ECD

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Talk mainly based on MB, A. Tsobanjan: arXiv:0911.4950

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Evaluating the dynamics

Testing the semiclassical limit of a quantum theory or deriving potential predictions requires a handle on properties of physical semiclassical states.

Different problems arise in this context:

- $\rightarrow$ Semiclassicality (definition, classes of states)
- $\rightarrow$ Physicality and dynamics (implementing and solving constraints)

Especially in quantum gravity, long evolution times are often involved. While initial (kinematical) states may be chosen to be of special (coherent) form, evolved states can easily become much, much, much, . . . more general.

Here: State properties in deparameterizable systems.

Deparameterizable systems

Constraint of the form

$$C_{\rm nonrel} = p_+ + H(\alpha, p_\alpha) = 0$$

or

$$C_{\rm rel} = p_+^2 - H(\alpha, p_\alpha)^2 = 0$$

for canonical variables (α, p_α) and (β_+, p_+). No terms in the constraint depend on β_+, which will play the role of time.

Reduced phase space quantization:

- $\rightarrow$ Choice of time β_+ before quantization.
- $\rightarrow$ Quantum dynamics determined by Schrödinger evolution with $\hat{H}(\alpha, p_\alpha)$ or $\pm|\hat{H}(\alpha, p_\alpha)|$.
- $\rightarrow$ Easy to define initial (“kinematical”) coherent states.

“Relativistic” harmonic oscillator

$$\hat{C} = \hat{p}_+^2 - \hat{p}_\alpha^2 - \hat{\alpha}^2$$

Physical states solve

$$i\hbar \frac{\partial}{\partial \beta_+} \Psi(\alpha, \beta_+) = \pm\sqrt{\hat{p}_\alpha^2 + \hat{\alpha}^2}\,\Psi(\alpha, \beta_+)$$

Solutions:

$$\Psi_\pm(\alpha, \beta_+) = \sum_{n=0}^{\infty} c_n \varphi_n(\alpha) \exp(\mp i \lambda_n \beta_+ / \hbar)$$

with harmonic oscillator eigenstates $\varphi_n(\alpha)$ and $\lambda_n = \sqrt{(2n+1)\hbar}$. Constants c_n determined by initial values.

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Physical semiclassical states

Harmonic oscillator ("kinematical") coherent states at $\beta_+ = 0$:

$$c_n = \exp\left(-\frac{|z|^2}{2}\right)\frac{z^n}{\sqrt{n!}} \quad , \quad z \in \mathbb{C}$$

