Cyl}(\mathcal{A}^G_{\hat{\Lambda}}) \simeq C(\mathcal{A}^G)$ where $\mathcal{A}^G$ is the space of generalised connections

The quantum algebra of LQG

Holonomy C^*-algebra and actions

- $(\varphi, \Phi) \in \mathrm{Diff}(\mathcal{P})$ a **graph diffeomorphism**

$$\alpha_{(\varphi,\Phi)} f(\mathfrak{h}_\Lambda) = f_\Gamma(\mathfrak{h}_\Gamma(\Phi(\gamma))) \text{ for } f \in \mathrm{Cyl}(\mathcal{A}^G_{\hat{\Lambda}})$$

- $\sigma \in \mathfrak{B}(\mathcal{P})$ a **bisection** of a path groupoid

$$\alpha_\sigma f(\mathfrak{h}_\Lambda) = f(R_\sigma \mathfrak{h}_\Lambda) = f_{\Gamma_1}(\mathfrak{h}_{\Gamma_1}(\gamma\sigma(t(\gamma)))) = f_{\Gamma_1}(\mathfrak{h}_{\Gamma_1}(\gamma \circ \gamma_0))$$

  if $\sigma(t(\gamma)) = \gamma_0$ and $\mathfrak{h}_\Lambda \in \mathcal{A}^G_{\hat{\Lambda}}$.

- $\rho_S \in \mathrm{Map}_S(\Gamma, G)$ (**exponentiated fluxes**) then there is an action

$$\alpha_L(\rho_S) f(\mathfrak{h}_\Lambda) = f_\Gamma(\rho_S(\gamma)\mathfrak{h}_\Gamma(\gamma))$$

The quantum algebra of LQG

Holonomy C^*-algebra and mean

- $\exists$ a unique mean on $\mathrm{Cyl}(\mathcal{A}^G_{\hat{\Lambda}})$ given by

$$\omega(f) = \int_{G^N} f_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\gamma_1), ..., \mathfrak{h}_{\Gamma_i}(\gamma_N))\, \mathrm{d}\,\mu_{\Gamma_i}(\mathfrak{h}_{\Gamma_i}(\gamma_1), ..., \mathfrak{h}_{\Gamma_i}(\gamma_N))$$

  which is invariant under $\mathrm{Diff}(\mathcal{P})$, $\mathfrak{B}(\mathcal{P})$ and $\mathrm{Map}_S(\Gamma_i, G)$
- $\exists$ a multiplication representation of $\mathrm{Cyl}(\mathcal{A}^G_{\hat{\Lambda}})$ on the Ashtekar-Lewandowski Hilbert space $\mathcal{H}_{AL} = L^2(\mathcal{A}_{\hat{\Lambda}}, \mu_{AL})$: $M(f)\psi = f\psi$
- But only $\mathrm{Map}_S(\Gamma, G)$ are implementable as a point-continuous automorphic action.

$(G^{|V\cap\Gamma|}, \mathrm{Cyl}(\mathcal{A}^G_{\hat{\Lambda}}), \alpha_L(\rho_S))$ is a C^*-dynamical system

The quantum algebra of LQG
Reduced holonomy-flux cross product C^*-algebra

The **reduced holonomy-flux cross product C^*-algebra**

$$C^*(G^{|V_{\Gamma\cap S}|},\mathrm{Cyl}(\mathcal{A}_{\hat{\Lambda}}^G))$$

is constructed from holonomy and exponentiated flux operators.

The integrated representation of $C^*(G^{|V_{\Gamma\cap S}|},\mathrm{Cyl}(\mathcal{A}_{\hat{\Lambda}}^G))$ and of the limit $\varinjlim_{\mathcal{P}_\Gamma\Sigma\in\mathcal{P}}$ -algebra is not (graph) diffeomorphism invariant

☹

The quantum algebra of LQG

Apparently, these technics from AQFT do not work in a diffeomorphism invariant Loop Quantum Gravity approach ...

The quantum algebra of LQG

Holonomy groupoid C^*-algebra

The idea:

- Continuous function algebra over the holonomy groupoid $\mathrm{Hol}_{\hat{\Lambda}}(\Sigma)$ associated to a principal fibre bundle $P(\Sigma,\pi)$

  $=$ **holonomy groupoid C^*-algebra** $C(\mathrm{Hol}_{\hat{\Lambda}}(\Sigma))$

- natural action of infinitesimal objects like flux, connection, curvature

  $=$ actions of families of local bisections

$$(t,v) \mapsto \exp(tX(v))$$

on $\mathrm{Hol}_{\hat{\Lambda}}(\Sigma)$, where $\exp : \Gamma A\,\mathrm{Hol}_{\hat{\Lambda}}(\Sigma) \to \Gamma\,\mathrm{Hol}_{\hat{\Lambda}}(\Sigma)$

Conclusion

A modification of the holonomy-flux algebra is very hard if diffeomorphism invariance and background independence is required.

Formulation in terms of groupoids

1. allows a mathematical definition of graph changing operators, like graph diffeomorphism, composition of edges
2. allows action of infinitesimal objects like forms and curvature
3. but background independence is broken

Outlook

1. extend the concept of holonomy groupoids associated to principal fibre bundles to orthonormal frame bundles
2. introduce a covariant formalism in the sense of functors between the category of holonomy groupoids and Lie groupoid morphisms and the category of C^*-algebras of elementary variables and faithful $*$-homomorphisms
   (AQFT)

Thank you for your attention