Conformal submersions with totally umbilical fibers

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Plan

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Submersion

Definition

Let M, B be manifolds (smooth, connected), $\dim M \geq \dim B$.
A mapping

$$\pi : M \xrightarrow{onto} B$$

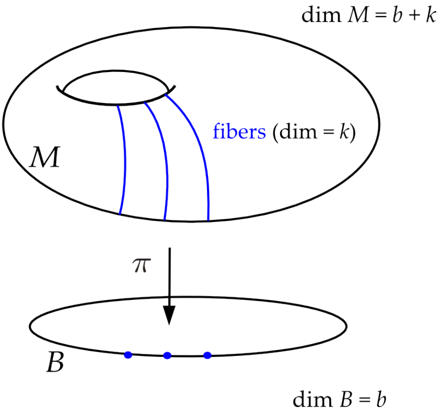
is called submersion , when:

1. π is smooth
2. $\text{rank of } \pi = \dim B$ at every point of M

Fibers

Let $\pi : M \to B$ be a submersion.

- For $q \in B$ the set $\pi^{-1}\{q\}$ is called a fiber of submersion π over q.
- Fibers of submersion π are submanifolds of M.



Vertical vectors

Let M be a Riemannian manifold.
Let $\pi : M \to B$ be a submersion.

Definition

Vectors $V \in T_pM$ tangent to the fibers of submersion, i.e., such that

$$\pi_{\star,p} V = 0$$

are called vertical at a point p

- Let $\mathcal{V}_p$ denote the linear subspace of vertical vectors at $p \in M$
- Distribution $p \mapsto \mathcal{V}_p$ is called vertical
- Let $\mathcal{V}$ denote orthogonal projection onto the vertical distribution

Horizontal vectors

Let M be a Riemannian manifold.
Let $\pi : M \to B$ be a submersion.

Definition

Vectors from orthogonal complement of $\mathcal{V}_p$ are called horizontal at p.

- Let $\mathcal{H}_p$ denote the linear subspace of horizontal vectors at $p \in M$
- Distribution $p \mapsto \mathcal{H}_p$ is called horizontal
- Let $\mathcal{H}$ denote orthogonal projection onto the horizontal distribution

Conformal submersion

Definition

Submersion $\pi : (M, g) \to (B, g_B)$ is called conformal if there exists a function $f \in C^\infty(M)$ such that

$$\forall_{p \in M} \forall_{X,Y \in \mathcal{H}_p} \quad e^{2f(p)} g(X, Y) = g_B(\pi_{\star,p} X, \pi_{\star,p} Y)$$

- We call f the dilation of a conformal submersion.
- Note that $\pi : (M, g \cdot e^{-2f}) \to (B, g_B)$ is then a Riemannian submersion , i.e.:

$$\forall_{X,Y \in \mathcal{H}_p} \quad (g \cdot e^{-2f})(X, Y) = g_B(\pi_{\star,p} X, \pi_{\star,p} Y)$$

- Also, for all horizontal X, Y and vertical V, we have

$$(\mathcal{L}_V g)(X, Y) = 2(Vf) g(X, Y)$$

Extrinsic geometry of fibers

Definition

Fibers of submersion π are called (totally) umbilical if there exists a horizontal vector field H on (M, g) such that

$$\mathcal{H}\nabla_V W = g(V, W)H$$

for all vertical vectors V, W on M.
We call H the mean curvature field of fibers.

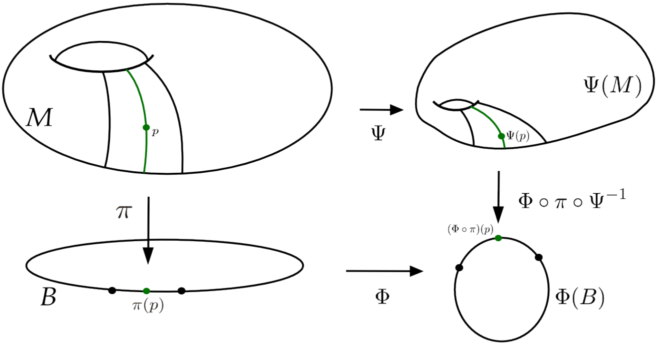
Examples of umbilical submanifolds:

- (conformal) spheres in (conformal) spheres
- curves in any manifold
- intersections of planes and space forms in space forms

Umbilical fibers with $H = 0$ are called totally geodesic .

