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Quantum scalar field in loop quantum gravity with spherical symmetry

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Overview:

We consider a spherically symmetric quantum scalar field coupled to spherically symmetric quantum gravity.

Since the problem has a constraint algebra with structure functions, we will use the "uniform discretizations" approach.

In previous work we have constructed the state corresponding to the vacuum and show that it results in a quantum state peaked around flat space (minus a deficit angle) for the gravitational variables and a state closely resembling the Fock vacuum for the scalar field.

We will use this state to compute propagators for the scalar field.

Context:

Loop quantum gravity is being extended systematically to situations of greater and greater complexity:

-Loop quantum cosmology (lots of people).

-Spherically symmetric vacuum gravity (our previous work)

-Gowdy cosmologies (Madrid group).

-1+1 field theories (RRI).

In all these cases, however, one has never had to confront “the problem of dynamics”, namely, that the constraint algebra of gravity has structure functions.

In loop quantum cosmology there is only one constraint with trivial algebra.
In the spherically symmetric case, special gauge fixings were used that rendered the constraint algebra Abelian. In the Gowdy case, the issue was avoided by polymerizing only partially the variables.

Our own work last year consisted in studying spherically symmetric gravity coupled to a spherically symmetric scalar field in the loop representation.
We chose to polymerize the gravitational variables but not the scalar field.

We wrote the discretized master constraint for the model and using a variational technique sought to minimize it.

The variational technique required a trial state. We chose a state that was a direct product of the Fock vacuum for the scalar field times Gaussians for the gravitational variables centered around the classical solution (flat space minus a solid deficit angle).

The minimum we found was not zero and did not have a continuum limit. However, even for relatively large lattice spacings (many orders of magnitude larger than the Planck scale) the master constraint was very small. This indicates that one has a theory with fundamental discreteness that approximates general relativity well.

Today we would like to build on these results by polymerizing the scalar field and studying what kinds of corrections one gets for the propagators.

We start with the Hamiltonian for a scalar field in flat space-time (we ignore the deficit angle since it changes things little)

$$H = \int d^3x \left(P_\phi{}^2 + (\phi')^2 \right)$$

Discretizing Rewriting

$$H = \sum_i H(i) = \sum_i \frac{P_\phi(i)^2}{2\epsilon} + \frac{(\phi(i+1) - \phi(i))^2}{2\epsilon} = \sum_i \frac{P_\phi(i)^2}{2\epsilon} - \frac{(\phi(i+1) + \phi(i-1) - 2\phi(i))\,\phi(i)}{2\epsilon},$$

Polymerizing the field or the momentum?

Two options: polymerize the field or polymerize the momentum. Will explore both. Similar results. However, polymerizing the momentum yields a theory that in the continuum limit is not polymeric. Starting from the Hamiltonian we just discussed,

$$H = \sum_i \frac{P_\phi(i)^2}{2\epsilon} - \frac{(\phi(i+1) + \phi(i-1) - 2\phi(i))\,\phi(i)}{2\epsilon},$$

One polymerizes

$$H = \sum_i \frac{\sin^2\left(\beta P_\phi(i)\right)}{2\beta^2\epsilon} - \frac{(\phi(i+1) + \phi(i-1) - 2\phi(i))\,\phi(i)}{2\epsilon},$$

And since $P_{continuum}=P_\phi/\varepsilon$ in the continuum limit the first term becomes P_ϕ^2

Since final theory has fundamental discreteness, this may not be an issue?

Polymerizing the field

$$H=\sum_i H(i)=\sum_i \frac{P_\phi(i)^2}{2\epsilon}+\frac{(\phi(i+1)-\phi(i))^2}{2\epsilon}=\sum_i \frac{P_\phi(i)^2}{2\epsilon}-\frac{(\phi(i+1)+\phi(i-1)-2\phi(i))\,\phi(i)}{2\epsilon},$$

$$H=\sum_i\left(\frac{P_\phi(i)^2}{2\epsilon}-\frac{\sin\left(\beta\left(\phi(i+1)+\phi(i-1)-2\phi(i)\right)\right)\sin(\beta\,\phi(i))}{2\epsilon\beta^2}\right).$$

Many choices

To keep things simple, we will expand in β.

$$H=H_0+H_{\rm int}$$

$$H_{\rm int}=\sum_i\frac{1}{2\epsilon}\left(\frac{1}{6}\phi(i)\left(\phi(i+1)+\phi(i-1)-2\phi(i)\right)^3+\frac{1}{6}\phi(i)^3\left(\phi(i+1)+\phi(i-1)-2\phi(i)\right)\right)\beta^2.$$

Appropriateness of using the vacuum obtained without polymerizing the field

The “vacuum” state was obtained minimizing the master constraint without polymerizing the field.