such that

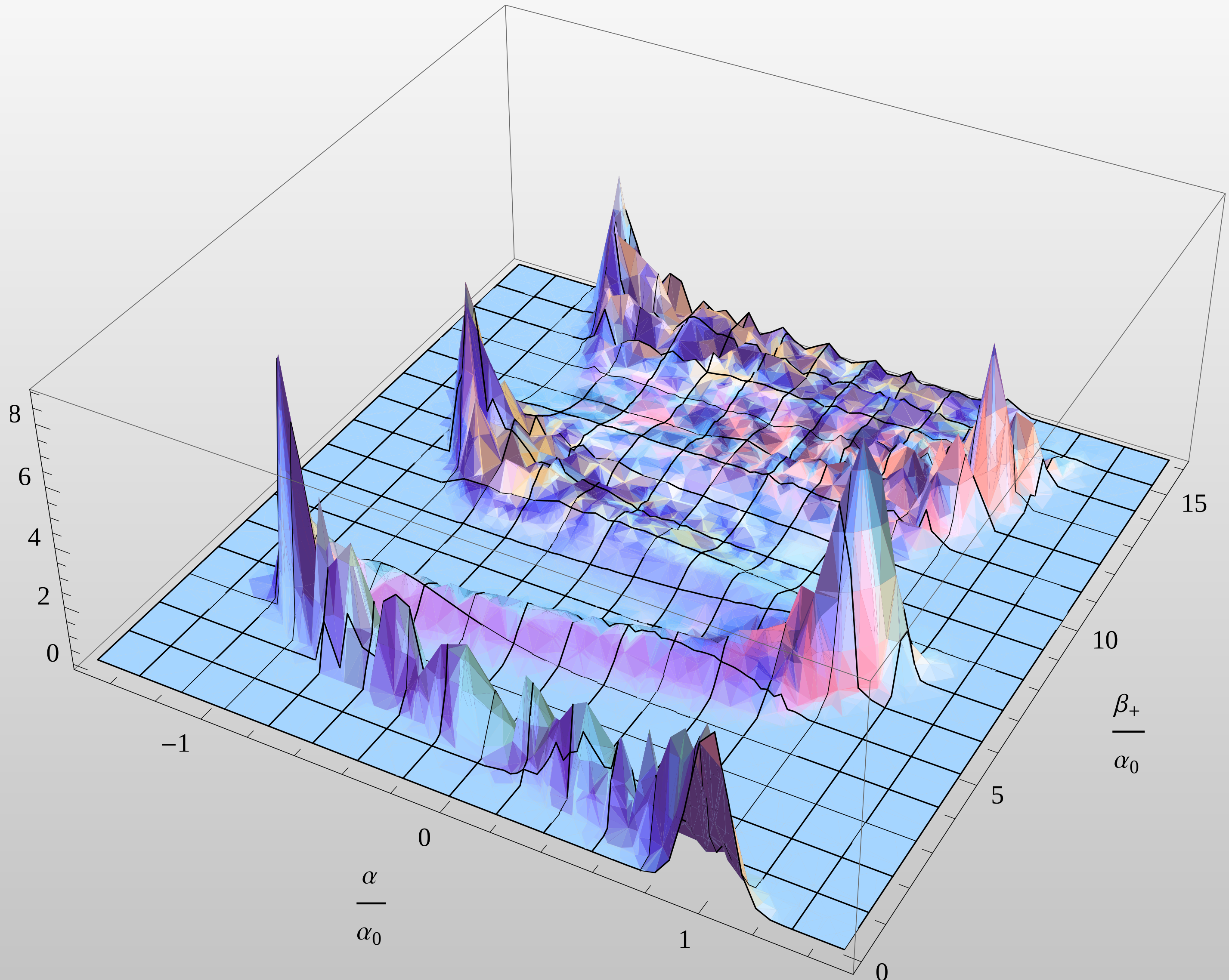
$$\Psi(\alpha, 0) = (2/\pi)^{1/4} \exp\left(-\frac{1}{2}(|z|^2 - z^2 + 2\alpha^2 - 4iz\alpha)\right)$$

Evolving state: coefficients

$$c_n e^{-i\lambda_n \beta_+/\hbar} = \frac{1}{\sqrt{n!}} e^{-|z|^2/2} z^n \exp(-i\sqrt{2n+1}\beta_+/\sqrt{\hbar})$$