Conformal maps and umbilical submanifolds

Umbilical submanifolds remain such after transforming the ambient manifold conformally



If:

- π is a conformal submersion with umbilical fibers
- Ψ, Φ are conformal diffeomorphisms

then $\Phi \circ \pi \circ \Psi^{-1}$ is a conformal submersion with umbilical fibers

Conformal submersions with umbilical fibers - examples

1. If $\pi : (M, g_M) \to (B, g_B)$ is a Riemannian submersion with umbilical fibers , then:
   - $\pi : (M, e^{2h} \cdot g_M) \to (B, g_B)$ for a smooth function h
   - $\pi \circ \Phi$ for a conformal diffeomorphism Φ

   are conformal submersions with umbilical fibers.
2. Projection onto any factor of a twisted product $(M \times N, e^{2\alpha} g_M + e^{2\beta} g_N)$ is a conformal submersion with umbilical fibers.
3. For a group G of conformal transformations of a manifold M , projection onto its space of orbits $G \backslash M$ may be a conformal submersion.

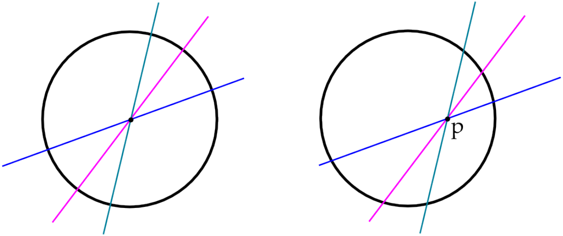
Modifications of Riemannian submersions other than explicit conformal diffeomorphisms usually do not lead to conformal submersions.

Example

Let $\sigma : \left(\mathbb{R}^{2n} \setminus \{0\}\right) \to \mathbb{C}P^n$ be the canonical projection and denote by S_p^{n-1} the $(n-1)$-dimensional unit sphere with the center at p.
The Hopf fibration $\pi_0 = \sigma|S_0^{n+1}$ is a Riemannian submersion with totally geodesic fibers. On the other hand,

$$\pi_p = \sigma|S_p^{n+1} \text{ for } 0 < \|p\| < 1$$

is not a conformal submersion.



Tensor A

Let $\pi : (M, \langle\cdot,\cdot\rangle) \to (B, \langle\cdot,\cdot\rangle_B)$ be a conformal submersion with umbilical fibers of dilation f , i.e.:

$$\langle X, X\rangle = \|X\|^2 = e^{2f}\|\pi_\star X\|_B^2 = e^{2f}\langle\pi_\star X, \pi_\star X\rangle_B^2$$

for all horizontal X.
We define a tensor field

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F$$

For horizontal X, Y we have:

$$A_X Y = \frac{1}{2}\mathcal{V}[X, Y] - \langle X, Y\rangle\mathcal{V}\,\mathrm{grad}\, f$$

Sectional curvatures

For horizontal X, Y we have:

$$A_X Y = \frac{1}{2}\mathcal{V}[X, Y] - \langle X, Y\rangle \mathcal{V}\,\mathrm{grad}\, f$$

- Let $\sec_M, \sec_B$ denote sectional curvatures on M, B, resp. Then for X, Y - horizontal and U - vertical , we have:

$$\begin{aligned} \sec_M(X, U) &= -\langle \nabla_U \mathcal{V}\,\mathrm{grad}\, f, U\rangle + \langle A_X U, A_X U\rangle \\ &\quad + \langle \nabla_X H, X\rangle - \langle H, X\rangle^2 - 2\langle \mathcal{V}\,\mathrm{grad}\, f, U\rangle^2 \end{aligned}$$

  (where H is the mean curvature field of fibers)
- Also, when X, Y are horizontal and orthonormal :

$$\begin{aligned} \sec_M(X, Y) &= e^{-2f} \sec_B(\pi_\star X, \pi_\star Y) - \mathrm{hess}\, f(X, X) - \mathrm{hess}\, f(Y, Y) \\ &\quad + \|\mathrm{grad}\, f\|^2 - (Xf)^2 - (Yf)^2 - 3\langle A_X Y, A_X Y\rangle \end{aligned}$$

Notation

Let

- $\{X_1, \ldots, X_b\}$ - an orthonormal base of horizontal distribution
- $\{U_1, \ldots, U_k\}$ - an orthonormal base of vertical distribution
- $\{Y_1, \ldots, Y_b\}$ - an orthonormal base of tangent space of B

We denote

mixed scalar curvature of M by
$$K_{mix} = \sum_{i=1}^{k} \sum_{j=1}^{b} \mathrm{sec}_M(U_i, X_j)$$

horizontal scalar curvature of M by
$$K_{\mathcal{H}} = \sum_{i,j=1}^{b} \mathrm{sec}_M(X_i, X_j)$$

scalar curvature of B by
$$K_B = \sum_{i,j=1}^{b} \mathrm{sec}_B(Y_i, Y_j)$$

laplacian of f along fibers of π by
$$\Delta^{\mathcal{V}} f = \sum_{i=1}^{k} \langle \nabla_{U_i} \mathcal{V} \, \mathrm{grad}\, f, U_i \rangle$$

while Δ denotes the laplacian on M.