To check the appropriateness of continuing to use this state in the case of a polymerized field, we computed the expectation value of the master constraint with the field polymerized in the state obtained without polymerizing. Expanding to leading order in the polymerization parameter the result obtained is

$$H = \int_{\varepsilon}^{L} dr \frac{(\ell_{Planck})^3}{\varepsilon r^2} + \beta^2 \frac{(\ell_{Planck})^5}{\varepsilon r^4} \ln(\frac{\pi r}{\varepsilon})^2$$

Which shows that the state still yields a very small value for the master constraint for the theory even with a polymerized scalar field.

Wish to study the propagator,

$$G^{(2)}(j,t,k,t') = G^{(0)}(j,t,k,t') + \frac{i^2}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \sum_{j'=-N}^{N}\sum_{k'=-N}^{N} \langle 0| T\left(\phi(j,t)\phi(k,t')H_{\text{int}}(j',t_1)H_{\text{int}}(k',t_2)\right), |0\rangle$$

Where:

$$G^{(0)}(j,t,k,t') = \sum_{n=-N}^{N}\sum_{n'=-N}^{N} \frac{2}{N}\sin\left(\frac{j\pi n}{N}\right)\sin\left(\frac{k\pi n'}{N}\right) G^{(0)}(n,t,n',t').$$

$$G^{(0)}(n,t,n',t') = \langle 0|T\left(\phi(n,t),\phi(n',t')\right)|0\rangle = D(n,t,t')\left(\delta_{n,n'} - \delta_{-n,n'}\right),$$

$$D(n,t,t') = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{d\omega}{\epsilon}\frac{1}{\omega^2 - \omega(n)^2 + i\sigma}\exp\left(-i\omega(t'-t)\right)$$

$$\omega(n) = |2\sin(\pi n/(\epsilon N))/\epsilon|$$

To compute the propagator, one first transforms the interaction Hamiltonian to momentum space and then performs the sums using identities for summations of products of sine functions.

$$
\begin{aligned}
H_{\rm int}(j',t_1) \; = \; & \sum_{n,m,p,q=-N}^{N} \left\{ \frac{1}{12}\beta^2\epsilon^5 \sin\left(\frac{\pi j' n}{N}\right)\omega(m)^2 \sin\left(\frac{\pi j' m}{N}\right)\phi(m,t_1)\right. \\
& \times\omega(p)^2 \sin\left(\frac{\pi j' p}{N}\right)\phi(p,t_1)\omega(q)^2 \sin\left(\frac{\pi j' q}{N}\right)\phi(q,t_1) \\
& + \frac{1}{12}\beta^2\epsilon \sin\left(\frac{\pi j' n}{N}\right)\phi(n,t_1)\sin\left(\frac{\pi j' m}{N}\right)\phi(m,t_1) \\
& \left. \times \sin\left(\frac{\pi j' p}{N}\right)\phi(p,t_1)\omega(q)^2 \sin\left(\frac{\pi j' q}{N}\right)\phi(q,t)\right\}.
\end{aligned}
$$

Going to the momentum representation

$$
\begin{aligned}
\Delta(n,m,p,q) \; \equiv \; & \sum_{j'=-N}^{N} \frac{4}{N^2}\sin\left(\frac{\pi j' n}{N}\right)\sin\left(\frac{\pi j' m}{N}\right)\sin\left(\frac{\pi j' p}{N}\right)\sin\left(\frac{\pi j' q}{N}\right) \\
= \; & \frac{1}{2N}\left[\delta_{n+m,p+q}+\delta_{n+p,m+q}+\delta_{n+q,m+p}+\delta_{n+m+p+q}-\delta_{n,m+p+q}-\delta_{m,n+p+q}-\delta_{p,n+m+q}-\delta_{q,n+m+p}\right].
\end{aligned}
$$