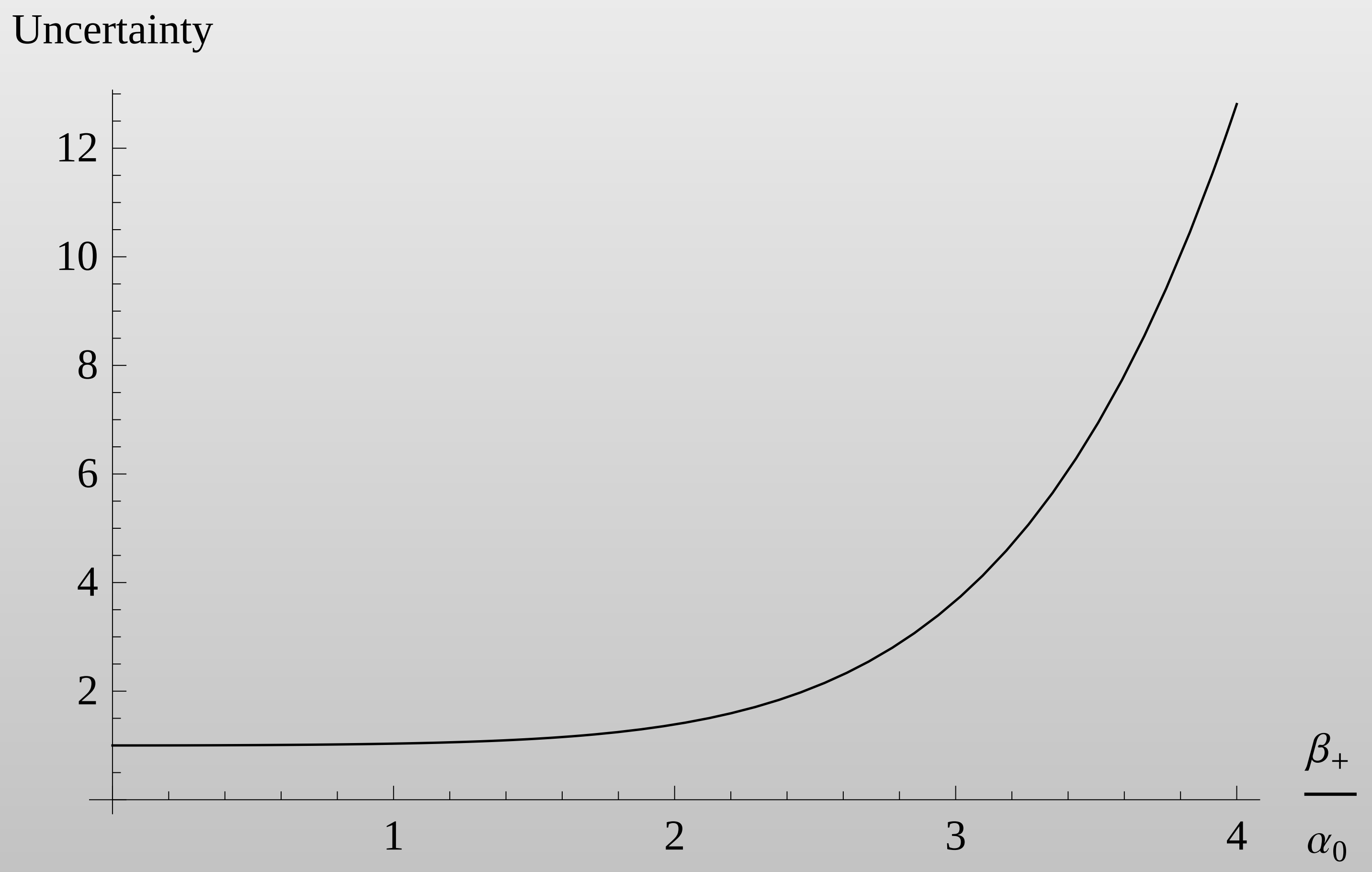
non-coherent for $\beta_+ \neq 0$.

