Flat effective spaces and spinfoams

Florian Girelli
University of Sydney

Scheme

(1) Certain types of scalar field theory in flat effective spaces are GFTs.

(2) Relate these field theories to spinfoam models

Usual scalar field theory

Haar measure

momentum conservation = translation invariance

* Consider a scalar field theory:

$$\int [dp]^2\, \phi(p_1)\mathcal{K}(p_1)\phi(p_2)\delta(p_1+p_2) - \frac{\lambda}{3!}\int [dp]^3\, \phi(p_1)\phi(p_2)\phi(p_3)\delta(p_1+p_2+p_3)$$

* Momentum space is an abelian group
  * Group element generated from abelian Lie algebra

$$e^{ip_1\cdot X}e^{ip_2\cdot X} = e^{i(p_1+p_2)\cdot X} \qquad [X_\mu, X_\nu] = 0$$

  * The addition is commutative and associative, with inverse $\quad p_{tot} = p_1 + p_2, \quad -p$
* Momentum space is a "coset" space, from which we determine Lorentz action.

$$R^4 \sim ISO(3,1)/SO(3,1)$$

Flat effective space

- Flat space with quantum gravity scale
  - Poincaré symmetries should be consistent with the existence of this scale (no symmetry breaking)
  - Construct it by introducing a momentum Planck scale: Planck mass.

$$M_p = \frac{1}{L_p}, \quad \hbar = c = 1$$

Introduce Planck scale

Abelian group

Introduce curvature

Non-abelian group

Addition is non-commutative
but associative

K-loop

Addition is non-commutative
and non-associative

$$e^{ik_1\cdot X}e^{ik_2\cdot X} = e^{i(k_1\oplus k_2)\cdot X} \quad [X_\mu, X_\nu] = C^\alpha_{\mu\nu}X_\alpha$$

$$p^\mu = m_p\, f(k, m_p)\, k^\mu \quad p_{tot} = p_1 \oplus_{m_p} p_2$$

Momentum is a coordinate on the manifold. There will be in general many patches.

K-loop

They can be constructed from group factorization.

$$G \sim L\,.\,H \;\rightarrow\; g = a\,h \quad a \in L,\, h \in H$$

$$ab = g \equiv (a \cdot b)\, h_{ab}, \quad a, b \in L$$

$$(a \cdot b)^{-1} = a^{-1} \cdot b^{-1} \quad \forall a, b \in L.$$

They satisfy weak version of commutativity and associativity

$$a \cdot b = h_{ab}\,(b \cdot a)\, h_{ab}^{-1},$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot (h_{ab}\, c\, h_{ab}^{-1}).$$

Well-known example: space of 3d speeds in Special Relativity

$$SO(3,1) = L\,.\,SO(3)$$

$$e^{i\eta_1 \vec{b}_1 \cdot \vec{N}} e^{i\eta_2 \vec{b}_2 \cdot \vec{N}} = e^{i\eta \vec{b} \cdot \vec{N}} R(b_1, b_2)$$

$$\vec{v} = \tanh \eta\, \vec{b}$$

$$\vec{v_1} \oplus \vec{v_2} = \frac{1}{1 + \frac{\vec{v_1}.\vec{v_2}}{c^2}} \left(\vec{v_1} + \frac{1}{\gamma_1}\vec{v_2} + \frac{1}{c^2}\frac{\gamma_1}{1+\gamma_1}(\vec{v_1}.\vec{v_2})\vec{v_1} \right)$$

Scalar field theory in semi-classical space

momentum conservation

We consider a field theory defined over a non-abelian group or a K-loop.

Haar measure

$$\int [dp]^2\, \phi(p_1)\mathcal{K}(p_1)\phi(p_2)\delta(p_1 \oplus p_2) - \frac{\lambda}{3!}\int [dp]^3\, \phi(p_1)\phi(p_2)\phi(p_3)\delta(p_1 \oplus (p_2 \oplus p_3))$$

$$\int [dg]^2\, \phi(g_1)\mathcal{K}(g_1)\phi(g_2)\delta(g_1 g_2)$$
$$-\frac{\lambda}{3!}\int [dg]^3\, \phi(g_1)\phi(g_2)\phi(g_3)\delta(g_1 g_2 g_3)$$

<u>Group case</u>

$$\int [da]^2\, \phi(a_1)\mathcal{K}(a_1)\phi(a_2)\delta(a_1 \cdot a_2)$$
$$-\frac{\lambda}{3!}\int [da]^3\, \phi(a_1)\phi(a_2)\phi(a_3)\delta(a_1 \cdot (a_2 \cdot a_3))$$

<u>K-loop case</u>

- Check that Lorentz invariance is there.
- Check that modified momentum conservation is related to deformed translations.