$$
\sum_{j'=-N}^{N} H_{\rm int}(j',t_1) = \frac{1}{4!N}\sum_{n,m,p,q=-N}^{N}\phi(n,t_1)\phi(m,t_1)\phi(p,t_1)\phi(q,t_1)\left[\left(\omega(m)^2\omega(p)^2\epsilon^4+1\right)\epsilon\,\omega(q)^2\right]\beta^2\Delta(n,m,p,q),
$$

$$
\begin{aligned}
G^{(2)}(n_1,t_1,n_2,t_2) \; = \; & G^{(0)}(n_1,t_1,n_2,t_2) + \frac{i^2}{2!}\langle 0|T(\phi(n_1,t_1)\phi(n_2,t_2) \\
& \times\frac{1}{4!N}\int_{-\infty}^{\infty}dt' \sum_{n,m,p,q=-N}^{N} :\phi(n,t')\phi(m,t')\phi(p,t')\phi(q,t'): f(n,m,p)\beta^2\Delta(n,m,p,q) \\
& \times\frac{1}{4!N}\int_{-\infty}^{\infty}dt'' \sum_{n',m',p',q'=-N}^{N} :\phi(n',t'')\phi(m',t'')\phi(p',t'')\phi(q',t''): f(n',m',p')\beta^2\Delta(n',m',p',q')|0\rangle
\end{aligned}
$$

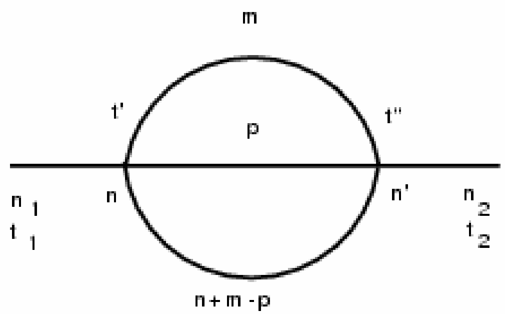
$$
f(m,p,q) = \left[\left(\omega(m)^2\omega(p)^2\epsilon^4+1\right)\epsilon\,\omega(q)^2\right].
$$

Rewriting the previous expression using Wick's theorem,

$$G^{(2)}(n_1,t_1,n_2,t_2) \;=\; G^{(0)}(n_1,t_1,n_2,t_2) - \frac{W}{N^2}\sum_{m,p=-N}^{N}\int_{-\infty}^{\infty} dt'dt''\,[D(n_1,t_1,t')D(m,t',t'')$$
$$\times\; D(p,t',t'')D(n+m-p,t',t'')D(n_2,t'',t_2)]\,f(m,p,n+m-p)\beta^4\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right),$$

or, graphically,

$-\frac{1}{2}$



If one Fourier transforms in time, and approximates the summations: $\sum_{m=1}^{N} \to \frac{L}{\pi}\int_{\pi/L}^{\pi/\epsilon} dp$

$$
\begin{aligned}
G^{(2)}(n_1,\omega_1,n_2,\omega_2) \;=\; & \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2 - p(n_1)^2 + i\sigma}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \\
& -\frac{W}{N^2}\frac{1}{2}\frac{4\pi i}{\epsilon\left(\omega_1^2 - p(n_1)^2 + i\sigma\right)}\frac{1}{\pi^2}\int_{-\infty}^{\infty} d\omega' d\omega'' \left[\int_{-\pi/\epsilon}^{-\pi/L} + \int_{\pi/L}^{\pi/\epsilon}\right] dp_1 dp_2 \\
& \times \frac{4\pi i}{\epsilon\left((\omega')^2 - {p_1}^2 + i\sigma\right)}\frac{4\pi i}{\epsilon\left((\omega'')^2 - {p_2}^2 + i\sigma\right)} \\
& \times \frac{4\pi i}{\epsilon\left(\left(\omega_1 - \omega' - \omega''\right)^2 - p(n_1 - p_1 - p_2)^2 + i\sigma\right)}\tilde{f}^2(p_1,p_2,p(n_1)-p_1-p_2)\beta^4 \\
& \times \frac{4\pi i}{\epsilon\left(\omega_2^2 - \omega(n_2)^2 + i\sigma\right)}\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)
\end{aligned}
$$

where

$$
\tilde{f}(p_1,p_2,p(n_1)+p_1-p_2) = \left(\epsilon^4\left({p_1}^2{p_2}^2\right)+1\right)\left(p(n_1)+p_1-p_2\right)^2\epsilon^2
$$

And carrying out an expansion in pε, the integrals can be computed explicitly for the lowest orders.