Wave function

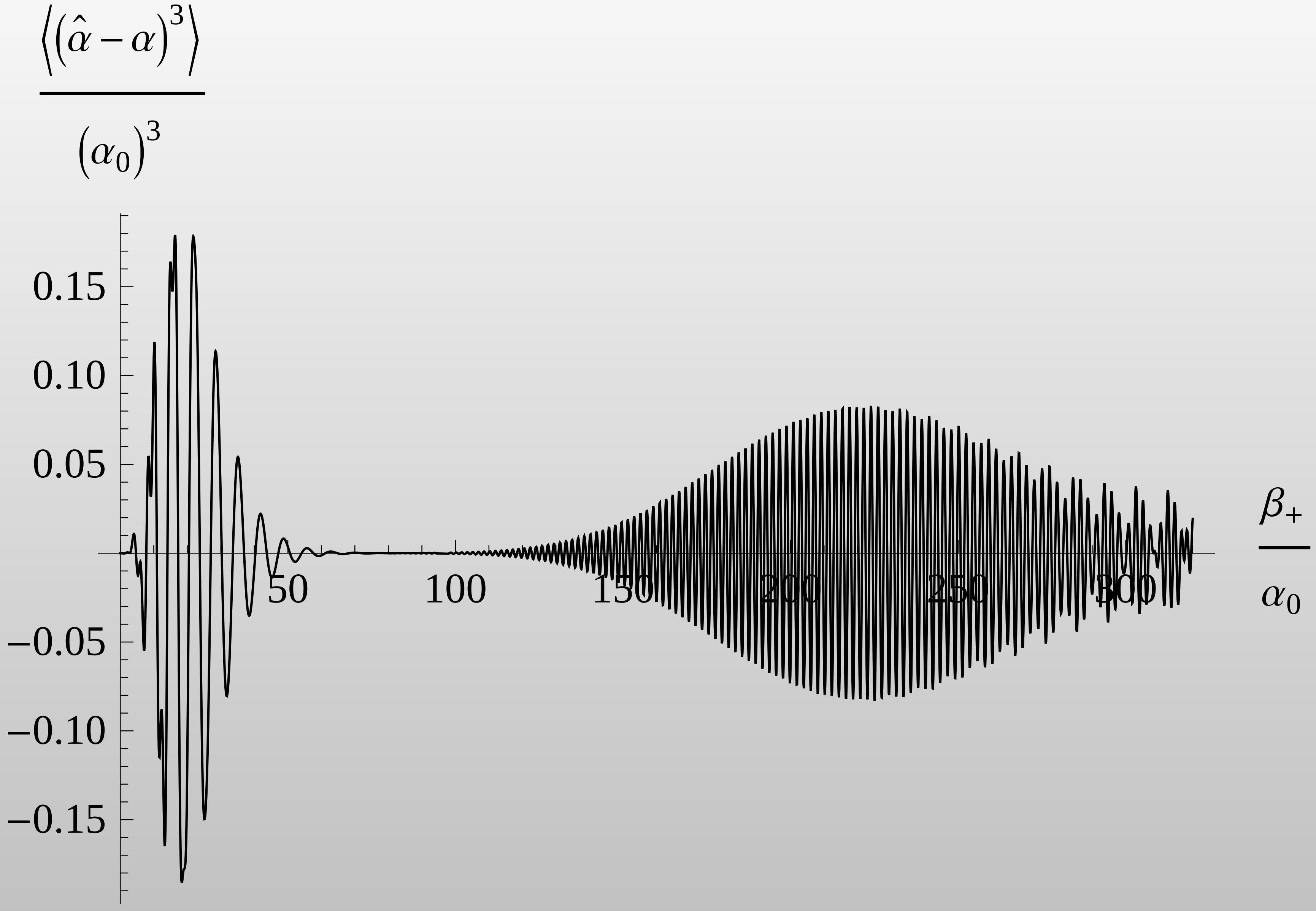


Uncertainty

$$(\Delta\alpha)^2(\Delta p_\alpha)^2 - \Delta(\alpha p_\alpha)^2 \geq \frac{\hbar^2}{4}$$



Skewness



$\frac{\left\langle (\hat{\alpha}-\alpha)^3 \right\rangle}{(\alpha_0)^3}$
0.15
0.10
0.05
−0.05
−0.10
−0.15
50
100
150
200
250
300
$\frac{\beta_+}{\alpha_0}$

Deparameterized quantization

- → No gauge choices, but frame fixed before quantization.
- → Square root.
- → Initial coherent states very special.
  Can change dramatically when evolved.

  Gaussian: Probe 1-dimensional freedom in
  infinite-dimensional state space.

Effective canonical dynamics (ECD)

New perspective:

- $\longrightarrow$ Map quantum system to “classical-type” system: dynamical systems methods.
- $\longrightarrow$ Deals with full quantum system; implements all requirements (constraints, physical inner product).
- $\longrightarrow$ Equations normally hopeless, but tractable approximation methods exist.
- $\longrightarrow$ Reproduces low-energy effective action when applied to anharmonic oscillator (including “mass renormalization”).

[MB, A. Skirzewski: math-ph/0511043, hep-th/0606232]

Setup

Describe (density) state by its moments:

$$\Delta(O_1 \dots O_n) := \left\langle \prod_{i=1}^{n} (\hat{O}_i - \langle \hat{O}_i \rangle) \right\rangle_{\mathrm{Weyl}}$$

for $O_i \in \{\alpha, p_\alpha, \beta_+, p_+\}$.

Phase space from

$$\{\langle \hat{A} \rangle, \langle \hat{B} \rangle\} := \frac{\langle [\hat{A}, \hat{B}] \rangle}{i\hbar}$$

When restricted to maximal order n of moments, (degenerate) Poisson structures arise.