Mixed scalar curvature formula

Let $\pi : M \to B$ be a conformal submersion with umbilical fibers. Then:

$$\begin{aligned} K_{mix} &= -b\Delta^{\mathcal{V}} f + \sum_{i,j=1}^{b} \|A_{X_j} X_i\|^2 + k \operatorname{div} H \\ &\quad + k(k-1)\|H\|^2 - 2b\|\mathcal{V} \operatorname{grad} f\|^2 \end{aligned}$$

(where $b = \dim B$, $k = \dim M - \dim B$)

Horizontal scalar curvature formulae

Let $\pi : M \to B$ be a conformal submersion with umbilical fibers. Then:

$$\begin{aligned} K_{\mathcal{H}} &= e^{-2f}(K_B \circ \pi) - 3\sum_{i,j=1}^{b} \|A_{X_i}X_j\|^2 \\ &\quad -2(b-1)\Delta f + 2(b-1)\Delta^{\mathcal{V}} f \\ &\quad -2(b-1)k \cdot (Hf) \\ &\quad +(b-1)(b-2)\|\mathcal{H}\,\mathrm{grad}\, f\|^2 + b(b+2)\|\mathcal{V}\,\mathrm{grad}\, f\|^2, \end{aligned}$$

Also, from the mixed scalar curvature formula we can obtain

$$\begin{aligned} K_{\mathcal{H}} + 3K_{mix} - e^{-2f}(K_B \circ \pi) &= -2(b-1)\Delta f - (b+2)\Delta^{\mathcal{V}} f + 3k\,\mathrm{div}\, H \\ &\quad +3k(k-1)\|H\|^2 - 2(b-1)k\langle H, \mathrm{grad}\, f\rangle \\ &\quad +(b-1)(b-2)\|\mathcal{H}\,\mathrm{grad}\, f\|^2 \\ &\quad +b(b-4)\|\mathcal{V}\,\mathrm{grad}\, f\|^2 \end{aligned}$$

Integration

Lemma

Let M and B be compact and oriented , and let $\pi : M \to B$ be a conformal submersion.
Let Ω_M and Ω_B denote Riemannian volume forms of M and B, resp. Then

$$\Omega_M(x) = e^{bf(x)}\Omega_F(x) \wedge (\pi^\star \Omega_B)(x),$$

where Ω_F restricted to any fiber of π is the Riemannian volume form of that fiber.

Hence for any smooth function ϕ on M

$$\int_M \phi(x)\,\Omega_M(x) = \int_B \left(\int_{\pi^{-1}(y)} e^{bf(x)}\phi(x)\,\Omega_F(x) \right)\,\Omega_B(y)$$

In particular, if fibers of M are closed manifolds

$$\int_M \Delta^{\mathcal{V}} f\,\Omega_M = -b\int_M \|\mathcal{V}\,\mathrm{grad}\, f(x)\|^2\,\Omega_M.$$

Integral formula for mixed scalar curvature

Let $\pi : M \to B$ be a conformal submersion with umbilical fibers from a closed, oriented manifold M. Then

$$\begin{aligned}\int_M K_{mix}\,\Omega_M &= b(b-1)\int_M \|\mathcal{V}\,\mathrm{grad}\, f\|^2\,\Omega_M \\ &\quad + \int_M \sum_{i=1}^{b}\sum_{\substack{j=1\\ j\neq i}}^{b} \|\mathcal{V}[X_i, X_j]\|^2\,\Omega_M \\ &\quad + k(k-1)\int_M \|H\|^2\,\Omega_M\end{aligned}$$

Hence

$$\int_M K_{mix}\,\Omega_M \geq 0$$

Integral formula for horizontal scalar curvature

Proposition

Let $\pi : M \to B$ be a conformal submersion with totally umbilical fibers. Assume that M is closed, oriented and one of the following holds:

- $\dim B = 1$
- $\frac{2}{3}(2 \dim B + 1) \leq \dim M \leq 2 \dim B - 2$
- *fibers of π are totally geodesic*

Then

$$\int_M \left(K_{\mathcal{H}} + 3K_{mix} - e^{-2f}(K_B \circ \pi)\right) \Omega_M \geq 0.$$

Note that the lowest pair of values satisfing the second condition is

$$\dim M = 6, \dim B = 4.$$

Signs of curvature

Proposition

Let M be a closed, oriented manifold of non-positive sectional curvatures and let B be a non-flat manifold of non-negative scalar curvature . Then there exist no conformal submersions with totally umbilical fibers from M onto B.

In particular, there exist no Riemannian submersions with totally geodesic fibers from M onto B (Escobales, 1975).

