① $w_H(\lambda) = \det\begin{pmatrix} 3-\lambda & 2-i \\ 2+i & 7-\lambda \end{pmatrix} = (3-\lambda)(7-\lambda) - (4+1) = 21 - 10\lambda + \lambda^2 - 5 = \lambda^2 - 10\lambda + 16 = (\lambda-2)(\lambda-8)$

$\Delta = 100 - 64 = 36$

a) $Sp\, H = \{2, 8\}$

- $\lambda = 2$

$\begin{pmatrix} 1 & 2-i \\ 2+i & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} v_1 + (2-i)v_2 = 0 & \longrightarrow v_1 = -v_2(2-i) \longrightarrow v_{\lambda=2} = v_2\begin{pmatrix} i-2 \\ 1 \end{pmatrix} \\ (2+i)v_1 + 5v_2 = 0 & \longrightarrow -(2+i)(2-i)v_2 + 5v_2 = 0 \longrightarrow 0 = 0 \end{cases}$

- $\lambda = 8$

$\begin{pmatrix} -5 & 2-i \\ 2+i & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -5v_1 + (2-i)v_2 = 0 \\ (2+i)v_1 - v_2 = 0 \end{cases} \Rightarrow \begin{cases} -5v_1 + (2-i)(2+i)v_1 = 0 \longrightarrow 0 = 0 \\ v_2 = (2+i)v_1 \longrightarrow v_{\lambda=8} = v_1\begin{pmatrix} 1 \\ i+2 \end{pmatrix} \end{cases}$

b) wielomian aproksymujący $p(\lambda) = \alpha_0 + \alpha_1\lambda$

$\begin{cases} e^{i\pi} = \alpha_0 + 2\alpha_1 & , e^{i\pi} = -1 \\ e^{4i\pi} = \alpha_0 + 8\alpha_1 & , e^{4i\pi} = 1 \end{cases} \Rightarrow \begin{cases} \alpha_0 + 2\alpha_1 = -1 \\ \alpha_0 + 8\alpha_1 = 1 \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{1}{3} \\ \alpha_0 = -\frac{5}{3} \end{cases} \Rightarrow p(\lambda) = -\frac{5}{3} + \frac{1}{3}\cdot\lambda \Rightarrow e^{\frac{2\pi}{2}H} = -\frac{5}{3}\cdot\mathbb{1} + \frac{1}{3}H$

$e^{\frac{i\pi}{2}H} = \begin{pmatrix} -\frac{5}{3} & 0 \\ 0 & -\frac{5}{3} \end{pmatrix} + \begin{pmatrix} 1 & \frac{1}{2}(2-i) \\ \frac{1}{3}(2+i) & \frac{7}{3} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3}(2-i) \\ \frac{1}{3}(2+i) & \frac{2}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -2 & 2-i \\ 2+i & 2 \end{pmatrix}$

② $\langle p|q\rangle = \int_0^\infty p(x)q(x)e^{-x}dx$, $\int_0^\infty x^n e^{-x}dx = n!$, $V = \langle 1, x, x^2\rangle$

$\langle 1|1\rangle = 1 \longrightarrow e_0 = 1$

$e_1 = \dfrac{x - \langle 1|x\rangle 1}{\|x - \langle 1|x\rangle 1\|} = \dfrac{x-1}{\|x-1\|}$, $\|x-1\|^2 = \int_0^1 (x-1)^2 e^{-x}dx = \int_0^1 (x^2-2x+1)e^{-x}dx = 2-2+1 = 1 \Rightarrow e_1 = x-1$

$e_2 = \dfrac{x^2 - \langle 1|x^2\rangle 1 - \langle x-1|x^2\rangle(x-1)}{\|x^2 - \langle 1|x^2\rangle 1 - \langle x-1|x^2\rangle(x-1)\|} = \dfrac{x^2 - 4(x-1) - 2}{\|x^2-4(x-1)-2\|} = \dfrac{x^2-4x+2}{\|x^2-4x+2\|}$, $\|x^2-4x+2\|^2 = \int e^{-x}(x^2-4x+2)^2 dx =$

$= \int e^{-x}(x^4 - 8x^3 + 16x^2 + 4 + 4(x^2-4x))$

$e_2 = \frac{1}{2}(x^2-4x+2)$

$= 4! - 8\cdot 3! + 16\cdot 2! + 4 + 4\cdot 2 - 16 = 52 - 48 = 4$

$e = (e_0, e_1, e_2) \longleftarrow$ baza ortonormalna w V

③ $Tv = \begin{pmatrix} x_1 - 3x_2 + 4x_3 \\ -2x_1 + 5x_2 - x_3 \end{pmatrix}$ $\begin{cases} Te_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ Te_2 = \begin{pmatrix} -3 \\ 5 \end{pmatrix} \\ Te_3 = \begin{pmatrix} 4 \\ -1 \end{pmatrix} \end{cases} \Rightarrow (T)^f_e = \begin{pmatrix} 1 & -3 & 4 \\ -2 & 5 & -1 \end{pmatrix}$, $\operatorname{Ker} T = \{v \in \mathbb{R}^3 : Tv = 0\}$

