A spin foam model for general Lorentzian 4–geometries

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FC, Jeff Hnybida, arXiv:1002.1959 [gr-qc]

Outline

1. Motivation
2. Quantum simplicity constraints for general Lorentzian geometries
3. Spin foam model
4. Summary

Motivation

Main innovations of the last years

EPRL Engle, Livine, Pereira, Rovelli, Nucl.Phys.B799,2008

- master constraint

→ EPRL model

- correct coupling between 4–simplices
- relation to canonical LQG

Coherent states Livine, Speziale, Phys.Rev.D76:084028,2007

- simplicity constraints on expectation values
- geometric understanding of intertwiners

→ FK model Freidel, Krasnov, Class.Quant.Grav.25:125018,2008

Motivation 1

It has been shown by explicit comparison that EPRL and FK model are very similar, but . . .

Question

How exactly are the master constraint and coherent state approach related?

Motivation 2

There is a Lorentzian EPRL model, but . . .

Geometries are restricted

In the Lorentzian EPRL model all triangles are spacelike.

3d analogy

3d triangulation, where all links are spacelike.

Coupling to Maxwell field

There is always a local frame in which the field is purely magnetic!

What we found

1. Simplicity constraints of EPRL can be equivalently understood in terms of conditions on coherent states.
2. Using this method we extended the EPRL model to general Lorentzian 4–geometries.

Quantum simplicity constraints for general Lorentzian geometries

Transition from BF theory to gravity

Action of BF theory:

$$S = \int J \wedge F = \int \left(B \wedge F + \frac{1}{\gamma} \star B \wedge F \right)$$

Impose simplicity constraints such that B becomes

$$B = \star(E \wedge E) \, .$$

Convenient to call the total bivector J, since it corresponds to the generator of SO(1,3) in the spin foam model.

Classical simplicity constraints

Simplicity constraint: $\exists$ unit four–vector U such that

$$U \cdot \star B = 0 \,.$$

From this it follows that

$$\star B = E_1 \wedge E_2 \,, \qquad U \cdot E_1 = U \cdot E_2 = 0 \,,$$

or equivalently

$$B = A\, U \wedge N \,, \qquad |N^2| = 1 \,, \quad U \cdot N = N \cdot E_1 = N \cdot E_2 = 0 \,.$$

A is the area of the parallelogram spanned by E_1 and E_2.

Classical simplicity constraints

In a discrete setting, these quantities assume the following meaning:

$\star B$	area bivector of triangle
$E_1,\ E_2$	edges of triangle
N	unit normal vector of triangle
U	unit normal vector of tetrahedron

Classical simplicity constraints

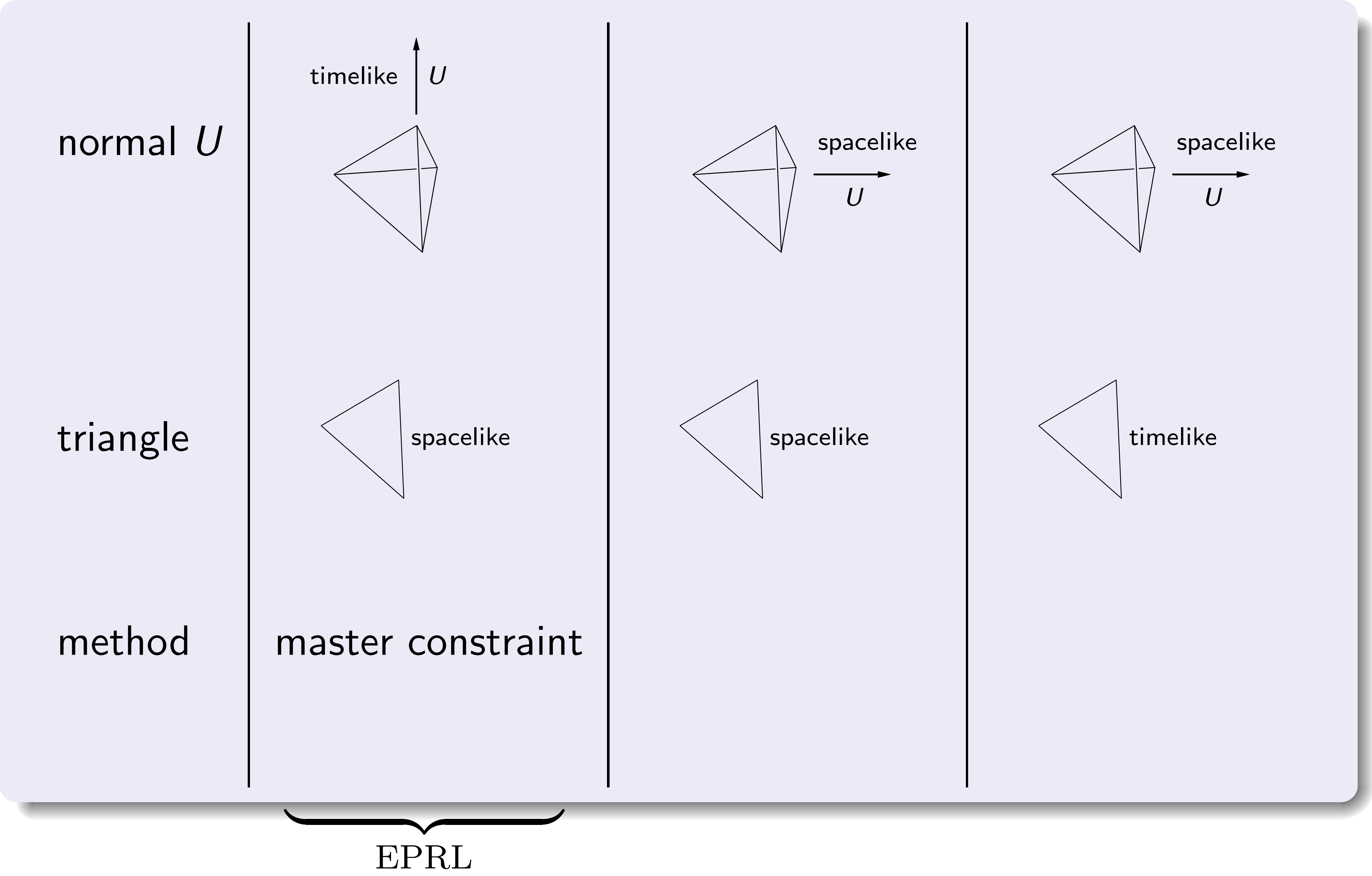
Express B in terms of the total bivector J:

$$B = \frac{\gamma^2}{\gamma^2 + 1} \left(J - \frac{1}{\gamma} \star J \right)$$

Starting point for quantization:

$$\boxed{U \cdot \left(J - \frac{1}{\gamma} \star J \right) = 0}$$

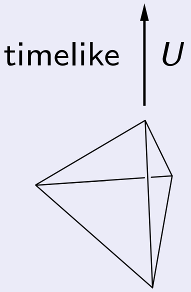
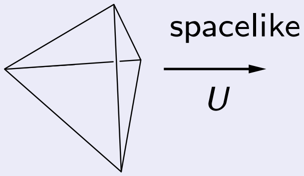
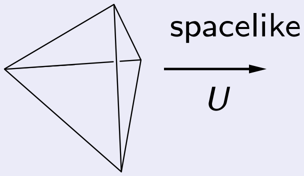
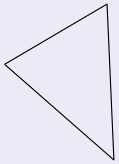
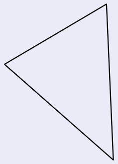
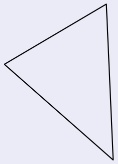
Different cases



normal U	timelike U	spacelike U	spacelike U
triangle	spacelike	spacelike	timelike
method	master constraint		

EPRL

Different cases

normal U	 timelike U	 spacelike U	 spacelike U
triangle	 spacelike	 spacelike	 timelike
method	master constraint coherent states		

EPRL

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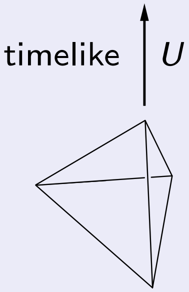
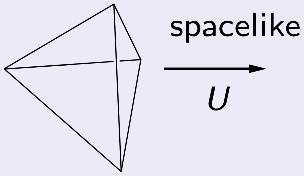
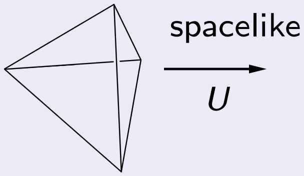
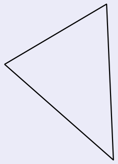
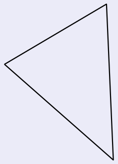
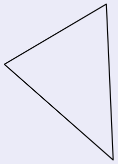
normal U	timelike U 	 spacelike U	 spacelike U
triangle	 spacelike	 spacelike	 timelike
method	master constraint coherent states	coherent states	

EPRL

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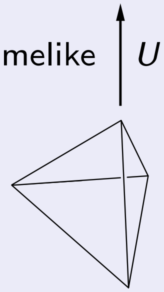
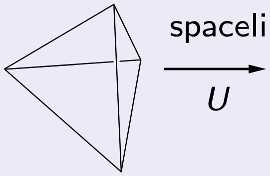
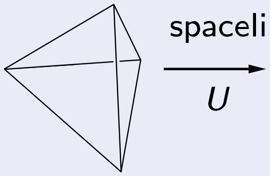
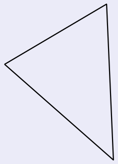
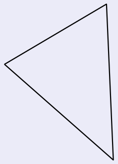
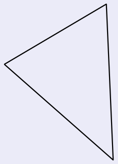
normal U	 timelike U	 spacelike U	 spacelike U
triangle	 spacelike	 spacelike	 timelike
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EPRL