Proof

Since M has non-positive sectional curvature and $\int_M K_{mix} \geq 0$ we have $K_{mix} = 0$ and $\langle R(X, V)V, X \rangle = 0$ for all horizontal X and vertical V.
From the formula

$$\begin{aligned} \int_M K_{mix}\, \Omega_M &= b(b-1) \int_M \|\mathcal{V}\operatorname{grad} f\|^2\, \Omega_M \\ &\quad + \int_M \sum_{i=1}^{b} \sum_{\substack{j=1 \\ j\neq i}}^{b} \|\mathcal{V}[X_i, X_j]\|^2\, \Omega_M \\ &\quad + k(k-1) \int_M \|H\|^2\, \Omega_M \end{aligned}$$

we have $\mathcal{V}\operatorname{grad} f = 0$ and $\mathcal{V}[X, Y] = 0$ for all horizontal X, Y.
It follows that $A = 0$.

Proof - continued

We have for any vertical V:

$$0 = \langle R(V,H)H, V\rangle - \langle A_H V, A_H V\rangle = \langle \nabla_H H, H\rangle - \langle H, H\rangle^2$$

Hence at maximum of $\langle H, H\rangle$ we have $0 = H\langle H, H\rangle = \langle H, H\rangle^2$ and it follows that $H = 0$ everywhere on M.

Proof - continued

Horizontal scalar curvature equation yields:

$$-2(b-1)\Delta f = K_{\mathcal{H}} + 3K_{mix} - e^{-2f}(K_B \circ \pi) - (b-1)(b-2)\|\mathcal{H}\,\mathrm{grad}\, f\|^2 \leq 0$$

Hence $f = const$ (as superharmonic function on closed manifold M) and for some X, Y we have

$$\sec_M(X, Y) = e^{-2f} \sec_B(\pi_\star X, \pi_\star Y) > 0$$

contrary to the assumption $\sec_M \leq 0$.

Conformal submersion with totally geodesic fibers...

Proposition

Let $\pi : M \to B$ be a conformal submersion with closed, connected, totally geodesic fibers .
Then on every fiber there exists a point at which

$$K_{mix} = \sum_{i,j=1}^{b} \|\mathcal{V}[X_i, X_j]\|^2$$

Also, if at all points of some fiber we have $K_{mix} \leq 0$, then in fact on that fiber: $K_{mix} = 0, A = 0$ and $f = const$.

Proof.

$$K_{mix} - \sum_{i,j=1}^{b} \|\mathcal{V}[X_i, X_j]\|^2 = -b\Delta^{\mathcal{V}} f - b\|\mathcal{V}\,\mathrm{grad}\, f\|^2 = -be^{-f}\Delta^{\mathcal{V}} e^f$$

...onto a 2-dimensional manifold

Proposition

Let $\pi : M \to B$ *be a conformal submersion with totally geodesic fibers .*
Assume that M *is closed , and* $\dim B = 2$.
Then, if $\int_B K_B > 0$,

$$\min\{volume\ of\ fiber\} \leq \frac{\int_M (K_{\mathcal{H}} + 3K_{mix})}{\int_B K_B}$$

and if $\int_B K_B < 0$,

$$\max\{volume\ of\ fiber\} \geq \frac{\int_M (K_{\mathcal{H}} + 3K_{mix})}{\int_B K_B}.$$

Conformal Killing fields...

Definition

We call a vector field V on (M, g) a conformal Killing field , if:

$$\mathcal{L}_V g = 2(Vf)g$$

for some function f on M.

If $\mathcal{L}_V g = 0$, we call V a Killing field.

… with integral curves being geodesics

Proposition

Let (M, g) be a closed manifold of sectional curvature which is either: everywhere non-negative or everywhere non-positive and let

$$\pi : (M, g) \to (B, g_B)$$

be a conformal submersion with fibers being geodesics . Suppose that there exists a nowhere vanishing conformal Killing field V tangent to fibers of π.

Then V is a Killing field and there exists a function ϕ on B such that

$$\pi : (M, g) \to (B, g_B \cdot e^{2\phi})$$

is a Riemannian submersion .

"Example" (Gudmundsson)

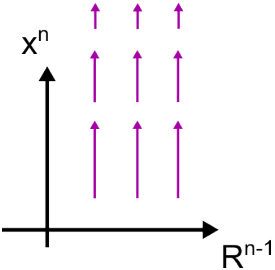
Consider

$$H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, x_n > 0\}$$

with Riemannian metric $\frac{1}{x_n^2} g_n$, where g_n is the Euclidean metric on $\mathbb{R}^n$.