Lorentz action

Examples of relevant groups

3d: SU(2), SL(2,R), AN2, AN2'
4d: AN3, AN3'

Lie algebras

$$[X_i, X_j] = \frac{1}{m_p}\epsilon_{ij}^k X_k$$

$$[X_0, X_i] = -\frac{1}{m_p} X_i \;\; i = 1, 2, ..$$

$$[X_1, X_\mu] = -\frac{1}{m_p} X_\mu \;\; \mu = 0, 2, ..$$

Iwasawa decomposition

$$SO(4,1) \sim AN_3 \cdot SO(3,1) \sim AN_3 \bowtie SO(3,1)$$ (Kowalski-Glikman)

$$SO(3,2) \sim AN_3'.SO(3,1) \sim AN_3' \bowtie SO(3,1)$$ (Girelli-Livine)

Cartan and other decompositions to generate K-loops

$$\mathrm{SO}(p,1) \sim L' \,.\, \mathrm{SO}(p) \qquad \mathrm{SO}(p,1) \sim L \,.\, \mathrm{SO}(p-1,1)$$

Momentum space is isomorphic to the coset space (which can have a group structure)

$$M \sim G/H$$

Space-time

Plane-wave $e^{ip\cdot x}$ $\qquad e^{ip_1\cdot x} * e^{ip_2\cdot x} \equiv e^{i(p_1 \oplus p_2)\cdot x}$

Fourier transform $\hat{\phi}(x) = \int [dp]\, \phi(p) e^{ip\cdot x}$

$\widehat{\phi \circ \psi} = \hat{\phi} * \hat{\psi}$ Convolution product on group and K-loop is dual to star product

Non-commutative space

$$X_\mu f(x) = x_\mu * f(x) \quad \begin{cases} [X_\mu, X_\nu] = \dfrac{1}{m_p} C^\alpha_{\mu\nu} X_\alpha & \text{Lie algebra type} \\ [X_\mu, X_\nu] = \dfrac{1}{m_p^2} J_{\mu\nu} & \text{Snyder space (K-loop) (Girelli-Livine)} \end{cases}$$

Non-commutative field action

$$\int [dx] \left((\partial_\mu \phi) * (\partial_\mu \phi)(x) - m^2 \phi * \phi(x) + \frac{\lambda}{3!} (\phi * (\phi * \phi))(x) \right)$$

Action of the translations $\hat{\phi}(x) \to \hat{\phi}(x+\epsilon) \quad \phi(p) \to e^{ip\cdot\epsilon}\phi(p)$

$$\phi(p_1)\phi(p_2) \to e^{i(p_1\oplus p_2)\cdot\epsilon}\phi(p_1)\phi(p_2) = e^{ip_1\cdot\epsilon} * e^{ip_2\cdot\epsilon}\phi(p_1)\phi(p_2)$$

Scalar field theory in kappa-Minkowski

Scalar field theory in kappa-Minkowski is a GFT.

Freidel, Kowalski-Glikman
Girelli, Livine, Oriti

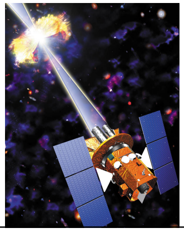
Momentum addition is non-commutative

$$p_1^0 \oplus p_2^0 = p_1^0 + p_2^0, \quad p_1^i \oplus p_2^i = p_1^i + e^{-\frac{p_1^0}{m_P}} p_2^i$$

One needs to be careful with Lorentz action and choice of coordinates

Freidel, Kowalski-Glikman
Arzano, Kowalski-Glikman, Walkus

Many arguments in favor of kappa-Minkowski space as flat effective space.



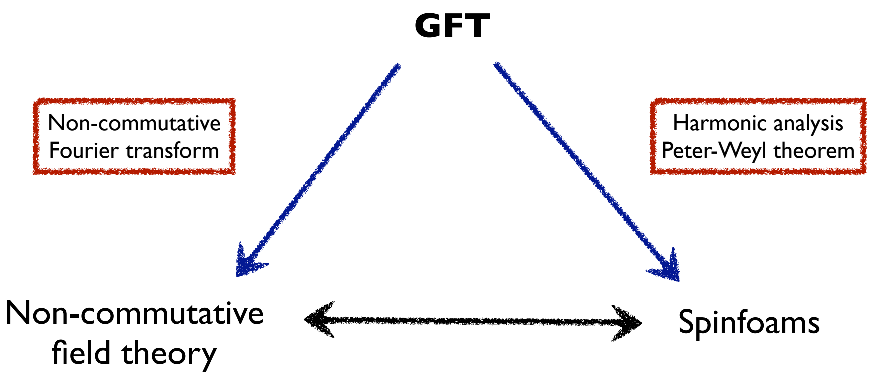
Amelino-Camelia, Smolin, Starodubtsev
Kowalski-Glikman, Starodubtsev
Girelli, Livine, Oriti

Some facts

✱ I call a scalar field theory defined on a group "a Group Field Theory" (**GFT**). A GFT is a non commutative field theory.

