Twistors, amplitudes and gravity

From twistor strings to quantum gravity?

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Based on JHEP10(2005)009 (hep-th/0507269), 0706.1941, joint with Martin Wolf, hep-th/0808.3907, joint with Dave Skinner.

1. Intro to twistor theory up to 1980.
2. Twistor-strings and scattering amplitudes.
3. Twistor actions as string field theories (non-perturbative formulation).
4. Plebanski action as space-time form of such action.
5. The Einstein Chern-Simons twistor action for the ASD sector.
6. Derivation of the gravity MHV amplitude.
7. The gravity twistor action (tentative).

Quantum gravity requires a pregeometry for space-time.

Penrose's Proposal:

Twistor space is the fundamental arena for physics.

Flat correspondence:

- Complex space-time $\mathbb{M} = \mathbb{C}^4$, coords $x^{AA'}$, $_{A=0,1,A'=0',1'}$ flat metric $\mathrm{d}s^2 = \mathrm{d}x^{AA'}\mathrm{d}x^{BB'}\varepsilon_{AB}\varepsilon_{A'B'}$, $\varepsilon_{AB} = \varepsilon_{[AB]}$ etc..
- Twistor space $\mathbb{T} = \mathbb{C}^4$, coords $Z^\alpha = (\omega^A, \pi_{A'})$, $_{\alpha=(A,A')}$. Projective twistor space

$$\begin{aligned}\mathbb{PT} &= \{\mathbb{T} - 0\}/\{Z \sim \lambda Z, {\scriptstyle \lambda\in\mathbb{C}^*}\} = \mathbb{CP}^3 \\ &= \text{degrees of freedom of massless spinning particle}\end{aligned}$$

- Incidence relation

$$\omega^A = i x^{AA'}\pi_{A'}.$$

$\{\text{Point } x \in \mathbb{M}\} \longleftrightarrow \{L_x = \mathbb{CP}^1 \subset \mathbb{PT}\}$, hgs coords $\pi_{A'}$.

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Massless fields of helicity $n/2 \leftrightarrow$ twistor 'functions' of homogeneity $-n-2$

$$f(Z) \in H^1(\mathbb{PT}', \mathcal{O}(-n-2)), \qquad f(\lambda Z) = \lambda^{-n-2} f(Z)$$

Linearized gravity:

$$(h_2(Z), h_{-6}(Z)) \in H^1(\mathcal{O}(2)) \oplus H^1(\mathcal{O}(-6)).$$

ASD part, Weyl$^+ = 0 \leftrightarrow h_2(Z)$, SD part, Weyl$^- = 0 \leftrightarrow h_{-6}(Z)$

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Main ideas: We deform $\mathscr{PT}$ by plate tectonics or changing $\bar\partial$.

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The googly problem and quantization

Googly problem: only have ASD data, need SD part also.

Quantize anyway!

- But, how do we quantize complex structures?
  For fuzzy points, quantize $\mathbb{CP}^1$s as maps $\mathbb{CP}^1 \to \mathbb{PT}$.
- $\rightsquigarrow$ **Holomorphic string theory:** Payoff, quantization of the $\mathbb{CP}^1$s leads to quantization of the complex structures. But
- **Anomalies:** String theory requires Calabi-Yau.
  Witten: Super-Twistor space $\mathbb{PT}_{[4]} = \mathbb{CP}^{3|4}$ is Calabi-Yau.
- **Bonus:** Such string theories depend on both $\mathbb{C}$-structure *and* 'B-field' $\partial b \in H^1(\mathbb{PT}_{[4]}, \Omega^2_{\mathrm{cl}})$ — encodes googly data.
- For $\mathcal{N} = 4$, ASD Yang-Mills multiplet contains SD part also.
- Obtain data for *full* $N = 4$ conformal supergravity or Yang-Mills.

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Twistor-string theories (Witten, Berkovits) simplest is a 'nonlinear β-γ system' (M & Skinner hep-th/0807.2276). Quantize fields on $\mathbb{CP}^1$, or Σ, a general Riemann surface:

$$Z : \Sigma \to \mathbb{PT}_{[4]}\,, \qquad Y \in Z^*(T^*\mathbb{PT}_{[4]}) \otimes \Omega^{1,0}(\Sigma)$$

$$S = \mathrm{Re} \int_\Sigma Y_\alpha \bar\partial Z^\alpha$$

Yang-Mills: String theory naturally couples to a bundle E via worldsheet spinors Σ with values in E.