The final result is,

$$G^{(2)}(n_1,\omega_1,n_2,\omega_2) \;=\; G^{(0)}(n_1,\omega_1,n_2,\omega_2) + \left[\frac{\alpha_1\beta^4}{\epsilon^2} + \beta^4\alpha_2 p(n_1)^2\right] \frac{4\pi i}{\epsilon}\frac{\delta(\omega_1-\omega_2)\,(\delta_{n_1,n_2}-\delta_{n_1,-n_2})}{\left(\omega_1^2 - p(n_1)^2 + i\sigma\right)^2}$$

$$\approx \;\; \frac{4\pi i}{\epsilon}\frac{1}{\omega_1^2 - p(n_1)^2\,(1+\alpha_2\beta^4) - \frac{\alpha_1\beta^4}{\epsilon^2} + i\sigma}\,(\delta_{n_1,n_2}-\delta_{n_1,-n_2})\;\;\delta(\omega_1-\omega_2)$$

Mass term

Shift in momentum

Generically there will be higher powers of p (we only kept the lowest) in the denominator. Therefore the resulting propagator is within the class considered recently by Hořava in his “Gravity at the Lifshitz point”,

$$\frac{1}{\omega^2 - c^2\mathbf{k}^2 - G(\mathbf{k}^2)^z},$$

Polymerizing the momentum

$$H = \sum_i \frac{P_\phi(i)^2}{2\epsilon} - \frac{(\phi(i+1) + \phi(i-1) - 2\phi(i))\,\phi(i)}{2\epsilon},$$

$$H = \sum_i \frac{\sin^2(\beta P_\phi(i))}{2\beta^2\epsilon} - \frac{(\phi(i+1) + \phi(i-1) - 2\phi(i))\,\phi(i)}{2\epsilon},$$

Expanding in β

$$H_{\rm int}(i) = -\frac{1}{6\epsilon}\beta^2 P_\phi(i)^4.$$

$$G^{(2)}(j,t,k,t') = G^{(0)}(j,t,k,t') + \frac{i^2}{2!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \sum_{j'=-N}^{N} \sum_{k'=-N}^{N} \langle 0| T\,(\phi(j,t)\phi(k,t') H_{\rm int}(j',t_1) H_{\rm int}(k',t_2)),|0\rangle$$

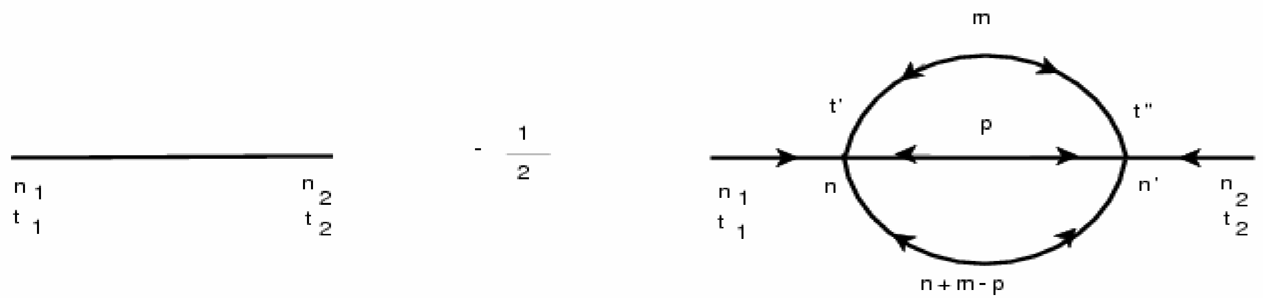
$$\sum_{j'=-N}^{N} H_{\rm int}(j',t_1) = -\frac{1}{12\epsilon N}\sum_{n,m,p,q=-N}^{N} P_\phi(n,t')P_\phi(m,t')P_\phi(p,t')P_\phi(q,t')\Delta(n,m,p,q)\beta^2$$