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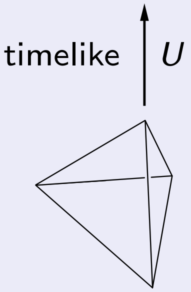
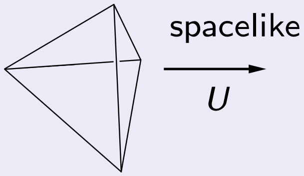
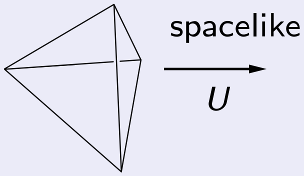
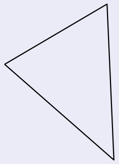
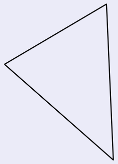
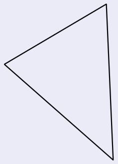
normal U	 timelike U	 spacelike U	 spacelike U
triangle	 spacelike	 spacelike	 timelike
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EPRL

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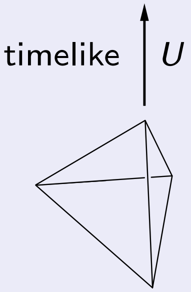
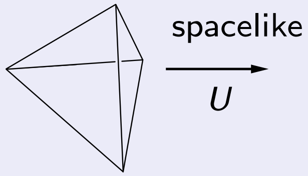
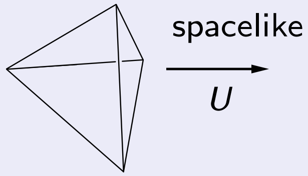
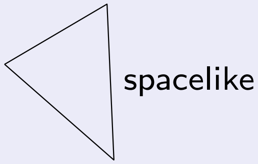
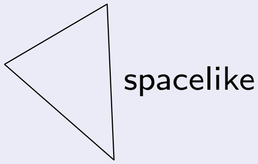
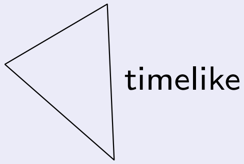
normal U			
triangle			
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EPRL

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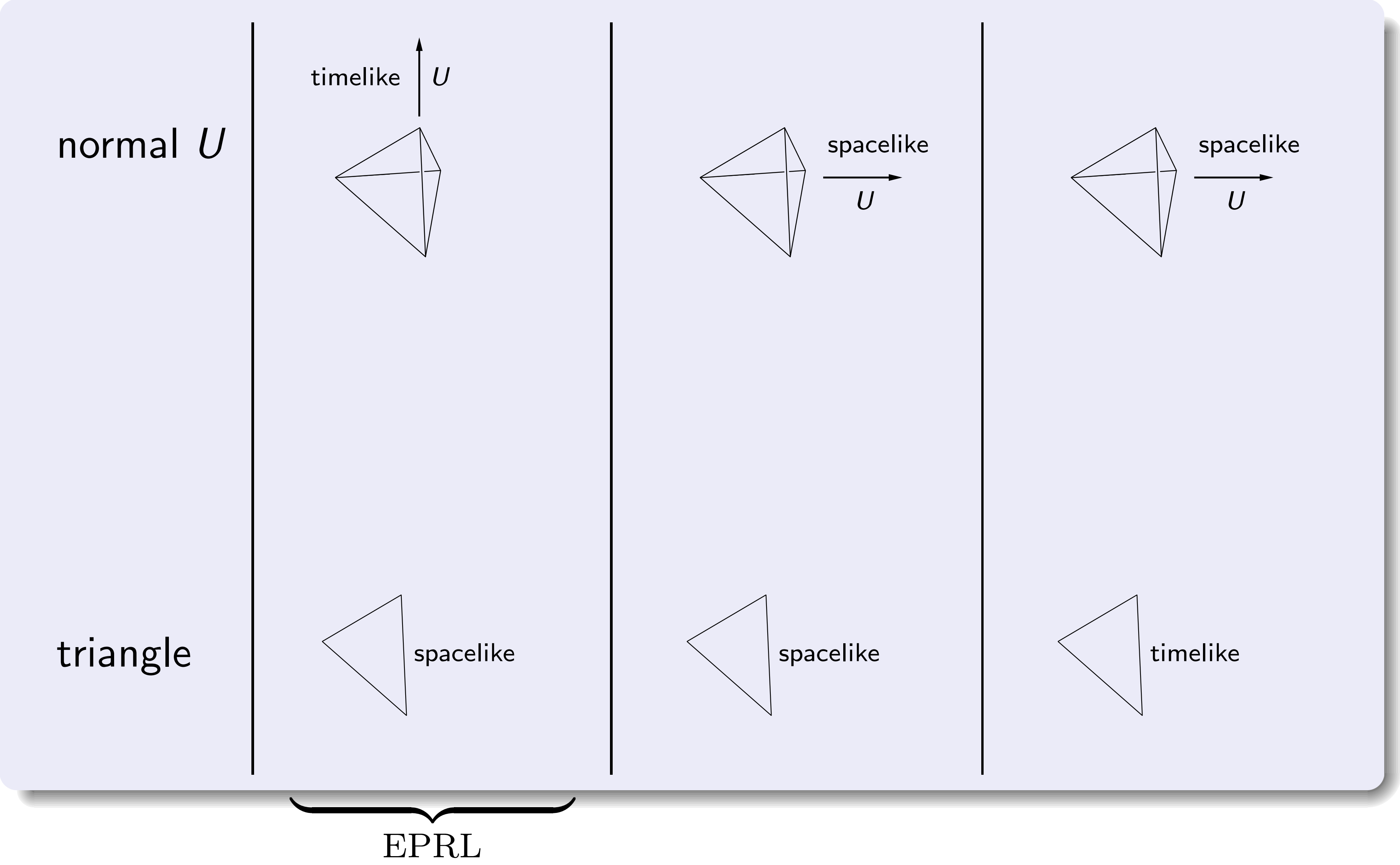