Application: Can calculate scattering amplitudes for $N = 4$ Super-Yang-Mills coupled to Conformal Supergravity

$$\text{amplitude} = \text{path-integral over hol. curves in } \mathbb{PT}_{[4]}$$

degree $d \leftrightarrow \#$ of $+$ helicity particles, genus $g = \#$ loops.

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$$Z = (\omega, \pi, \eta) \in \mathbb{T} := \mathbb{C}^{4|0} \times \mathbb{C}^{0|4}.$$

Amplitudes functions of null supermomenta $P = (\tilde{\pi}_A, \pi_{A'}, \eta)$.

Half-Fourier transform amplitudes from momentum space

$$\mathcal{A}(Z_a) = \int A(P_a) \prod_{a=1}^{n} \mathrm{d}^2\tilde{\pi}_a \, \mathrm{e}^{i\tilde{\pi}_a \cdot \omega_a}, \quad a = 1, \ldots, n, \text{ \# of particles.}$$

i.e., just in variables $\tilde{\pi} \leftrightarrow \omega$, ($\pi, \eta$ unchanged).
Turns analytic structure into geometric structure
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- Amplitude $= 0$ if particles all have $-$ helicity or single $+$.
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The MHV amplitude

Nair (1986)

Parke-Taylor formula for $\mathcal{N} = 4$ super Yang-Mills MHV amplitude becomes twistor-string path integral for worldsheet instantons of degree one

$$\mathcal{A}(Z_a) = \int_{\mathbb{R}^{4|8}\times\mathbb{R}^n} \frac{\mathrm{d}^{8|8}(Y_0 Y_1)}{\mathrm{Vol}\,\mathrm{GL}(2)} \prod_{a=1}^{n} \frac{\delta^{3|4}(Z_a, Z(\sigma_a))\mathrm{d}\sigma_a}{\sigma_a - \sigma_{a-1}}$$

restricts Z_a to lie on the line $Z(\sigma) = Y_0 + \sigma Y_1$.

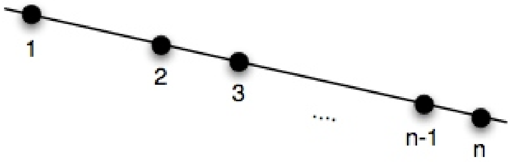

Figure: *Twistor support of MHV amplitude on line in $\mathbb{PT}$.*

Tree-level N^{m}MHV: (Roiban, Spradlin & Volovich, 2004)

$$\mathcal{A}(Z_a) = \int_{\mathscr{M}_{0,n}(\mathbb{PT},d)} \frac{\prod_{r=0}^{d} \mathrm{d}^{4|4} Y_r \cdot}{\text{vol GL}(2)} \prod_{a=1}^{n} \frac{\delta^{3|4}(Z_a, Z(\sigma_a))\, \mathrm{d}\sigma_a}{(\sigma_a - \sigma_{a-1})}$$

where $Z(\sigma)$ now has degree $1 + m$

$$Z(\sigma) \;=\; \sum_{r=0}^{d} Y_r \sigma^r : \mathbb{CP}^1 \to \mathbb{PT}\,, \quad \sigma \in \mathbb{C} \subset \mathbb{CP}^1$$

Conjecture: N^{m}MHV amplitudes at g-loops $\leftrightarrow$ integrals over space of degree d maps of genus g curves to twistor space with

$$d = m + 1 + g$$

(Proved now for leading singularities of pure $\mathcal{N} = 4$ YM $\leftrightarrow$ nodal curves, Bullimore, M. & Skinner hep-th/0912.0539.)

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Twistor-string for Einstein (super-)gravity?

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$$S([g]) = \int (\mathrm{Weyl})^2$$

Conformal SUGRA unphysical

- ghosts
- wrong equs of motion,
- ...

want Einstein SUGRA at most.

Current programme: Construct twistor-action for Einstein gravity and reverse engineer corresponding twistor-string.

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The existing non-trivial twistor-strings contain Yang-Mills and *Conformal* supergravity, i.e., spacetime action

$$S([g]) = \int (\text{Weyl})^2$$

Conformal SUGRA unphysical

- ghosts
- wrong equs of motion,
- ...

want Einstein SUGRA at most.

Current programme: Construct twistor-action for Einstein gravity and reverse engineer corresponding twistor-string.

The twistor actions = twistor-string field theories.