Poisson brackets

Results in $\{\langle\hat{\alpha}\rangle, \langle\hat{p}_\alpha\rangle\} = 1$, $\{\langle\hat{\alpha}\rangle, \Delta(\cdots)\} = 0 = \{\langle\hat{p}_\alpha\rangle, \Delta(\cdots)\}$,

$$\begin{aligned}
\{\Delta(\alpha^a p_\alpha^b), \Delta(\alpha^c p_\alpha^d)\} &= a!b!c!d! \sum_{r=0}^{\infty} (\hbar^2/4)^r \sum_{s=0}^{2r-a-b} (-1)^{r+s} \frac{1}{s!(2r+1-s)!} \\
&\quad \times \frac{\Delta(\alpha^{a+b+c-2r-1} p_\alpha^{a+b+c+d-4r-2})}{(c-2r-1+s)!(a-s)!(d-s)!(b-2r-1+s)!} \\
&\quad + ad\Delta(\alpha^{a-1} p_\alpha^b)\Delta(\alpha^c p_\alpha^{d-1}) - bc\Delta(\alpha^a p_\alpha^{b-1})\Delta(\alpha^{c-1} p_\alpha^d)
\end{aligned}$$

Example: second order moments

$$\begin{aligned}
\{(\Delta\alpha)^2, \Delta(\alpha p_\alpha)\} &= 2(\Delta\alpha)^2 \\
\{(\Delta\alpha)^2, (\Delta p_\alpha)^2\} &= 4\Delta(\alpha p_\alpha) \\
\{\Delta(\alpha p_\alpha), (\Delta p_\alpha)^2\} &= 2(\Delta p_\alpha)^2
\end{aligned}$$

Effective constraints

Physical states $\langle\hat{C}\rangle = 0$, but also

$$C_{\mathrm{pol}} := \langle(\widehat{\mathrm{pol}} - \langle\widehat{\mathrm{pol}}\rangle)\hat{C}\rangle = 0$$

for polynomials $\widehat{\mathrm{pol}}$ in basic operators. Infinitely many first-class constraints for infinitely many variables. (Ordering!)

Semiclassical states: $\Delta(\alpha^a p_\alpha^b) \sim \hbar^{(a+b)/2}$*.*

Gaussian: $\Delta(\alpha^a p_\alpha^b) = \frac{\sigma^{b-a} a! b!}{2^{(3a+b)/2}(a/2)!(b/2)!} \hbar^{(a+b)/2}$ if a, b even

Fixed order of $\hbar$:
Finitely many equations and variables; solve order by order.

“Relativistic” harmonic oscillator

Second order in moments:

$$
\begin{aligned}
C &= \langle\hat{p}_+\rangle^2 - \langle\hat{p}_\alpha\rangle^2 - \langle\hat{\alpha}\rangle^2 + (\Delta p_+)^2 - (\Delta p_\alpha)^2 - (\Delta\alpha)^2 \\
C_{\beta_+} &= 2\langle\hat{p}_+\rangle\Delta(\beta_+ p_+) + i\hbar\langle\hat{p}_+\rangle - 2\langle\hat{p}_\alpha\rangle\Delta(\beta_+ p_\alpha) - 2\langle\hat{\alpha}\rangle\Delta(\beta_+\alpha) \\
C_{p_+} &= 2\langle\hat{p}_+\rangle(\Delta p_+)^2 - 2\langle\hat{p}_\alpha\rangle\Delta(p_+ p_\alpha) - 2\langle\hat{\alpha}\rangle\Delta(p_+\alpha) \\
C_\alpha &= 2\langle\hat{p}_+\rangle\Delta(p_+\alpha) - 2\langle\hat{p}_\alpha\rangle\Delta(\alpha p_\alpha) - i\hbar\langle\hat{p}_\alpha\rangle - 2\langle\hat{\alpha}\rangle(\Delta\alpha)^2 \\
C_{p_\alpha} &= 2\langle\hat{p}_+\rangle\Delta(p_+ p_\alpha) - 2\langle\hat{p}_\alpha\rangle(\Delta p_\alpha)^2 - 2\langle\hat{\alpha}\rangle\Delta(\alpha p_\alpha) + i\hbar\langle\hat{\alpha}\rangle
\end{aligned}
$$

Some kinematical moments must be complex.