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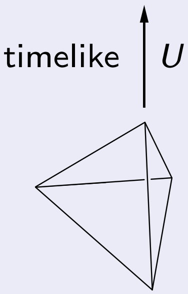
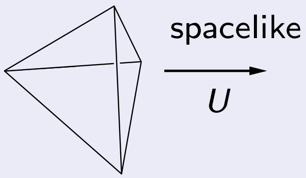
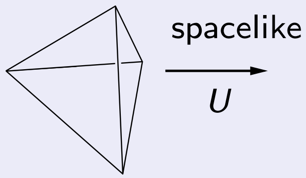
normal U	 timelike U	 spacelike U	 spacelike U
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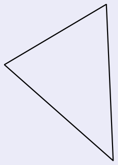
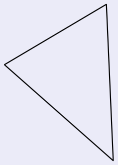
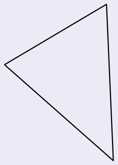
triangle	 spacelike	 spacelike	 timelike

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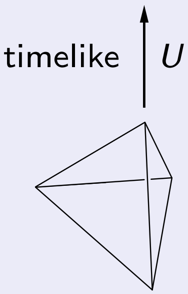
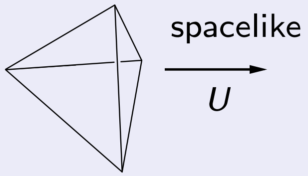
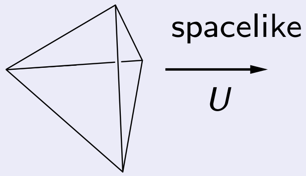
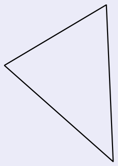
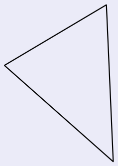
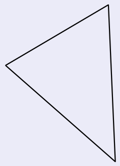
normal U			
gauge–fix	$U = (1, 0, 0, 0)$	$U = (0, 0, 0, 1)$	$U = (0, 0, 0, 1)$
little group	SO(3)	SO(1,2)	SO(1,2)
triangle	 spacelike	 spacelike	 timelike

EPRL

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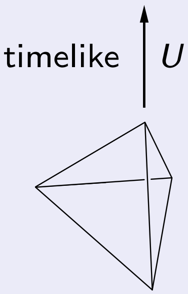
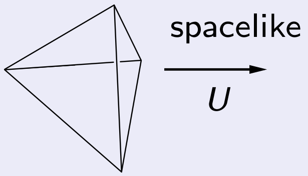
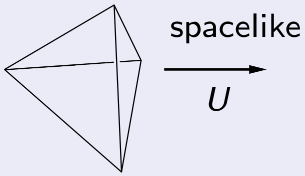
normal U			
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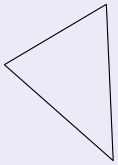
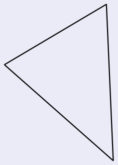
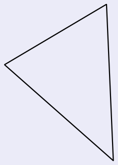
triangle			

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{EPRL}}$

Representation theory

	$\mathrm{SL}(2,\mathbb{C})$	SU(2)
generators	$J^i,\ K^i$	$J^1,\ J^2,\ J^3$
Casimirs	$C_1 = \vec{J}^2 - \vec{K}^2$ $C_2 = -4\vec{J}\cdot\vec{K}$	$\vec{J}^2$
unitary irreps	$\mathcal{H}_{(\rho,n)}$	$\mathcal{D}_j$
	$\rho \in \mathbb{R},\ n \in \mathbb{Z}_+$	$j \in \mathbb{Z}_+/2$
	$C_1 = \frac{1}{2}(n^2 - \rho^2 - 4)$ $C_2 = \rho n$	$\vec{J}^2 = j(j+1)$

Representation theory

	SU(2)	SU(1,1)	
generators	$J^1,\ J^2,\ J^3$	$J^3,\ K^1,\ K^2$	
Casimirs	$\vec{J}^2$	$Q = (J^3)^2 - (K^1)^2 - (K^2)^2$	
		discrete series	continuous series
unitary irreps	$\mathcal{D}_j$	$\mathcal{D}_j^{\pm}$	$\mathcal{C}_s^{\epsilon}$
	$j \in \mathbb{Z}_+/2$	$j = \frac{1}{2}, 1, \frac{3}{2} \ldots$	$j = -\frac{1}{2} + is$, $0 < s < \infty$
	$\vec{J}^2 = j(j+1)$	$Q = j(j-1)$	$Q = -s^2 - \frac{1}{4}$