- For YM and conformal gravity can write action as

$$S_{\rm Full} = S_{\rm ASD} + S_{\rm MHV}$$

  $S_{\rm ASD}$ contains kinetic terms & interactions of ASD sector, $S_{\rm MHV}$ contains remaining interactions of full theory.
- We reformulate on twistor space where ASD sector is of Chern-Simons type (reflecting complete integrability).

Two special gauges

1. leads directly to space-time action
2. axial on twistor space, **not** accessible from space-time.
   - linearizes ASD sector, reduces $S_{\rm ASD}$ to kinetic terms; $\leftrightarrow$ complete integrability of ASD sector.
   - $S_{\rm MHV} =$ generating function for MHV amplitudes.
   - leads to MHV formalism as Feynman rules: scalar propagators and MHV amplitudes as vertices.

Twistor actions; a non-perturbative approach

hep-th/0507269, 0604040, 0702035

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Maximal Helicity Violating (MHV) amplitudes for gravity

As before, MHV amplitudes are $++---\cdots-$;
using spinor helicity notation:

$$\langle 1\,2\rangle := p_{1A}p_2^A\,,\ [1\,2] := p_{1A'}p_2^{A'}\,,\ \langle 1|2|3] = p_{1A}P_2^{AA'}p_{3A'}\,.$$

we have

$$M(1^+,2^-,3^-,\ldots,n{-}1^-,n^+) = \delta^4\left(\sum_i P_i\right)\frac{[1\,n]^8}{[1\,n-1][n-1\,n][n\,1]}$$
$$\left\{\frac{1}{\Pi_{i=1}^n[i\,i+1]}\prod_{k=2}^{n-1}\frac{\langle k|P_{k+1}+\cdots+P_{n-1}|n]}{[kn]} + \Pi_{(2,\ldots,n-2)}\right\}\,.$$

- Conjectured by Berends, Giele & Kuijf (1988) using Kawai Llewellyn Tye string theory relations between gravity and Yang-Mills.

Bjerrum-Bohr et. al. (2006) argue that all gravity amplitudes are generated by MHV vertices and scalar propagators.

- Similar results true and proved for Yang-Mills and conformal gravity.
- Big simplification; difficult to see from space-time action.
- Uses recursion relations & asymptotics of amplitudes.
- Proof is controversial; asymptotics break down at 12 points, Elvang & Freedman.

Chiral action for expansion about ASD sector

Abou-Zeid, Hull hep-th/0511189

Use Plebanski-Palatini formulation with variables (on $\mathscr{M}^4$):

- $e^{AA'}$, tetrad of 1-forms s.t.

$$\mathrm{d}s^2 = e^{AA'} \odot e^{BB'} \varepsilon_{AB}\varepsilon_{A'B'} \,,$$

- $\Gamma_{A'B'} = \Gamma_{(A'B')}$ the SD spin connection 1-forms.

Action

$$S = \int_{\mathscr{M}} \Sigma^{A'B'} \left(d\Gamma_{A'B'} + \kappa^2 \, \Gamma_{A'}^{C'} \wedge \Gamma_{B'C'} \right) \,,$$

where $\Sigma^{A'B'} = e_A^{(A'} \wedge e^{B')A}$.

Field equations:

$\kappa^2 \Gamma_{A'B'} =$ primed spin connection 1-form; Ricci$= 0$.

The ASD sector and perturbations around it

ASD sector: Set $\kappa = 0$, $S_{ASD} = \int_{\mathscr{M}} \Sigma^{A'B'} \mathrm{d}\Gamma_{A'B'}$, $\rightsquigarrow$ field equs

$$\mathrm{d}\Sigma^{A'B'} = 0 \Rightarrow \text{ metric is ASD, and}$$

$$\mathrm{d}\Gamma_{A'B'} \wedge \mathrm{e}^{AA'} = 0\,, \Rightarrow \mathrm{d}\Gamma_{A'B'} = \psi_{A'B'C'D'} \Sigma^{C'D'}$$

and $\psi_{A'B'C'D'}$ is linearized SD Weyl spinor on ASD background.

Remaining interactions:

$$S_{MHV} = \int_{\mathscr{M}} \kappa^2 \Sigma^{A'B'} \wedge \Gamma_{A'C'} \wedge \Gamma^{C'}_{B'}\,.$$

- All $-$ amplitude $= 0 \leftrightarrow$ consistency of ASD sector.
- One $+$, rest $-$ amplitude $= 0 \leftrightarrow$ integrability of ASD sector.
- MHV amplitude $\leftrightarrow \Gamma = \Gamma_1 + \Gamma_2$, asymptotic plane waves on ASD background, also aymptotic to sum of plane waves.