$Tv = 0 \longrightarrow \begin{pmatrix} 1 & -3 & 4 \\ -2 & 5 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Rightarrow$

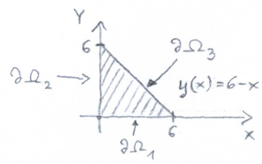
$\Rightarrow \begin{cases} v_1 - 3v_2 + 4v_3 = 0 \\ -2v_1 + 5v_2 - v_3 = 0 \longrightarrow v_3 = -2v_1 + 5v_2 \end{cases} \longrightarrow -7v_1 + 17v_2 = 0 \longrightarrow v_1 = \frac{17}{7}v_2 \longrightarrow v_3 = \frac{1}{7}v_2 \Rightarrow \operatorname{Ker} T = \left\langle \begin{pmatrix} \frac{17}{7} \\ 1 \\ \frac{1}{7} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 17 \\ 7 \\ 1 \end{pmatrix} \right\rangle$

④ a) $\lim_{(x,y)\to(0,0)} y^3\cos\left(\frac{1}{x^2+y^2}\right) = 0$ bo dla dowolnych ciągów $(x_n, y_n) \longrightarrow (0,0)$: $0 \leqslant |y_n|^3 \left|\cos\left(\frac{1}{x_n^2+y_n^2}\right)\right| \leqslant |y_n|^3$ więc z tw. o trzech ciągach $f(x_n, y_n) \longrightarrow 0$.

b) $\nabla_e f(0,0) = \lim_{t\to 0^+} \frac{1}{t}\left(f(0+t\cos\alpha, 0+t\sin\alpha) - f(0,0)\right) = \lim_{t\to 0^+} \dfrac{f(t\cos\alpha, t\sin\alpha)}{t} = \lim_{t\to 0^+} \dfrac{t^3\sin^3\alpha\cos\left(\frac{1}{t^2\cos^2\alpha + t^2\sin^2\alpha}\right)}{t} =$

$= \lim_{t\to 0^+} t^2\sin^3\alpha\cos\left(\frac{1}{t^2}\right) = 0$

⑤ $f(x,y) = x^2y(4-x-y)$, $x \geqslant 0$, $y \geqslant 0$, $x+y \leqslant 6$



na $\partial\Omega_1$: $f(x,0) = 0$

na $\partial\Omega_2$: $f(0,y) = 0$

na $\partial\Omega_3$: $g(x) = f(x, y(x)) = x^2(6-x)(-2)$

$\begin{cases} g'(x) = 6x^2 - 24x = 6x(x-4) \\ g''(x) = 12x - 24 \end{cases}$

$g'(x) = 0 \Leftrightarrow x = 0 \lor x = 4$

$g''(4) = 24 \longrightarrow$ minimum $\longrightarrow f(4,2) = -64$

$g''(0) = -24 \longrightarrow$ maksimum $\longrightarrow f(0,6) = 0$

$\left.\begin{aligned} \partial_x f &= 2xy(4-x-y) - x^2y = xy[2(4-x-y) - x] = 0 \\ \partial_y f &= x^2(4-x-y) - x^2y = x^2[4-x-2y] = 0 \end{aligned}\right\} \Rightarrow \begin{cases} 3x+2y = 8 \\ x+2y = 4 \end{cases}$ bo ekstr. lokalne $\Leftrightarrow \begin{cases} x > 0 \\ y > 0 \\ x+y < 6 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases} \longrightarrow (2,1)$ – punkt podejrzany o ekstremum

$\partial_x^2 f = 2y(4-x-y) - 2xy - 2xy$

$\partial_y^2 f = -x^2 - x^2 = -2x^2$

$\partial_x\partial_y f = 2x(4-x-y) - x^2 - 2xy$

$\Rightarrow M = \begin{pmatrix} \partial_x^2 f & \partial_x\partial_y f \\ \partial_x\partial_y f & \partial_y^2 f \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ -4 & -8 \end{pmatrix}$

$\det M = 32 > 0$

$\begin{cases} f(2,1) = 4(4-2-1) = 4 \\ f(0,0) = 0 \\ f(6,0) = 0 \\ f(0,6) = 0 \\ f(4,2) = -64 \end{cases} \Rightarrow \begin{cases} \inf_\Omega f(x,y) = -64 & \text{min. wartość} \\ \sup_\Omega f(x,y) = 4 & \text{max. wartość} \end{cases}$