Continuum approximation of microscopic quantum dynamics: lessons from condensed matter systems and analogue gravity models

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Plan of the tour

- General discussion/motivation
- Crystals
- Bose-Einstein condensation
- Large N and Phase Transitions
- Matrix models
- Conclusion
First of all...

I will always have in mind LQG, spinfoams, simplicial gravity, etc.

I will try to explore the implications that the analysis of CM systems can have for these scenarios.
Background

- QG scenarios based generally on discrete microscopic constituents (even just as tools, e.g. regulators)
- The microscopic dynamics is assigned
- The case of small number of constituents is understood (e.g. asymptotics & single 4-simplex)

Ref.: all the other talks
• Continuum semiclassical gravity is the endpoint of the analysis.

• What is the macroscopic/continuum/classical limit of QG?

• Context in which the same problems arise: Condensed Matter (CM)
The Problem in CM

- In CM one is asking a very basic question: *Given a set of a certain number of quantum particles in a given region of space, what is the ground state of the system.*

- As a side issue: how to pass from discrete to continuum

- The microscopic dynamics (atomic physics) is known exactly (at least in principle).

\[ \hat{H} = \sum_I -\frac{\hbar^2 \nabla_I^2}{2M_I} + \sum_{ie} -\frac{\hbar^2 \nabla_{ie}^2}{2m_e} + \frac{1}{2} \sum_{ie \neq je} V_{ee}(x_{ie} - x_{je}) + \frac{1}{2} \sum_{I \neq J} V_{IJ}(x_I - x_J) + \sum_{ie,I} V_{Ie}(x_{ie} - x_J) \]
How to solve the problem?

• No general rules, few analytical techniques
• Some methods (mean field approx, symmetries...) variously applied and improved
• Sometimes they turn out to be wrong
• Experiments and numerical approaches are providing new insights and challenges for the development of accurate theoretical models
I: warm up with crystalline solids
Crystals

- A very simple example (but already complicated enough) of CM system is represented by crystals
- Described by regular lattices with atoms/ions sitting on the nodes
- Classification of all the possible lattices based on symmetries (crystallographic groups)
Continuum limit: Elasticity

- Natural problem: elasticity properties of certain macroscopic bodies. Other possibilities (optics, thermal, etc.)
- On large scales, the discrete lattice is replaced with a continuum density function (coarse graining).
- All the microphysics is encoded and summarized into some macroscopic parameters. Ideally, they could be computed from Van der Waals forces.
Elasticity/2 : collective fields

\[ \sigma_{ij} \rightarrow F_i(V) = \int_V dV \partial_j \sigma_{ij} \quad \text{Stress tensor} \]

Displacement from equilibrium configuration

\[
\begin{align*}
  x^i &\rightarrow x^i + \xi^i(x) \\
  dx^i &\rightarrow dx^i + \frac{\partial \xi^i(x)}{\partial x^j} dx^j \\
  ds^2 &\rightarrow ds^2 + 2\frac{\partial \xi^i(x)}{\partial x^j} dx^j dx^k \delta_{ik} + O(\xi^2)
\end{align*}
\]

\[ u_{ij} = \frac{1}{2} \left( \frac{\partial \xi_i}{\partial x^j} + \frac{\partial \xi_j}{\partial x^i} \right) \quad \text{Strain tensor} \]

Mechanical properties encoded into the free energy

\[
F = F_0 + \int_V \sigma_{ij} u^{ij} d^3x
\]

Constitutive relations

\[ \sigma_{ij} = \lambda_{ijhk} u^{hk} \]
In general the structure of the elastic modulus tensor will depend on the microscopic structure (symmetries of the lattice).

**Isotropic case**

\[ \lambda_{ijhk} = a \delta_{ij} \delta_{hk} + b \left( \delta_{ih} \delta_{jk} + \delta_{ik} \delta_{jh} \right) \]

This tensor encodes all the macroscopic mechanical properties of the continuum (e.g. sound waves).
Comments

• There are no theorems that prove that the ground state of the system should be a crystal (plausibility arguments)

• Some intuition come from atomic physics (He vs Li, but see Hg)

• This intuition is not enough!