SU(2) decomposition of $\mathrm{SL}(2,\mathbb{C})$ irrep

Canonical basis

$$\mathcal{H}_{(\rho,n)} \simeq \bigoplus_{j=n/2}^{\infty} \mathcal{D}_j$$

$$\mathbb{1}_{(\rho,n)} = \sum_{j=n/2}^{\infty} \sum_{m=-j}^{j} |\Psi_{j\,m}\rangle \langle\Psi_{j\,m}|$$

SU(1,1) decomposition of $\mathrm{SL}(2,\mathbb{C})$ irrep

$$\mathcal{H}_{(\rho,n)} \simeq \left(\bigoplus_{j>0}^{n/2} \mathcal{D}_j^+ \oplus \int_0^\infty \mathrm{d}s\; \mathcal{C}_s^\epsilon\right) \oplus \left(\bigoplus_{j>0}^{n/2} \mathcal{D}_j^- \oplus \int_0^\infty \mathrm{d}s\; \mathcal{C}_s^\epsilon\right)$$

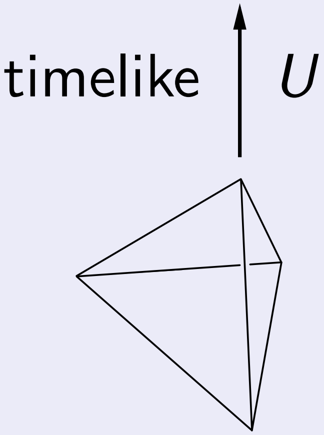
$$\begin{aligned}
\mathbb{1}_{(\rho,n)} &= \sum_{j>0}^{n/2} \sum_{m=j}^{\infty} \left|\Psi^+_{j\,m}\right\rangle \left\langle \Psi^+_{j\,m}\right| + \sum_{j>0}^{n/2} \sum_{-m=j}^{\infty} \left|\Psi^-_{j\,m}\right\rangle \left\langle \Psi^-_{j\,m}\right| \\
&+ \int_0^\infty \mathrm{d}s\; \mu_\epsilon(s) \sum_{\pm m=\epsilon}^{\infty} \left|\Psi^{(1)}_{s\,m}\right\rangle \left\langle \Psi^{(1)}_{s\,m}\right| \\
&+ \int_0^\infty \mathrm{d}s\; \mu_\epsilon(s) \sum_{\pm m=\epsilon}^{\infty} \left|\Psi^{(2)}_{s\,m}\right\rangle \left\langle \Psi^{(2)}_{s\,m}\right|
\end{aligned}$$

(see chapter 7 in Rühl's book)

General scheme for quantization

- Translate bivectors of triangles to quantum states in irreps
- Simplicity constraint → constraints on states
- Four quantum states → tetrahedron

First case: normal U timelike



In the gauge $U = (1, 0, 0, 0)$, the simplicity constraint takes the form

$$\vec{J} + \frac{1}{\gamma}\vec{K} = 0$$

The little group is SU(2), so we use states of the SU(2) decomposition!

Coherent state method

We look for quantum states that mimic classical bivectors as closely as possible.

Conditions

1. The expectation value of the bivector operator is simple.*
2. The uncertainty in the bivector is minimal.

* inspired by FK model

Coherent state method for SU(2) case

In the SU(2) case, we require the existence of quantum states such that

$$\frac{\Delta J}{|\vec{J}|} = O\left(\frac{1}{\sqrt{|\vec{J}|}}\right)$$

$$\langle\vec{J}\rangle + \frac{1}{\gamma}\langle\vec{K}\rangle = O(1)$$

$$\frac{\Delta K}{|\vec{K}|} = O\left(\frac{1}{\sqrt{|\vec{K}|}}\right)$$

$\langle\ \rangle$ denotes the expectation value w.r.t. the state, and $|\vec{J}| \equiv |\langle\vec{J}\rangle|$ etc.

SU(2) Coherent states

The first condition leads to SU(2) coherent states:

$$|j\, g\rangle \equiv D^j(g)|j\, j\rangle\,, \qquad g \in \mathrm{SU}(2)\,,$$

$$|j\, \vec{N}\rangle \equiv D^j(g(\vec{N}))|j\, j\rangle\,, \quad \vec{N} \in S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1)\,.$$

Perelomov, Comm.Math.Phys.26,1972

Simplicity of expectation values

The second condition

$$\langle \vec{J} \rangle + \frac{1}{\gamma} \langle \vec{K} \rangle = O(1)$$

gives

$$j + \frac{1}{\gamma}\left(-j \frac{\rho\, n}{4j(j+1)}\right) = 0$$

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$$4\gamma j(j+1) = \rho n$$

or equivalently

$$\begin{aligned} \langle \vec{J}^2 \rangle &= \frac{1}{\gamma^2}\langle \vec{K}^2 \rangle + O(|\vec{J}|)\,, \\ \langle \vec{J}^2 \rangle &= -\frac{1}{\gamma}\langle \vec{J} \cdot \vec{K} \rangle\,. \end{aligned}$$