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On $\mathbb{PT}$ have Poisson structure $\{\,,\,\} = \varepsilon^{AB}\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\omega^B}$.
Fields $\Sigma^{A'B'}$ and $\Gamma_{A'B'}$ correspond respectively to

$$h \in \Omega^1(2)\,, \quad \text{and} \quad B = B_A \mathrm{d}\omega^A \in \Omega^{1,1}_V(-4)$$

h defines almost complex structure $\bar\partial_h = \bar\partial + \{h, \cdot\}$

- Action

$$S_{\mathrm{ASD}}[h,B] = \int_{\mathbb{PT}} \left\{\bar\partial h + \{h,h\}\,,\cdot\right\} \lrcorner\, B \wedge \mathrm{d}^3 Z$$

- with field equations

$$\left\{\bar\partial h + \{h,h\},\cdot\right\} = 0\,, \quad \bar\partial_h\left(\frac{\partial}{\partial\omega^A}B^A\right) = 0\,.$$

- $\Rightarrow$ integrability $\bar\partial_h^2 = 0$ and $\mathrm{d}B \in H^1_{\bar\partial_h}(\mathscr{PT}, \mathcal{O}(-6)) \otimes \mathrm{d}^2\omega$.

Complete integrability: Choose $\mathscr{I}$ coordinates ('axial gauge')
$\rightsquigarrow \{\{h,h\},\cdot\} \lrcorner\, B = 0$; equations become linear.

Twistor action for ASD gravity

M. & Wolf hep-th/0706.1941

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A direct calculation for the MHV amplitude

Sub integral formula $\Gamma_{A'B'} = \int_{\mathbb{CP}^1_x} \pi_{A'}\pi_{B'} B \wedge [\pi\, \mathrm{d}\pi]$ into S_{MHV}

$$S_{MHV} = \int_{\mathscr{M}\times\mathbb{CP}^1_x\times\mathbb{CP}^1_x} \mathrm{d}\omega_1^A \mathrm{d}\omega_{2A} [\pi_1\, \mathrm{d}\pi_1][\pi_2\, \mathrm{d}\pi_2][\pi_1\, \pi_2]^2 B_1 \wedge B_2$$

here for $x \in \mathscr{M}$, $\mathbb{CP}^1_x$, is corresponding $\bar{\partial}_h$-holom $\mathbb{CP}^1$ in $\mathscr{P}\mathscr{T}$. Choose momentum eigenstates for B_i, $i = 1, 2$ and let h be a sum of momentum eigenstates; integrate and expand to get

$$M(1^+, 2^-, 3^-, \ldots, n{-}1^-, n^+) = \delta^4\left(\sum_i P_i\right) \frac{[1\, n]^8}{[1\; n-1][n-1\; n][n\, 1]}$$
$$\left\{ \frac{1}{\Pi_{i=1}^n [i\; i+1]} \prod_{k=2}^{n-1} \frac{\langle k | P_{k+1} + \cdots + P_{n-1} | n]}{[kn]} + \Pi_{(2,\ldots,n-2)} \right\} ;$$

the BGK MHV formula derived from first principles.

MHV Twistor action for gravity

- This formulation of anti-self-dual gravity works 'off-shell' and can include interactions of full gravity and more.
- The MHV vertices are generated by $S_{MHV}[B, h]$.
- So for full theory set

$$S_{Full}[B, h] = S_{\mathrm{ASD}}[B, h] + S_{MHV}[B, h]$$

an action that $\rightsquigarrow$ the MHV formalism perturbatively.

- $\mathcal{N} = 8$ version possible; $S_{ASD\,[8]}$, has holomorphic Chern-Simons form for group of complex poisson diffeos.

Problems

- Off-shell, but gauge fixed; doesnt express full symmetries.
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- Self-contained proof of the BGK MHV gravity amplitudes.
- Twistor gravity action that yields Gravity MHV formalism.

Outlook

- Twistor actions for Yang-Mills and conformal gravity are **not** gauge fixed and **do** give self-contained proof of MHV formalism for Yang-Mills and conformal gravity;
- so full gauge invariant twistor action should exist; if MHV formalism is incomplete, it would give completion.
- Action $\stackrel{?}{\leftrightarrow}$ string field theory for an Einstein twistor-string?
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Thank You!