• Allotropic forms: for C you have, for instance, both diamond (insulator) and graphite (conductor)
Coarse graining

- Crystals, but also fluids, are inhomogeneous on scales small enough.
- To go for the continuous representation one should perform a coarse graining.
- Find a scale $L$, such that any quantity, averaged over a cell of size $L$, has small fluctuations.
- This scale is statistical in nature, not necessarily dynamical.
In practice

$$\rho(x) \approx \sum_i m_i \delta^3(x - x_i)$$

$$\bar{\rho}_L(x_i) = \frac{1}{L^3} \int_{V_i} d^3x \rho(x)$$  \hspace{1cm} \text{Constant in each cell in which the body is partitioned}$$

$$\Delta \rho = \langle \rho^2 \rangle_{C_i} - \bar{\rho}_L^2(C_i)$$ \hspace{1cm} \frac{\Delta \rho}{\bar{\rho}^2} \ll 1$$

The effective continuum theory (e.g. differential equations for continuous density) is a theory for the coarse grained density, not for the microscopic one.
Comments

- Microscopic symmetries and vs macroscopic ones:
- Quantum mechanics not really relevant.
- Renormalization group: how the effective theory changes when we change the coarse graining
- Concrete example: take Regge calculus (but other discrete approaches as well) and try to address the problem of coarse graining.
- What is the effective action after coarse graining?
- What is the fate of symmetries?

Bahr and Dittrich (Regge calculus) 2009
Bombelli, Corichi, Winkler (How to reconstruct a manifold out of a graph) 2004, 2009
II: Inclusion of Quantum Mechanics
Bose Einstein Condensates

- Macroscopic system in which quantum mechanics is crucial
- At sufficiently low temperature, a system of bosons condenses.
- Macroscopic occupation number of the ground state (for single particle)

Pethick and Smith
Fetter and Walecka
Abrikosov, Gorkov, Dzyaloshinski
General framework

General setting: dilute, weakly interacting Bose gas

Formalism: second quantization

\[
\hat{\Psi} = \frac{1}{\sqrt{V}} \sum_k \hat{a}_k e^{i k \cdot x}
\]

Operators creating/destroying bosons (atoms)

\[
\hat{H}_0 = \int \hat{\Psi}^\dagger(x) \left( -\frac{\hbar}{2m} \nabla^2 - \mu + \frac{\kappa}{2} |\hat{\Psi}|^2 \right) \hat{\Psi}(x) d^3 x
\]

Condensation via mean field

\[
\hat{\Psi} \approx \psi \mathbb{I} + \hat{\chi} \quad \langle \Omega | \hat{\Psi} | \Omega \rangle = \psi \quad \text{condensate wavefunction}
\]

Gross-Pitaevski equation for the condensate (neglecting the fluctuations)

\[
i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi - \mu \psi + \kappa |\psi|^2 \psi
\]

Note that:

\[
\hat{\Psi} |0\rangle = 0 \quad \hat{\Psi} |\Omega\rangle \neq 0
\]

atomic Fock vacuum

ground state
More about MFA

• MFA out of coherent states, peakedness around a given classical complex field.

\[ |\Omega\rangle \approx \exp(z\hat{a}^\dagger_0)|0\rangle \quad |z|^2 = N_0 \]
\[ \hat{\Psi}(x)|\Omega\rangle = \sqrt{N_0}u_0(x)|\Omega\rangle = \psi(x)|\Omega\rangle \]

• Example of how the microscopic theory goes onto a continuum fluid description

\[ \psi = \sqrt{n_c}\exp(-i\theta/\hbar) \quad \text{Madelung representation} \]

• Gross-Pitaevski goes onto continuity and Euler equation

\[ |\psi(x)|^2 = n_c(x) \quad \vec{v} \propto \vec{\nabla}\theta \]
Comments

- The mean field approximation can be seen as the **guess** that the ground state is a coherent state.
- The continuum limit follows immediately.
- There is no obvious coarse graining scale.
- Coherent state just an approximation (presence of interactions). Beyond mean field methods required for certain experimental conditions.
- Coarse graining is needed for BEC (failure related to large fluctuations).
What about QG?

- Coherent state for QG as a route to classical spacetime?
- There is no coarse graining scale: is the use of semiclassical states enough to get continuum GR?
- LQG, spinfoam & semiclassical states
- What about GFT? The sum over the discretizations already implemented: what about coherent states?
- What are the physical consequences for excitations around semiclassical states? (i.e. are there collective d.o.f.?)
III: Large $N$
Thermodynamic limit...

- Thermodynamics: one ignores the microstates of the system, and tries to describe it via a small number of global variables (N, p, V, T, S...).

- Thermodynamic limit: one takes the limit of infinitely large system (e.g. N, V formally becoming infinite) keeping fixed intensive quantities (e.g. number density)
... and phase transitions

- The state of a macroscopic system is characterized by certain state functions.
- Phase transition: discontinuities of these functions
- Critical exponents: scaling laws describing the critical behavior

\[ f(t) \sim t^\alpha \quad t = \frac{T_c - T}{T_c} \]
Comments

• Phase transitions are strictly related to the thermodynamic limit. Absent for systems with finite number of d.o.f.