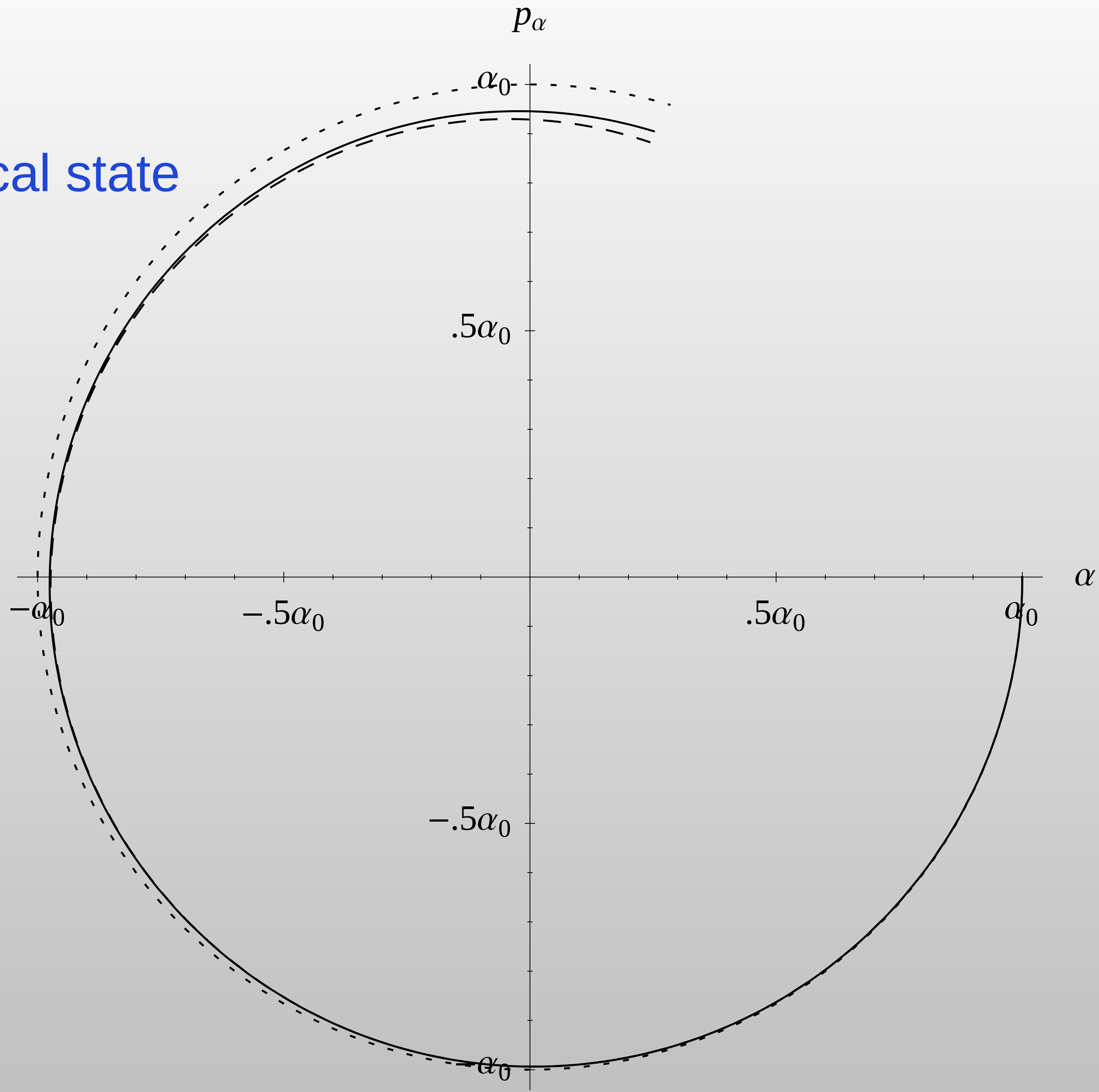
Deparameterized to $\langle\hat{p}_+\rangle = \pm H_Q$ with quantum Hamiltonian

$$
H_Q = \sqrt{\langle\hat{p}_\alpha\rangle^2 + \langle\hat{\alpha}\rangle^2}\left(1 + \frac{\langle\hat{\alpha}\rangle^2(\Delta p_\alpha)^2 - 2\langle\hat{\alpha}\rangle\langle\hat{p}_\alpha\rangle\Delta(\alpha p_\alpha) + \langle\hat{p}_\alpha\rangle^2(\Delta\alpha)^2}{2(\langle\hat{p}_\alpha\rangle^2 + \langle\hat{\alpha}\rangle^2)^2}\right)
$$