Minimal uncertainty in $\vec{K}$

The third condition involves the uncertainty in $\vec{K}$:

$$(\Delta K)^2 = \langle \vec{K}^2 \rangle - \langle \vec{K} \rangle^2 = \langle \vec{J}^2 \rangle - \frac{1}{2} C_1 - \langle K \rangle^2 \,.$$

By inserting the previous two eqns. this can be rewritten as

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$$\begin{aligned}
(\Delta K)^2 &= -\frac{1}{\gamma}(1-\gamma^2)\vec{J}\cdot\vec{K} - \frac{1}{2} C_1 + O(|\vec{J}|) \\
&= -\frac{\gamma}{4}\left[\left(1-\frac{1}{\gamma^2}\right) C_2 + \frac{2}{\gamma} C_1\right] + O(|\vec{J}|) \\
&= -\frac{\gamma}{4} B \cdot \star B + O(|\vec{J}|) \,.
\end{aligned}$$

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By inserting the previous two eqns. this can be rewritten as

$$\begin{aligned} (\Delta K)^2 &= -\frac{\gamma}{4} B \cdot \star B + O(|\vec{J}|) \\ &= \frac{1}{4}\Big(\rho - \gamma n\Big)\Big(\rho + \frac{n}{\gamma}\Big) + O(|\vec{J}|) \,. \end{aligned}$$

Result

Altogether we get the conditions

$$4\gamma j(j+1) = \rho n$$

$$\left(\rho - \gamma n\right)\left(\rho + \frac{n}{\gamma}\right) = 0$$

which have the approximate solution

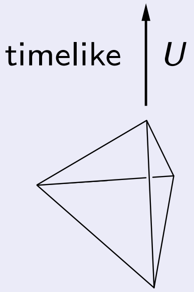
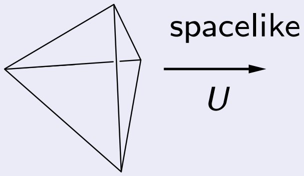
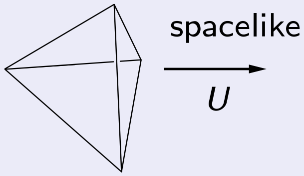
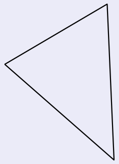
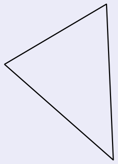
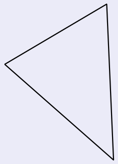
$$\rho = \gamma n \qquad\qquad j = n/2$$

These are the EPRL constraints!

Coherent state and master constraint method

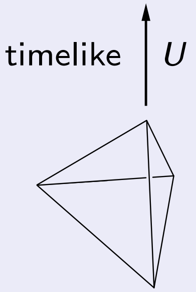
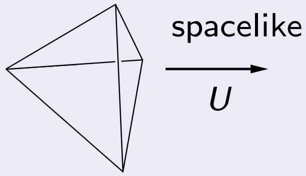
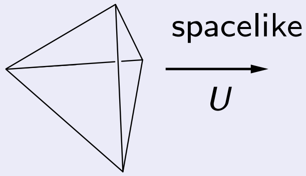
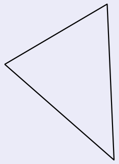
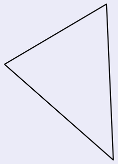
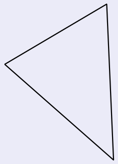
The EPRL constraints are equivalent to the existence of semiclassical simple bivector states!

New cases: normal U spacelike

normal U			
triangle	 spacelike	 spacelike	 timelike
method	master constraint coherent states		

EPRL

New cases: normal U spacelike

normal U	 timelike U	 spacelike U	 spacelike U
triangle	 spacelike	 spacelike	 timelike
method	master constraint coherent states		coherent states

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{EPRL}}$

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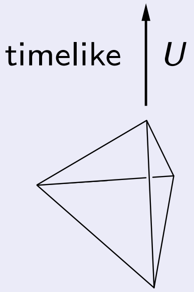
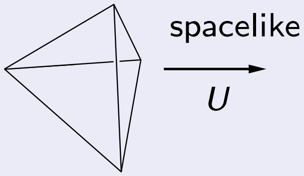
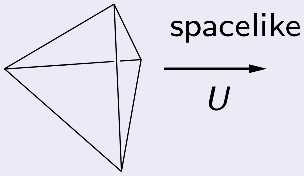
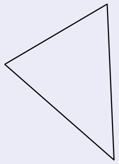
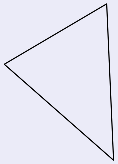
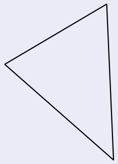
normal U	 timelike U	 spacelike U	 spacelike U
triangle	 spacelike	 spacelike	 timelike
method	master constraint coherent states		master constraint coherent states

$\underbrace{\hphantom{master constraint}}_{\text{EPRL}}$

Spacelike U

In the gauge $U = (0, 0, 0, 1)$, the simplicity constraint becomes

$$\boxed{\vec{F} + \frac{1}{\gamma}\vec{G} = 0}$$

where

$$\vec{F} \equiv \begin{pmatrix} J^3 \\ K^1 \\ K^2 \end{pmatrix} \qquad \text{and} \qquad \vec{G} \equiv \begin{pmatrix} K^3 \\ -J^1 \\ -J^2 \end{pmatrix}.$$

The little group is SU(1,1), so we use states of the SU(1,1) decomposition!