- Vector field $V = \frac{\partial}{\partial x_n}$ is a conformal Killing field and its integral curves are geodesics.
- V is tangent to fibers of conformal submersion $\pi : H^n \to \mathbb{R}^{n-1}$ defined by:

$$\pi(x_1, \ldots, x_{n-1}, x_n) = (x_1, \ldots, x_{n-1})$$



Proof

Submersion π is conformal, so for all horizontal X, Y we have:

$$e^{-2f} g(X, Y) = g_B(\pi_\star X, \pi_\star Y) \circ \pi$$

Because V is vertical, conformal Killing field, we have:

$$(\mathcal{L}_V g) = 2(Vf)g$$

Note that:

$$\begin{aligned} V \frac{e^{2f}}{g(V,V)} &= e^{2f} \frac{2Vf}{g(V,V)} - e^{2f} \frac{Vg(V,V)}{g(V,V)^2} \\ &= e^{2f} \left(\frac{2Vf}{g(V,V)} - \frac{(\mathcal{L}_V g)(V,V)}{g(V,V)^2} \right) \\ &= e^{2f} \left(\frac{2Vf}{g(V,V)} - \frac{g(V,V) \cdot 2Vf}{g(V,V)^2} \right) = 0, \end{aligned}$$

so function $\frac{e^{2f}}{g(V,V)}$ is constant along fibers.

Proof - continued

Denote:

$$e^{2\psi} \circ \pi = \frac{e^{2f}}{g(V,V)}.$$

For all horizontal X, Y:

$$e^{-2f} g(X,Y) = g_B(\pi_\star X, \pi_\star Y) \circ \pi$$

$$\frac{1}{g(V,V)} g(X,Y) = (e^{2\psi} \cdot g_B(\pi_\star X, \pi_\star Y)) \circ \pi$$

Hence $\pi : (M,g) \to (B, e^{2\psi} \cdot g_B)$ is a conformal submersion with dilation $\frac{1}{g(V,V)}$.

Proof - continued

Since integral curves of V are geodesics, for all horizontal X we have:

$$0 = (\mathcal{L}_V g)(X, V) = g(\nabla_X V, V) + g(\nabla_V V, X) = g(\nabla_X V, V)$$

Therefore:

$$\mathcal{H}\,\mathrm{grad}\, g(V, V) = 0.$$

Hence $\pi : (M, g) \to (B, e^{2\psi} \cdot g_B)$ is a conformal submersion with dilation $\frac{1}{g(V,V)}$ such that $\mathcal{H}\,\mathrm{grad}\, \frac{1}{g(V,V)} = 0$.

Such conformal submersions are called horizontally homothetic .

Proposition (Ou, Wilhelm)

Let (M, g) be a closed manifold of non-negative sectional curvature and let $\pi : (M, g) \to (B, g_B)$ be a conformal submersion of dilation e^{2f} such that $\mathcal{H} \operatorname{grad} e^{2f} = 0$. Then there exists a constant c such that

$$\pi : (M, g) \to (B, c \cdot g_B)$$

is a Riemannian submersion .

Proposition (Ou, Wilhelm)

Let (M, g) be a closed manifold of nonpositive sectional curvature and let $\pi : (M, g) \to (B, g_B)$ be a conformal submersion with totally geodesic fibers of dilation e^{2f} such that $\mathcal{H} \operatorname{grad} e^{2f} = 0$. Then there exists a constant c such that

$$\pi : (M, g) \to (B, c \cdot g_B)$$

is a Riemannian submersion .

Conformal submersions with fibers being geodesics

Proposition

Let M be closed and let $\pi : (M, g) \to (B, g_B)$ be a conformal submersion with fibers being geodesics. If one of the following conditions holds on M:

- $A_X Y = -A_Y X$ *for all horizontal* X, Y
- *There exists a nowhere-vanishing vertical conformal Killing field* V *and* M *is of non-negative curvature*
- *There exists vertical field* V *and horizontal field* X *such that* $g(R(X, V)V, X) \leq 0$
- *There exists a horizontal field* X *such that for all horizontal* Y *we have* $\mathcal{V}[X, Y] = 0$ *and* $g(R(X, V)V, X) \geq 0$ *,*

then we have $\mathcal{V}\,\mathrm{grad}\, f = 0$ *and there exists a function* ϕ *on* B *such that*

$$\pi : (M, g) \to (B, g_B \cdot e^{2\phi})$$

is a Riemannian submersion .

A conformal submersion with totally geodesic fibers

Example

Let $(M, g) = (\mathcal{H} \times \mathcal{V}, e^{2f} g_{\mathcal{H}} + e^{2\phi} g_{\mathcal{V}})$, where:

- $\mathcal{H}\, \mathrm{grad}\, \phi = 0$
- $\mathcal{V}\, \mathrm{grad}\, f \neq 0$.

Then $\pi : (M, g) \to (\mathcal{H}, g_{\mathcal{H}})$ is a conformal submersion with totally geodesic fibers .

