Mechanical control systems and their linearization

Witold RESPONDEK

Normandie Université
INSA de Rouen, France

based on joint work with
Sandra RICARDO, University of Vila Real, Portugal

Geometrical Methods in Physics
IM PAN, Warsaw, December 17, 2014

Aim

- To analyze compatibility of two structures of control systems: mechanical structure and linear structure
- To identify general control systems that are mechanical, that is, that admit a mechanical structure
- To describe equivariants of mechanical control systems

Outline

- Problem description
- Mechanical control systems
- Linearization preserving the mechanical structure
- Control systems that admit a mechanical structure
- Uniqueness of mechanical structure
- Linearization of mechanizable control systems
- Equivariants of mechanical control systems

Problem statement

- Assume that a control system Σ is equivalent to a mechanical control system $(\mathcal{MS})$

$$\Sigma \longleftrightarrow (\mathcal{MS})$$

- Assume that Σ is equivalent to a linear control system Λ

$$\Sigma \longleftrightarrow \Lambda$$

- Question: Are the linear and mechanical structures of Σ compatible, i.e., is Σ equivalent to a linear mechanical control system $(\mathcal{LMS})$?

$$\Sigma \longleftrightarrow (\mathcal{LMS})$$

- Two variants of our problem: we may wish $(\mathcal{MS})$ and $(\mathcal{LMS})$ to have equivalent mechanical structures or we may allow for non equivalent ones (the latter possibility being, obviously, related with the problem of (non)uniqueness of mechanical structures that a control system may admit).

- To make the problem precise: define the class of systems Σ, linear systems Λ, mechanical control systems $(\mathcal{MS})$, linear mechanical control system $(\mathcal{LMS})$, and the equivalence.

Notions

- We will consider smooth control-affine systems of the form

$$\Sigma : \quad \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z), \ \ z \in M$$

- Σ and $\tilde{\Sigma} : \ \dot{\tilde{z}} = \tilde{F}(\tilde{z}) + \sum_{r=1}^{m} u_r \tilde{G}_r(\tilde{z})$ on $\tilde{M}$ are (locally) state-space equivalent, shortly (locally) S-equivalent, if there exists a (local) diffeomorphism $\Psi : \ M \to \ \widetilde{M}$ such that $D\Psi(z) \cdot F(z) = \tilde{F}(\tilde{z})$ and $D\Psi(z) \cdot G_r(z) = \tilde{G}_r(\tilde{z})$, $1 \leq r \leq m$.
- Ψ preserves trajectories.
- Σ is S-linearizable if it is S-equivalent to a linear system of the form

$$\Lambda : \quad \dot{\tilde{z}} = A\tilde{z} + \sum_{r=1}^{m} u_r B_r.$$

Mechanical Control Systems

A *mechanical control system* ($\mathcal{MS}$) as a 4-tuple $(Q, \nabla, \mathfrak{g}_0, d)$, in which

(i) Q is an n-dimensional manifold, called *configuration manifold*;

(ii) ∇ is a symmetric affine connection on Q;

(iii) $\mathfrak{g}_0 = (e, g_1, \ldots, g_m)$ is an $(m+1)$-tuple of vector fields on Q;

(iv) $d : TQ \to TQ$ is a map preserving each fiber and linear on fibers.

defining the system that, in local coordinates (x, y) of TQ, reads

$$\begin{aligned} \dot{x}^i &= y^i \\ \dot{y}^i &= -\Gamma^i_{jk}(x)y^j y^k + d^i_j(x)y^j + e^i(x) + \sum_{r=1}^{m} u_r g^i_r(x). \end{aligned}$$

- Γ^i_{jk} are the Christoffel symbols of ∇ (Coriolis and centrifugal forces)
- the terms $d^i_j(x)y^j$ correspond to dissipative-type (or gyroscopic-type) forces acting on the system,
- e represents an uncontrolled force (which can be potential or not)
- $g_1, \ldots, g_m$ represent controlled forces.

Examples: planar rigid body

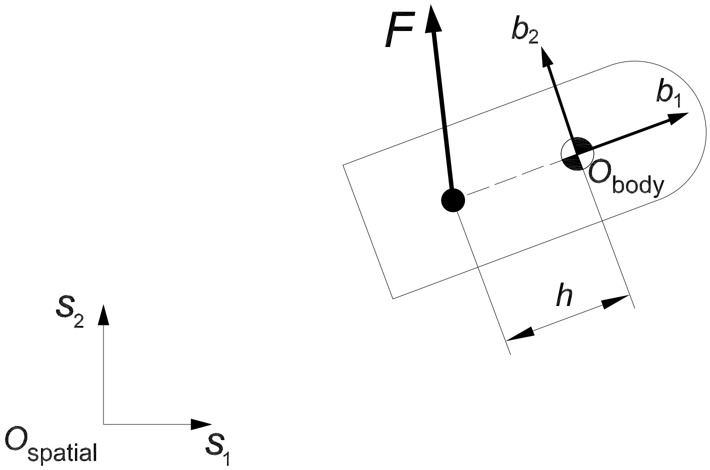

Figure: The planar rigid body

Examples: planar rigid body

- Configuration: $q = (\theta, x_1, x_2) \in \mathbb{S}^1 \times \mathbb{R}^2$, where

$$\begin{aligned} \theta &= \text{relative orientation of } \Sigma_{\text{body}} \text{ w.r.t. } \Sigma_{\text{spatial}} \\ (x_1, x_2) &= \text{position of the center of mass} \end{aligned}$$