$\vec{F}$ and $\vec{G}$ transform like 3d Minkowski vectors under SU(1,1).

Spacelike vs. timelike triangles

Classically, the normal $\vec{N}$ to the triangle is given by

$$A\begin{pmatrix} N^0 \\ N^1 \\ N^2 \end{pmatrix} = \gamma \begin{pmatrix} F^0 \\ F^2 \\ -F^1 \end{pmatrix} .$$

Hence

discrete series	$Q = \vec{F}^2 > 0$	$\longrightarrow$	$\vec{N}^2 = 1$	triangle spacelike
continuous series	$Q = \vec{F}^2 < 0$	$\longrightarrow$	$\vec{N}^2 = -1$	triangle timelike

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Master constraint method for timelike triangles

$B \cdot \star B = 0$ diagonal constraint

$M = (\star B)^{3i}(\star B)_{3i} = 0$ master constraint

In terms of $\vec{F}$ and $\vec{G}$ the master constraint becomes

$$\left(1 + \frac{1}{\gamma^2}\right) \vec{F}^2 - \frac{1}{2\gamma^2} C_1 - \frac{1}{2\gamma} C_1 = 0\,.$$

By inserting the diagonal constraint into this one obtains

$$4\gamma \vec{F}^2 = C_2\,.$$

Master constraint method for timelike triangles

In the case of the continuous series, the constraints are therefore

$$\left(\rho - \gamma n\right)\left(\rho + \frac{n}{\gamma}\right) = 0$$

$$-4\gamma\left(s^2 + \frac{1}{4}\right) = \rho\, n$$

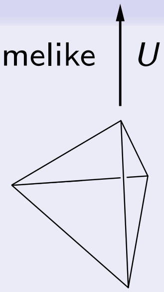
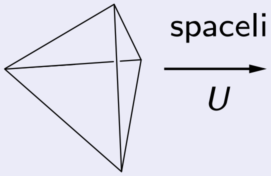
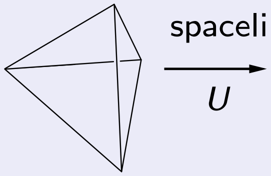
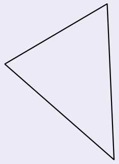
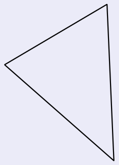
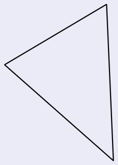
Solution

$\rho = -\frac{n}{\gamma}$ $\qquad$ $s^2 = \frac{1}{4}\left(\frac{n^2}{\gamma^2} - 1\right)$

Discrete area spectrum of timelike triangles

$$\boxed{A = \gamma\sqrt{-Q} = \gamma\sqrt{s^2 + 1/4} = n/2}$$

Table of constraints

normal U	 timelike U	 spacelike U	 spacelike U
triangle	 spacelike	 spacelike	 timelike
little group	SU(2)	SU(1,1)	SU(1,1)
relevant irreps	$\mathcal{D}_j$	$\mathcal{D}_j^{\pm}$	$\mathcal{C}_s^{\epsilon}$
constr. on (ρ, n)	$\rho = \gamma n$	$\rho = \gamma n$	$n = -\gamma\rho$
constr. on irreps	$j = n/2$	$j = n/2$	$s^2 + 1/4 = \rho^2/4$
area spectrum	$\gamma\sqrt{j(j+1)}$	$\gamma\sqrt{j(j-1)}$	$\gamma\sqrt{s^2 + 1/4} = n/2$

$\underbrace{\qquad\qquad\qquad}_{\text{EPRL}}$

Spin foam model

Spin foam model for general Lorentzian geometries

Complex:

- simplicial complex Δ: 4–simplex σ, tetrahedron τ, triangles t, . . .
- dual complex Δ^*: vertex v, edge e, face f, . . .

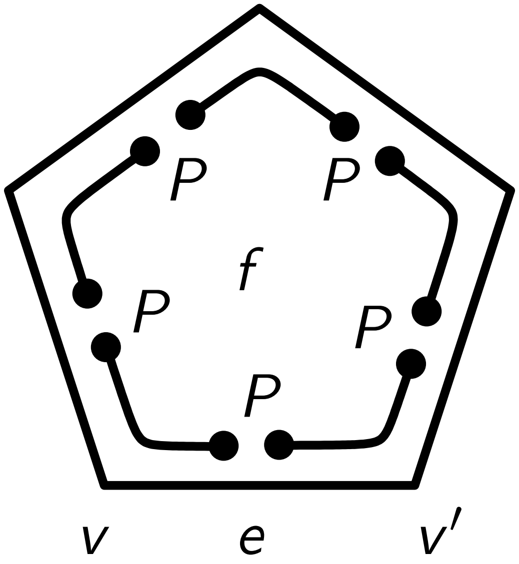
Variables (same as in EPRL):