Hopf fibration

Definition

Let $n \geq 1$ and let:

$$\sigma_{\mathbb{C}} : \mathbb{R}^{2n+2} \setminus \{0\} \to \mathbb{C}P^n,$$
$$\sigma_{\mathbb{H}} : \mathbb{R}^{4n+4} \setminus \{0\} \to \mathbb{H}P^n,$$
$$\sigma_{Ca} : \mathbb{R}^{16} \setminus \{0\} \to CaP^1$$

be canonical projections. We define Hopf fibrations as restrictions to the unit sphere of those projections:

$$\begin{aligned} \pi_1 &= \sigma_{\mathbb{C}}|S^{2n+1}, \\ \pi_2 &= \sigma_{\mathbb{H}}|S^{4n+3}, \\ \pi_3 &= \sigma_{Ca}|S^{15} \end{aligned}$$

Submersions from spheres

Theorem (Gromoll, Grove, Wilking)

Let π be a Riemannian submersion from a round sphere. Then up to isometries of the sphere and the image of submersion, π is one of the following Hopf fibrations :

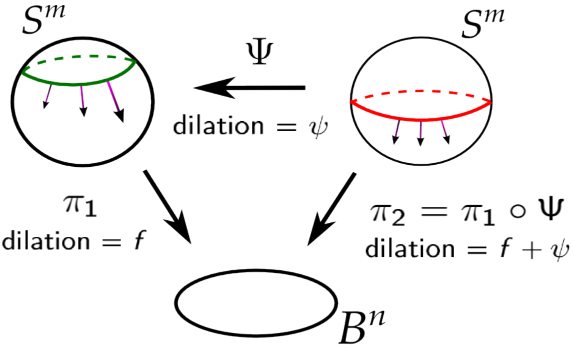
- $\pi_1 : S^{2n+1} \to \mathbb{C}P^n$ *(where* $\mathbb{C}P^1 \equiv S^2(\frac{1}{2})$*)*
- $\pi_2 : S^{4n+3} \to \mathbb{H}P^n$ *(where* $\mathbb{H}P^1 \equiv S^4(\frac{1}{2})$*)*
- $\pi_3 : S^{15} \to CaP^1$ *(where* $\mathbb{H}P^1 \equiv S^8(\frac{1}{2})$*)*

(all projective spaces considered with Fubini-Study metric)

Theorem (Heller)

The only (up to conformal diffeomorphisms) conformal submersion with circular fibers from S^3 is the Hopf fibration .

Conformal submersions from spheres



Dilation ψ of a conformal diffeomorphism Ψ is defined by the formula:

$$e^{-2\psi} g = \Psi^\star g$$

Note that ψ is dilation of conformal diffeomorphism of the sphere (S^m, g) if and only if (S^m, g), $(S^m, ge^{-2\psi})$ are isometric.

Theorem (Pina, Tenenblat)

Let (S^m, g) be the unit sphere with the usual metric. Metric $g\varphi^{-2}$ on S^m is isometric to g if and only if for all $y \in S^m \subset \mathbb{R}^{m+1}$

$$\varphi(y) = a + c + (a - c)y_{m+1} + \sum_{i=1}^{m} b_i y_i,$$

where a, c, b_i are real numbers such that $\varphi|S^m > 0$, or equivalently:

$$\left(\sum_{i=1}^{m} b_i^2 - 4ac\right) < 0.$$

Conformal submersions from spheres

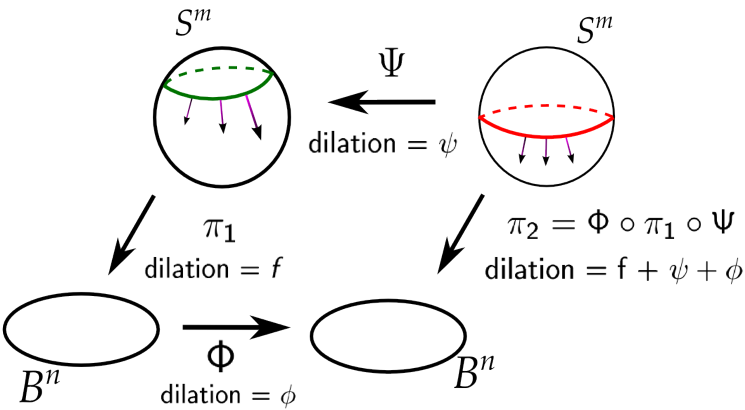
Proposition

Let $\pi : S^{n+k} \to B^n$ be a conformal submersion of dilation f, i.e.: $\|X\|^2 = e^{2f}\|\pi_\star X\|^2$ for horizontal X. If:

$$e^{f(y)} = a + c + (a-c)y_{n+k+1} + \sum_{i=1}^{n+k} b_i y_i,$$

for all $y \in S^{n+k} \subset \mathbb{R}^{n+k+1}$, where a, c, b_i are real numbers such that $\left(\sum_{i=1}^{n+k} b_i^2 - 4ac\right) < 0$, then