✱ The deformed symmetries can be obtained by algebraic methods from quantum groups (kappa-Poincare, Drinfeld double). For the Snyder case, this is a new quantum group!

✱ GFT of Boulatov (3d topological), Ooguri (4d topological), Barrett-Crane (4d quantum gravity) are non-commutative field theories. (Girelli, Livine -- Baratin, Oriti)



Livine, Perez, Rovelli (2d BF)
Girelli, Livine (3d QG)
Baratin, Oriti (3d, 4d QG)
Baratin, Girelli, Oriti

Questions

* GFT with non-trivial spin?
  * Gurau's GFT proposal to improve renormalization analysis.
  * For SU(2), use Drinfeld double: anyons

    (Bais, Muller, Schroers)
  * On kappa-Minkowski, this is not easy...
    * No agreement for fermions
    * Yang-Mills theories not under control
* What is the role of braiding in the spinfoam GFT context?
* Can we use the fact that spinfoam GFT and field theory in flat effective space are described by the same formalism to relate them?

Boulatov model,

Complex field defined on SU(2)xSU(2)xSU(2)

$\forall g \in \mathrm{SU}(2), \quad \phi(g_1 g, g_2 g, g_3 g) = \phi(g_1, g_2, g_3).$ $\quad \overline{\phi(g_1, g_2, g_3)} = \phi(g_3, g_2, g_1).$

Boulatov action

$$S_c[\phi] = \int [dg]^3 \, \phi_{123} \, \overline{\phi_{123}} - \frac{\lambda}{4} \int [dg]^6 \, \phi_{123} \, \overline{\phi_{543}} \, \phi_{526} \, \overline{\phi_{146}}$$

Solution of equation of motion (Livine)

$$\phi(g_3, g_2, g_1) = \frac{\lambda}{3!} \int [dg]^3 \, \phi(g_3, g_4, g_5) \phi(g_5, g_2, g_6) \phi(g_6, g_4, g_1).$$

$$\phi_f(g_1, g_2, g_3) = \sqrt{\frac{3!}{\lambda}} \int dg \, \delta(g_1 g) f(g_2 g) \delta(g_3 g).$$

$$if \;\; f \neq 0, \qquad \int f^2 = 1$$

Perturbations as matter

Perturbations $\Delta\phi(g_1, g_2, g_3) \equiv \psi(g_1 g_3^{-1}).$

Effective action for perturbations $S_{eff}[\psi] \equiv S_0[\phi_f + \psi] - S_0[\phi_f].$

Around trivial solution

$$S_0[\phi = \psi] = \frac{1}{2}\int dg\, \psi(g)\psi(g^{-1}) - \frac{\lambda}{4!}\int [dg]^4\, \psi(g_1)\psi(g_2)\psi(g_3)\psi(g_4)\delta(g_1g_2g_3g_4).$$

Around non-trivial solution (Livine-Fairbain)

$$S^{(f)}_{eff}[\psi] = \frac{1}{2}\int dg\, K(g)\psi(g)\psi(g^{-1}) - \sqrt{\frac{\lambda}{3!}}\left(\int f\right)\int [dg]^3\, \psi(g_1)\psi(g_2)\psi(g_3)\delta(g_1g_2g_3)$$
$$-\frac{\lambda}{4!}\int [dg]^4\, \psi(g_1)\psi(g_2)\psi(g_3)\psi(g_4)\delta(g_1g_2g_3g_4),$$

Choice of f leads to the right propagator K(g) $f = f_0 + f_1\chi^1, \;\; \mathcal{K}_f(g) = k_1[f]\, P^2(g) - \kappa^2 f_0^2$

This is the effective action obtained by Freidel-Livine:
scalar field theory in effective flat 3d space.

This works also for the other types of perturbations, if we consider the braiding associated to quantum double.

$$\boxed{\psi(g_1)\psi(g_2) \rightarrow \psi(\widetilde{g}_2)\psi(\widetilde{g}_1) \qquad \text{with} \quad \widetilde{g}_1 = g_2^{-1} g_1 g_2, \quad \widetilde{g}_2 = g_2, \quad \widetilde{g}_2\widetilde{g}_1 = g_1 g_2 \neq g_2 g_1.}$$

$$\boxed{\psi(g_1)\overline{\psi(g_2)} \rightarrow \overline{\psi(\widetilde{g}_2)}\psi(\widetilde{g}_1), \qquad \text{with} \quad \widetilde{g}_1 = g_1^{-1}, \, \widetilde{g}_2 = g_1^{-1} g_2^{-1} g_1, \quad \widetilde{g}_2^{-1}\widetilde{g}_1 = g_1^{-1} g_2.}$$