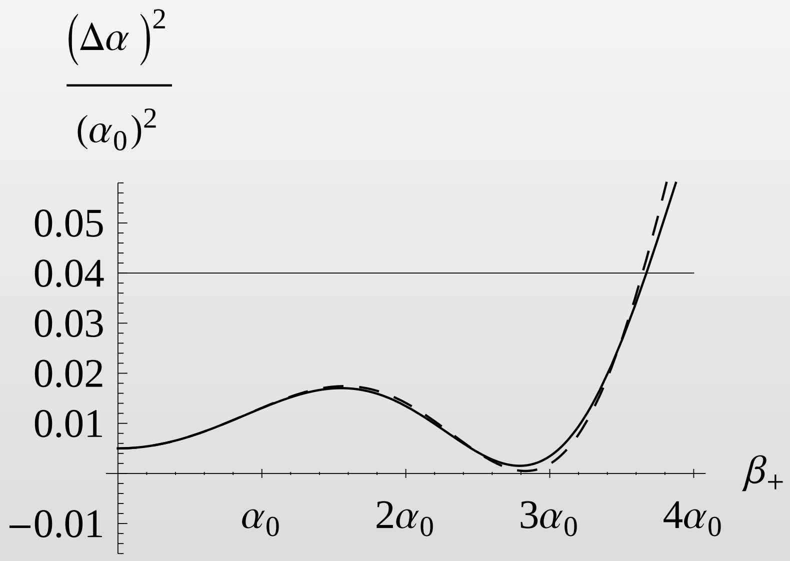
Reality imposed on Dirac observables $\langle\hat{\alpha}\rangle(\beta_+)$, $\langle\hat{p}_\alpha\rangle(\beta_+)$, $\Delta(\cdots)(\beta_+)$.

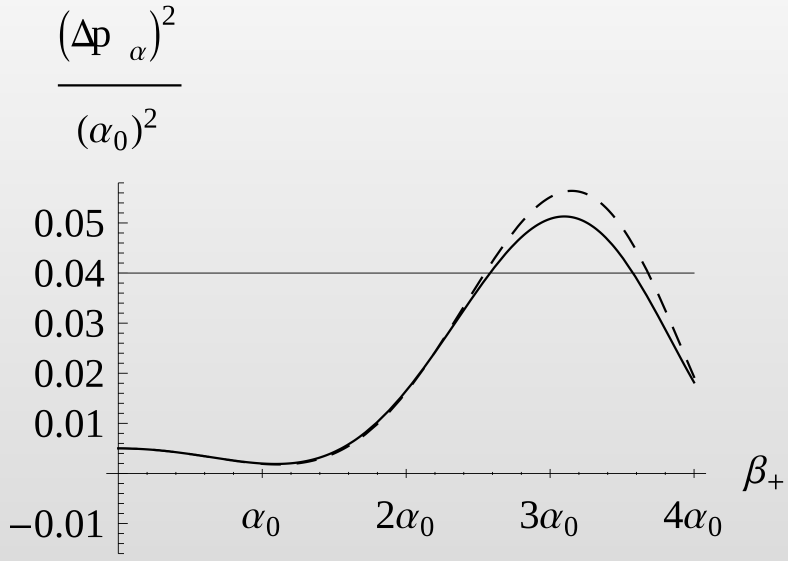
Comparison

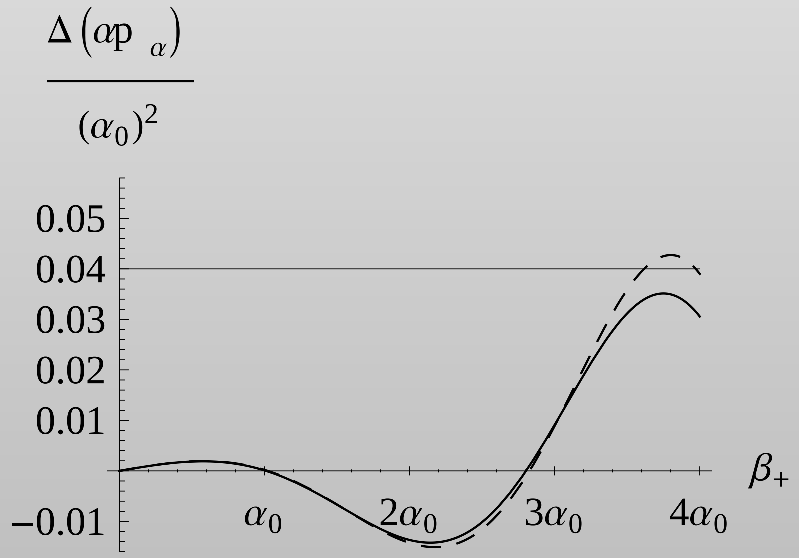
dotted: classical
solid: semiclassical state
dashed: effective



Second order moments







Advantages

→ Deal directly with relevant observables.