- *π is composition of the Hopf fibration and a conformal diffeomorphism Ψ of S^{n+k}*
- *fibers of π are totally umbilical*
- *$k \in \{1, 3, 7\}$, $B \in \{\mathbb{C}P^n, \mathbb{H}P^n, S^8\}$ (resp.)*



$$\mathcal{V}\,\mathrm{grad}(f + \psi + \phi) = \mathcal{V}\,\mathrm{grad}(f + \psi)$$

Proposition

Let $\pi : S^{n+k} \to B^n$ *be a conformal submersion of dilation* f*.*
If

$$\mathcal{V}\,\mathrm{grad}\, e^f = \mathcal{V}\,\mathrm{grad}\left(a + c + (a-c)y_{n+k+1} + \sum_{i=1}^{n+k} b_i y_i \right),$$

for all $y \in S^{n+k} \subset \mathbb{R}^{n+k+1}$*, where* a, c, b_i *are real numbers such that* $\left(\sum_{i=1}^{n+k} b_i^2 - 4ac \right) < 0$*, then*

- *fibers of* π *are totally umbilical and* $k \in \{1, 3, 7\}$*,*
- *there exist conformal diffemorphisms* Ψ *of* S^{n+k} *and* $\Phi : B^n \to \mathbb{H}P^n$ *such that* $\Phi \circ \pi \circ \Psi$ *is the Hopf fibration.*

Homogeneous conformal submersions from spheres

Proposition

Let $\pi : S^{2n+1} \to B$ be a conformal submersion.
If fibers of π are integral curves of a nowhere vanishing conformal Killing field U on S^{2n+1} such that:

$$\frac{Ug(U,U)}{g(U,U)} = U\left(a + c + (a-c)y_{2n+2} + \sum_{i=1}^{2n+1} b_i y_i\right),$$

for all $y \in S^{2n+1} \subset \mathbb{R}^{2n+2}$, where a, c, b_i are real numbers such that $\left(\sum_{i=1}^{2n+1} b_i^2 - 4ac\right) < 0$, then

- *B is $\mathbb{C}P^n$ with metric conformal to the standard one.*
- *There exists a conformal diffemorphism Ψ of S^{2n+1} such that $\pi \circ \Psi$ is the Hopf fibration.*

Envelopes

Definition

For all $s = (s_1, \dots, s_k)$ let M_s be a surface in $\mathbb{R}^n$ given by equation $F_s = 0$, where $F : \mathbb{R}^n \to \mathbb{R}$. An envelope of family of surfaces M_s is the set of points satisfying the following equations:

$$\begin{aligned} F_s &= 0 \\ \frac{\partial}{\partial s_1} F_s &= 0 \\ &\vdots \\ \frac{\partial}{\partial s_k} F_s &= 0 \end{aligned}$$

Envelopes of spheres

If every M_s is a sphere:

$$F_s(x) = \|x - x_0(s)\|^2 - r(s)^2$$

then the equations of the envelope are following:

$$\left.\|x - x_0(s)\|^2 - r(s)^2 = 0,\quad\right\}(n-1) - \text{sphere}$$

$$\left.\begin{array}{r} -2\langle \frac{\partial x_0}{\partial s_1}, x - x_0\rangle - 2r\frac{\partial r}{\partial s_1} = 0, \\ \vdots \\ -2\langle \frac{\partial x_0}{\partial s_k}, x - x_0\rangle - 2r\frac{\partial r}{\partial s_k} = 0 \end{array}\right\}(n-k)\text{-plane } \Sigma_s$$

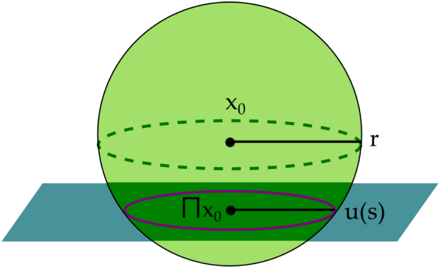
$$(n-1) - \text{sphere} \cap (n-k) - \text{plane} = (n-k-1) - \text{sphere}$$

For every s, the set of equations above defines $(n-k-1) - \text{sphere } \Gamma_s$.
The envelope is foliated by those spheres .

When is this foliation conformal?

Points of envelope

For 2-parameter family of 3-spheres in $\mathbb{R}^4$:
$s = (s_1, s_2)$ and every leaf Γ_s of the foliation is a circle .