• Clear case in which the macroscopic regime is qualitatively different from the few-body case

• Universality: at critical point many microscopically different systems have the same critical exponents (insensitivity to microphysics.)
Matrix models

- Hermitian $N \times N$ matrices

\[ M = M^\dagger \]

- Potential

\[ V(M) = \sum_{k=0}^{\infty} \frac{g_k}{k} \text{tr}(M^k) \]

- “Classical theory”: gauge field theory in zero dimensions

\[ U^\dagger U = \mathbb{I} \]
\[ M \rightarrow U^\dagger MU \]

\[ V'(M) = 0 \quad \text{Classical EOM} \]
Matrix models/2

• Statistical mechanics

\[ Z = \int dM \exp(-NV(M)) \]

\[ N \sim \beta = \frac{1}{k_B T} \]

\[ N \sim \frac{1}{\hbar} \]

• Feynman diagrams: perturbative expansion is linked to two dimensional compact surfaces (sum over all finite polygonulations)

• Proposal: use it as the definition of the partition function for two dimensional quantum gravity

Di Francesco, Ginsparg, Zinn-Justin

hep-th/9306153
Diagrammatics

Propagator

Cubic vertex

‘t Hooft NPB 1974
Diagrammatics/2
Diagrammatics/2
Matrix models/3

• The graphs have a weight which depends on the Euler characteristic of the corresponding dual triangulation
  \[ A(\Gamma) = N^{2-2h}a(g) \]

• Large N limit: only planar surfaces are contributing!

• But for the continuum limit one has to go for the double scaling limit

• Case in which the sum over the discretizations is not enough to get a very refined continuum limit.

• Besides the large N, also tuning of the coupling constants to some critical value.
Saddle point evaluation

- One can always diagonalize an Hermitian matrix

\[ M = U^\dagger \Lambda U \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \]
\[ V'(\lambda_i) = 0 \quad \forall i = 1, \ldots, N \]

- Large $N$/semiclassical limit of the partition function: saddle point approximation

\[ Z = \int dM \exp(-NV(M)) \]

- Crucial feature: effect of the measure of the path integral (genuinely “quantum” in nature)

\[ Z = \int \left( \prod_{i=1}^{N} d\lambda_i \right) \Delta^2(\lambda) \exp \left[ -N \sum_{j=1}^{N} V(\lambda_j) \right] \]

Vandermonde determinant (Faddeev-Popov)
“Hydrodynamics” of the eigenvalues

- Equations for the saddle point
  \[ \frac{1}{N} \sum_{j,j \neq i} \frac{1}{\lambda_i - \lambda_j} = V'(\lambda_i) \]

- Large N: the density of eigenvalues (collective field)
  \[ \rho(x) = \frac{1}{N} \sum_{k=1}^{N} \delta(x - \lambda_k) \quad \int dx \rho(x) = 1 \]

- The large N limit (the saddle point) as an equation for continuous \( \rho(\lambda) \)
  \[ 2 \int \frac{\rho(\lambda')}{\lambda - \lambda'} d\lambda' = V'(\lambda) \]

- The density of eigenvalues encodes the properties of the thermodynamical limit (large N expansion) and all the critical properties

Sakita, Jevicki 1980, 1981
Comments

- Matrix models: an example in which the “quantum” theory is qualitatively different from the classical case even in the semiclassical regime (single matrix vs 2D QG)

- Obvious relevance for GFT (generalization to Boulatov and Ooguri).

- What is the technical point that turns a GFT into a theory for gravity on macroscopic continuum spacetime?

- Ongoing programs in perturbative renormalization (Freidel, Gurau, Oriti), condensation in GFT (work in progress)?

- What are the implications for spinfoams/LQG? Is the continuum limit a sort of phase transition (thermodynamic limit involved)?
Wrapping up...

- Not covered topics like kinetic theory, quantum phase transitions, superconductivity, etc.

- Bottomline: “More is different”.  
  
  P.W. Anderson, 1972

- Knowing the microscopic d.o.f. and their dynamics is not enough even in the “simple” case of CM

- Even the fate of symmetries is not completely clear (crystal, QCD, diffeomorphism inv.).

- One has to use several strategies (numerical, heuristics, experiments) to circumvent the impossibility of doing analytical calculations.
The pessimistic slide

• In fact, besides the big problem of obtaining the continuum limit, we should face an additional challenge.

• The inverse problem: given the macroscopic dynamics (GR and modifications), what are the specific signatures of the underlying dynamics?

• Typical situation: the microscopic dynamics is (partially) washed away when we go for the macroscopic limit.
The optimistic slide

- After all, you can work on CM and you can understand a lot of the physical properties of the systems considered.

- Perhaps among the machineries elaborated for CM there is the right tool to attack the continuum semiclassical limit of QG.

Yes, we can!