- connection $g_e \in \mathrm{SL}(2,\mathbb{C})$
- irrep label $n_f \in \mathbb{Z}_+$

Additional variables:

- $U_e = (1,0,0,0)$ or $(0,0,0,1)$: normal of tetrahedron dual to e
- $\zeta_f = \pm 1$: spacelike/timelike triangle dual to f

Spin foam model for general Lorentzian geometries

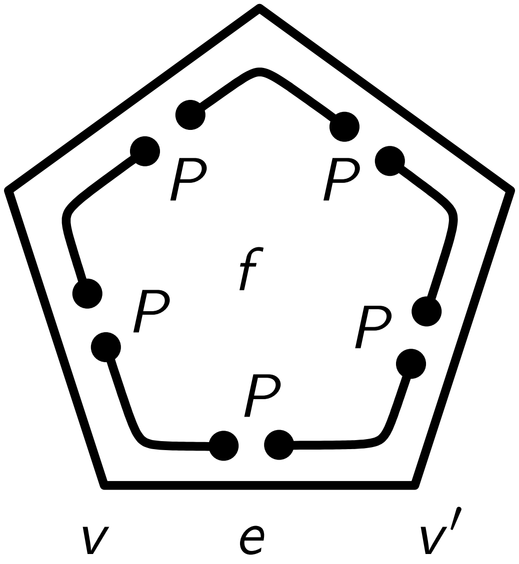


BF theory
$$A_f((\rho, n); g_{ev}) = \mathrm{tr}\left[\prod_{e\subset f} D^{(\rho,n)}(g_{ve})\mathbb{1}_{(\rho,n)} D^{(\rho,n)}(g_{ev'})\right]$$

$$\downarrow$$

$$A_f((\rho, n), \zeta; U_e; g_{ev}) = \lim_{\delta\to 0}\ \mathrm{tr}\left[\prod_{e\subset f} D^{(\rho,n)}(g_{ve}) P_{(\rho,n),\zeta,U_e}(\delta) D^{(\rho,n)}(g_{ev'})\right]$$

Spin foam model for general Lorentzian geometries



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$$\downarrow$$

$$A_f((\rho,n),\zeta;U_e;g_{ev}) = \lim_{\delta\to 0}\mathrm{tr}\left[\prod_{e\subset f} D^{(\rho,n)}(g_{ve})P_{(\rho,n),\zeta,U_e}(\delta)D^{(\rho,n)}(g_{ev'})\right]$$

Projector onto allowed irrep

The projector $P_{(\rho,n),\zeta,U_e}(\delta)$ projects onto the irreps permitted by the simplicity constraints.

Subtlety for continuous series:
states not normalizable → smearing with wavefunction required!

$$P^{\epsilon}_{s}(\delta) = \sum_{\alpha=1,2} \sum_{\pm m=\epsilon} \int\limits_0^{\infty} \mathrm{d}s' \; \mu_\epsilon(s') \, f_\delta(s'-s) \left|\Psi^{(\alpha)}_{s'\,m}\right\rangle \left\langle\Psi^{(\alpha)}_{s'\,m}\right|$$

Spin foam model for general Lorentzian geometries

Partition function

$$Z = \int_{\mathrm{SL}(2,\mathbb{C})} \prod_{ev} \mathrm{d}g_{ev} \sum_{n_f} \sum_{\zeta_f = \pm 1} \sum_{U_e} \prod_{f} (1+\gamma^{2\zeta_f}) n_f^2 \, A_f \Big((\zeta_f \gamma^{\zeta_f} n_f , n_f), \zeta_f ; U_e ; g_{ev} \Big)$$

Spin foam sum in terms of coherent states

Using completeness relations of coherent states the spin foam sum can be also written in terms of vertex amplitudes.

For example, in the case of the discrete series,

$$P_j^{\pm} = (2j-1) \int\limits_{\mathrm{SU}(1,1)} \mathrm{d}g \left|\Psi^{\pm}_{j\,g}\right\rangle \left\langle\Psi^{\pm}_{j\,g}\right| = (2j-1) \int\limits_{\mathbb{H}_\pm} \mathrm{d}^2 N \left|\Psi_{j\vec{N}}\right\rangle \left\langle\Psi_{j\vec{N}}\right| ,$$

where $\mathbb{H}_\pm$ is the upper/lower hyperboloid.

More details soon . . .

Summary of results

- coherent state derivation of EPRL constraints
    - based on correspondence between classical and quantum states
- extension of EPRL constraints to general Lorentzian geometries
    - normals of tetrahedra can be timelike and spacelike
    - triangles can be spacelike and timelike
- discrete area spectrum of timelike surfaces
- definition of associated spin foam model
- coherent states for timelike triangles (see paper)

Outlook

- Our results open the way to analyzing realistic Lorentzian geometries
  - corresponding to generic discretizations of smooth geometries
  - regions with timelike boundaries
  - black holes?
- Extension of results on EPRL model?
  - asymptotics, graviton propagator . . . ?
- canonical LQG on timelike surfaces?
- comparison with previous work on timelike surfaces

Perez, Rovelli
Alexandrov, Vassilevich
Alexandrov, Kadar