Every point of envelope is of the form:

$$x(s,t) = \Pi x_0(s) + u(s) \cdot (E_1(s)\cos t + E_2(s)\sin t)$$

where
Πx_0 is orthogonal projection of x_0 onto plane Σ_s ,
$E_1(s), E_2(s)$ are orthonormal basis of plane Σ_s
and $u(s)$ is the radius of circle Γ_s , we have: $u(s) = r^2 - \|x_0 - \Pi x_0\|^2$.

We assume that when t, u are the radial coordinates on a plane Σ_s, then s_1, s_2, t, u are coordinates in a neighbourhood of an envelope.
Let $x_0(s_1, s_2)$ be the surface of centers of spheres. We assume:

$$\forall_{s_1, s_2} \quad x_0(s_1, s_2) \in \mathbb{R}^3 \times \{0\}.$$

Then $E_2 = (0, 0, 0, 1)$ and vectors:

$$X_1 = \frac{\partial}{\partial s_1} + \frac{\partial u}{\partial s_1}\frac{\partial}{\partial u} + \frac{1}{u}\langle\frac{\partial \Pi x_0}{\partial s_1}, E_1\rangle \sin t \frac{\partial}{\partial t}$$

$$X_2 = \frac{\partial}{\partial s_2} + \frac{\partial u}{\partial s_2}\frac{\partial}{\partial u} + \frac{1}{u}\langle\frac{\partial \Pi x_0}{\partial s_2}, E_1\rangle \sin t \frac{\partial}{\partial t}$$

are horizontal , i.e. tangent to the envelope and orthogonal to leaves of foliation.

For vector field $V = \frac{\partial}{\partial t}$, tangent to leaves of foliation, we have:

$$(\mathcal{L}_V g)(X_i, X_j) = -a_1(i,j)\sin t - a_2(i,j)\sin(2t)$$

and

$$g(X_i, X_j) = a_0(i,j) + \frac{1}{2}a_2(i,j) + a_1(i,j)\cos t + \frac{1}{2}a_2(i,j)\cos(2t)$$

where:

$$a_0 = (\frac{\partial u}{\partial s_i})\cdot(\frac{\partial u}{\partial s_j}) + \langle\frac{\partial \Pi x_0}{\partial s_i}, \frac{\partial \Pi x_0}{\partial s_j}\rangle - \langle\frac{\partial \Pi x_0}{\partial s_i}, E_1\rangle \cdot \langle\frac{\partial \Pi x_0}{\partial s_j}, E_1\rangle$$

$$a_1 = u(s)\left(\langle\frac{\partial \Pi x_0}{\partial s_i}, \frac{\partial E_1}{\partial s_j}\rangle + \langle\frac{\partial \Pi x_0}{\partial s_j}, \frac{\partial E_1}{\partial s_i}\rangle\right) + \frac{\partial u}{\partial s_i}\langle\frac{\partial \Pi x_0}{\partial s_j}, E_1\rangle + \frac{\partial u}{\partial s_j}\langle\frac{\partial \Pi x_0}{\partial s_i}, E_1\rangle$$

and

$$a_2 = u(s)^2\langle\frac{\partial E_1}{\partial s_i}, \frac{\partial E_1}{\partial s_j}\rangle + \langle\frac{\partial \Pi x_0}{\partial s_i}, E_1\rangle \cdot \langle\frac{\partial \Pi x_0}{\partial s_j}, E_1\rangle$$

Definition

We call a foliation Riemannian when:

$$(\mathcal{L}_V g)(X, Y) = 0$$

for all vectors X, Y orthogonal to the foliation and all vector fields V tangent to the foliation.

Proposition

The foliation $\mathcal{F}$ of the envelope of family of 3-spheres by characteristic circles is Riemannian if and only if the surface of centers of spheres is a plane .

Denote:

$$\Phi(i,j) = \frac{(\mathcal{L}_V g)(X_i, X_j)}{g(X_i, X_j)} = \frac{-a_1(i,j)\sin t - a_2(i,j)\sin(2t)}{a_0(i,j) + \frac{1}{2}a_2(i,j) + a_1(i,j)\cos t + \frac{1}{2}a_2(i,j)\cos(2t)}.$$

The foliation $\mathcal{F}$ is conformal when either

- the function $\Phi(i,j)$ is the same for all pairs of $i, j \in \{1,2\}$

or

- $\Phi(1,1) = \Phi(2,2)$ and $(\mathcal{L}_V g)(X_1, X_2) = g(X_1, X_2) = 0$.

Proposition

Consider the envelope of 2-parameter family of 3-spheres . Let the surface of centers of spheres be a unit 2-sphere . Then the foliation is conformal if all the 3-spheres have the same radius $0 < r < 1$.
For all horizontal vectors X, Y and all vertical (tangent to leaves) fields V we have then:

$$(\mathcal{L}_V g)(X,Y) = \frac{-2r\sin t}{1 + r\cos t} \cdot g(X,Y).$$

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