$$\begin{aligned}
S_c[\phi = \psi(g_1 g_3^{-1})] &= \frac{1}{2}\int dg\, \psi(g)\overline{\psi(g)} - \frac{\lambda}{4!}\int [dg]^4\, \psi(g_1)\overline{\psi(g_2)}\psi(g_3)\overline{\psi(g_4)}\, \delta(g_1(g_2)^{-1}g_3(g_4)^{-1}),\\
S_c[\phi = \psi(g_1 g_2^{-1})] &= \frac{1}{2}\int dg\, \psi(g)\overline{\psi(g)} - \frac{\lambda}{4!}\int [dg]^4\, \psi(g_1)\psi(g_2)\overline{\psi(g_3)}\,\overline{\psi(g_4)}\, \delta(g_1(g_2)^{-1}g_3(g_4)^{-1}),\\
S_c[\phi = \psi(g_2 g_3^{-1})] &= \frac{1}{2}\int dg\, \psi(g)\overline{\psi(g)} - \frac{\lambda}{4!}\int [dg]^4\, \psi(g_1)\overline{\psi(g_2)}\,\overline{\psi(g_3)}\psi(g_4)\, \delta(g_1(g_2)^{-1}g_3(g_4)^{-1}).
\end{aligned}$$

The quantum double symmetry of the perturbations can be identified in the Boulatov GFT. (Girelli-Livine)

→ Meaning of the symmetry from the simplicial perspective?

→ Meaning of the perturbations from the simplicial perspective?

Extension to 4d?

Girelli, Livine, Oriti

In 4d, we are interested by a scalar field theory in kappa-Minkowski, ie a GFT on AN3. Spinfoams models in 4d are generated from the Ooguri GFT.

$$\phi(g_1,g_2,g_3,g_4) = \phi(g_1g,g_2g,g_3g,g_4g),$$

$$\begin{aligned} S_{4d} &= \frac{1}{2}\int [dg]^3\,\phi(g_1,g_2,g_3,g_4)\phi(g_4,g_3,g_2,g_1) \\ &-\frac{\lambda}{5!}\int [dg]^9\phi(g_1,g_2,g_3,g_4)\phi(g_4,g_5,g_6,g_7)\phi(g_7,g_3,g_8,g_9)\phi(g_9,g_6,g_2,g_{10})\phi(g_{10},g_8,g_5,g_1) \end{aligned}$$

Solution of equation of motion:

$$\phi^{(0)}(g_i) = \sqrt[3]{\frac{4!}{\lambda}}\int_{\mathrm{SO}(4,1)} dg\,\delta(g_1g)F(g_2g)\tilde{F}(g_3g)\delta(g_4g),$$

Direct generalization from 3d case: consider Ooguri's GFT on AN3 and consider the effective action for the perturbations. Careful with infinities!

$$F(g) = \alpha(v_4(g)+a)\vartheta(g), \qquad \tilde{F}(g) = \delta(g).$$

DSR propagator

Regularisation function

→ What is the meaning of BF with group AN3?

Extension to 4d?

Consider instead group SO(4,1), since AN3 is a subgroup.
Ooguri's GFT with SO(4,1): basis for Freidel-Starodubtsev action.

DSR propagator

$$\vartheta(g) \;=\; \chi(h)\theta(\Lambda). \qquad c = \int F. \qquad \alpha = (a+1)^{-1}$$

$$\frac{1}{2}\int \psi(g)\psi(g^{-1})\left[1 - 2c^2 - \vartheta^2(g)\frac{2c(a+v_4(g))^2}{(a+1)^2}\right] - c\left(\sqrt[3]{\frac{\lambda}{4!}}\right)\int \psi(g_1)..\psi(g_3)\,\delta(g_1..g_3)\,[c+F(g_3)]$$
$$-c\left(\sqrt[3]{\frac{\lambda}{4!}}\right)^2\int \psi(g_1)..\psi(g_4)\,\delta(g_1..g_4) - \frac{\lambda}{5!}\int \psi(g_1)..\psi(g_5)\,\delta(g_1..g_5),$$

This is a field theory on SO(4,1). It is not easy to disentangle the SO(3,1) and AN3 sectors in the interaction term.

→ Key idea in FS action: the separation of these two sectors. A better understanding of the quantisation of FS action in terms of spinfoams will help to get a better derivation.

Concluding questions

☑ Scalar field theory in flat effective spaces are **GFTs**.
☑ They can be identified as sub-sectors of spinfoam GFTs.

✱ Can we improve the perturbations approach in 3d and 4d to include other type of field (fermions, Yang-Mills)?
✱ First define these fields on the non-commutative spaces!
✱ Physical interpretation of the non-commutative field theories?
✱ What is the simplicial interpretation of the perturbations?

✱ What are the non-commutative features useful for the spinfoam approach: braiding, quantum symmetries?