$$
\begin{aligned}
G^{(2)}(n_1,t_1,n_2,t_2) &= G^{(0)}(n_1,t_1,n_2,t_2) + \frac{i^2}{2!}\langle 0|T(\phi(n_1,t_1)\phi(n_2,t_2)\\
&\times \frac{1}{12N}\int_{-\infty}^{\infty}\frac{dt'}{\epsilon}\sum_{n,m,p,q=-N}^{N} :P_\phi(n,t')P_\phi(m,t')P_\phi(p,t')P_\phi(q,t'):\beta^2\Delta(n,m,p,q)\\
&\times \frac{1}{12N}\int_{-\infty}^{\infty}\frac{dt''}{\epsilon}\sum_{n',m',p',q'=-N}^{N} :P_\phi(n',t'')P_\phi(m',t'')P_\phi(p',t'')P_\phi(q',t''):\beta^2\Delta(n',m',p',q')|0\rangle.
\end{aligned}
$$

$$
\begin{aligned}
G^{(2)}(n_1,t_1,n_2,t_2) &= G^{(0)}(n_1,t_1,n_2,t_2) - W\frac{\beta^4}{3N^2}\sum_{m,p,q,m',p',q'=-N}^{N}\int_{-\infty}^{\infty} dt'dt''D_{\phi P_\phi}(n_1,t_1,m,t')D_{P_\phi P_\phi}(p,t',p',t'')\\
&\times D_{P_\phi P_\phi}(q,t',q',t'')D_{P_\phi P_\phi}(m+p+q,t',m'+p'+q',t'')D_{P_\phi \phi}(m',t'',n_1,t_1) \qquad (64)
\end{aligned}
$$

where

$$
\begin{aligned}
D_{\phi\phi}(n_1,t_1,n_2,t_2) &= \frac{i}{\pi\epsilon}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2-\omega(n_1)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)(\delta_{n_1,n_2}-\delta_{n_1,-n_2})\\
D_{P_\phi\phi}(n_1,t_1,n_2,t_2) &= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\omega(n_1)\,d\omega}{\omega^2-\omega(n)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)(\delta_{n_1,n_2}-\delta_{n_1,-n_2})\\
D_{\phi P_\phi}(n_1,t_1,n_2,t_2) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\omega(n_1)\,d\omega}{\omega^2-\omega(n)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)(\delta_{n_1,n_2}-\delta_{n_1,-n_2})\\
D_{P_\phi P_\phi}(n_1,t_1,n_2,t_2) &= -\frac{i\epsilon}{\pi}\int_{-\infty}^{\infty}\frac{\epsilon\omega(n_1)^2\,d\omega}{\omega^2-\omega(n_1)^2+i\sigma}\exp\left(-i\omega(t_2-t_1)\right)(\delta_{n_1,n_2}-\delta_{n_1,-n_2})
\end{aligned}
$$



Just like before, we go to Fourier space in time, approximate summations by integrals and compute them expanding in power of pε

$$
\begin{aligned}
G^{(2)}(n_1,\omega_1,n_2,\omega_2) &= G^{(0)}(n_1,\omega_1,n_2,\omega_2) + \beta^4\alpha_2 p(n_1)^2 \frac{4\pi i}{\epsilon} \frac{\delta(\omega_1-\omega_2)\left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right)}{\left(\omega_1^2-p(n_1)^2+i\sigma\right)^2} \\
&= \frac{4\pi i}{\epsilon} \frac{1}{\omega_1^2-p(n_1)^2\left(1+\alpha_2\beta^4\right)+i\sigma} \left(\delta_{n_1,n_2}-\delta_{n_1,-n_2}\right) \ \delta(\omega_1-\omega_2)
\end{aligned}
$$

No mass term, but still of the Hořava form.

Summary:

- One can study spherically symmetric gravity coupled to a spherical scalar field using techniques of loop quantum gravity.
- The study has progressed to the point where we can compute propagators.
- The resulting propagators are within Horava's class of Lorentz violating theories.
- The scalar field may acquire a mass.
- The value of the parameter β is unclear.
- One may feel tempted to get a value for it say, by making it responsible for the Higgs mass, but we should recall that we are in a very limited context, spherical symmetry.
- The form of the Lorentz violation depends on how one polymerizes.