→ Minimal restrictions on states; preserve genericness.

→ If deparameterized, square roots only of numbers, not operators.

$$C = \langle \hat{p}_+ \rangle^2 - \langle \hat{p}_\alpha \rangle^2 - \langle \hat{\alpha} \rangle^2 + (\Delta p_+)^2 - (\Delta p_\alpha)^2 - (\Delta \alpha)^2 = 0$$

→ No need to choose time before quantization:
gauge and frame independence.

(Especially useful for cosmological perturbations.)

→ Local internal time possible (but tricky).

[Work in progress with P. Höhn, A. Tsobanjan; see also Rovelli 1990]

Application 1: Group coherent states

Number of constraints can often be reduced when spectral properties of $\hat{C}$ known.

→ Example: $\langle\psi|\hat{C}^2|\psi\rangle = 0$ implies $\hat{C}|\psi\rangle = 0$ if zero in discrete spectrum of $\hat{C}$. [A. Corichi: arXiv:0801.1119]

(Discrete: Have to get only close to zero, not exactly to zero.)

→ Group coherent states: non-canonical algebra of basic operators. Finite-dimensional quantum phase space.

Small number of constraints may suffice;
see simplicity constraint. [F. Conradi's talk]

Application 2: Quantum cosmology

Deparameterizable if free, massless scalar $\phi = \beta_+$.

- $\rightarrow$ Wheeler–DeWitt: $H = |qp|$ quadratic
- $\rightarrow$ Loop quantum cosmology: $H = |\mathrm{Im} J|$ linear in terms of non-canonical basic variable J (holonomy-like)

Harmonic cosmology: Equations for moments decouple; exact effective system; no quantum back-reaction; dynamical coherent states.

Loop effects: Simple effective dynamics in proper time

$$\left(\frac{\dot{\langle\hat{a}\rangle}}{\langle\hat{a}\rangle}\right)^2 = \frac{8\pi G}{3}\left(\rho\left(1-\frac{\rho}{\rho_{\mathrm{crit}}}\right)\right)$$

as seen numerically for this model. [A. Ashtekar, T. Pawlowski, P. Singh]

As special as harmonic oscillator, but basis for perturbations.

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Application 3: Ground states

Ground state of harmonic oscillator: Hamiltonian

$$H_Q = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 + \frac{1}{2m}(\Delta p)^2 + \frac{1}{2}m\omega^2(\Delta q)^2$$

Implies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\Delta q)^2 &= \frac{2}{m}\Delta(qp) \\
\frac{\mathrm{d}}{\mathrm{d}t}\Delta(qp) &= -m\omega^2(\Delta q)^2 + \frac{1}{m}(\Delta p)^2 \\
\frac{\mathrm{d}}{\mathrm{d}t}(\Delta p)^2 &= -m\omega^2\Delta(qp)
\end{aligned}$$

Stationary: $\Delta(qp) = 0$, thus $\Delta p = m\omega\Delta q$.

Minimal uncertainty: $\Delta q = \sqrt{\hbar/2m\omega}$ fixes spread.

Boundary Hamiltonian?

Try same procedure to fix shape and parameters of boundary states for graviton propagator.

Brown–York quasilocal energy: $\int(K - K_0)$, integrated trace of extrinsic curvature (minus background subtraction).

Quadratic in connection variables, of form $H = qp$: Harmonic cosmology per point, just like field theory as harmonic oscillators per point/mode.

But: no ground state in harmonic cosmology, except zero volume.

[Work in (really slow) progress; discussions with S. Speziale]

Conclusions

Effective canonical dynamics provides perturbation techniques for physical state properties.

Evaluates relevant properties, minimal requirements, generic class of states. Amenable to dynamical systems analysis.

Effective constraints: Applies to discrete as well as continuous spectrum. Many subtleties avoided.

Well-adapted to canonical quantum gravity: harmonic systems exist as basis for perturbation expansion.

In progress:
Computational/numerical tools [D. Brizuela], more on constrained systems, quantum field theory, loop quantum gravity.