- Equations of motion:

$$\begin{aligned} \ddot{\theta} &= -u_2 \frac{h}{J} \\ \ddot{x}_1 &= u_1 \frac{\cos\theta}{m} - u_2 \frac{\sin\theta}{m} \\ \ddot{x}_2 &= u_1 \frac{\sin\theta}{m} + u_2 \frac{\cos\theta}{m} \end{aligned}$$

- no d-forces
- The Christoffel symbols Γ^i_{jk} of the Euclidean metric $J \mathrm{d}\theta \otimes \mathrm{d}\theta + m(\mathrm{d}x_1 \otimes \mathrm{d}x_1 + \mathrm{d}x_2 \otimes \mathrm{d}x_2)$ vanish

Examples: robotic leg

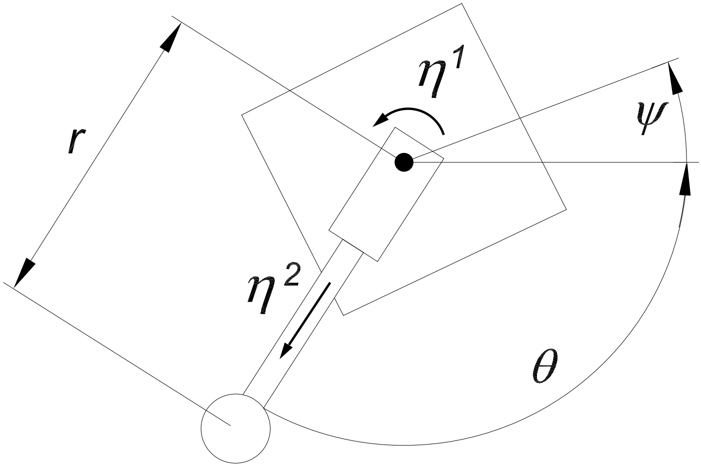

Figure: Robotic leg

Examples: robotic leg

- Configuration: $q = (r, \theta, \psi) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1$, where

$$\begin{aligned} r &= \text{extension of the leg} \\ \theta &= \text{ angle of the leg from an inertial reference frame} \\ \psi &= \text{angle of the body} \end{aligned}$$

- Equations of motion:

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 + \frac{1}{m}u_2 \\ \ddot{\theta} &= -\frac{2}{r}\dot{r}\dot{\theta} + \frac{1}{mr^2}u_1 \\ \ddot{\psi} &= -\frac{1}{J}u_1. \end{aligned}$$

- no d-forces
- The Christoffel symbols of the Riemannian metric $m\mathrm{d}r \otimes \mathrm{d}r + mr^2\mathrm{d}\theta \otimes \mathrm{d}\theta + J\mathrm{d}\psi \otimes \mathrm{d}\psi$ are $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$.

Vertical distribution and mechanical MS-equivalence

- Any mechanical control system $(\mathcal{MS})$ evolves on TQ and thus defines the *vertical distribution* $\mathfrak{V}$, of rank n, that is tangent to fibers T_qQ. In (x, y)-coordinates it is given by

$$\mathfrak{V} = \text{span}\ \left\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n}\right\}.$$

  Clearly, $\mathfrak{V}$ contains all control vector fields $g_r^i(x)\frac{\partial}{\partial y^i}$ of $(\mathcal{MS})$.

- Two mechanical systems $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$ are MS-equivalent if there exists a diffeomorphism φ between their configuration manifolds Q and $\tilde{Q}$ such that the corresponding control systems on the tangent bundles TQ and $T\tilde{Q}$ are S-equivalent via the extended point diffeomorphism $\Phi = (\varphi, \mathrm{D}\varphi \cdot y)^T$.

- The diffeomorphism Φ, establishing the MS-equivalence, maps the vertical distribution into the vertical distribution.

Linear Mechanical Control Systems

Systems that are simultaneously linear and mechanical form the class of Linear Mechanical Control Systems

$$(\mathcal{LMS}) \qquad \begin{array}{rcl} \dot{\tilde{x}} & = & \tilde{y}, \\ \dot{\tilde{y}} & = & D\tilde{y} + E\tilde{x} + \sum_{r=1}^{m} u_r b_r, \end{array}$$

where D and E are matrices of appropriate sizes.

Example

The mechanical system

$$(\mathcal{MS})_1 : \begin{array}{ll} \dot{x}^1 = y^1, & \dot{y}^1 = u, \\ \dot{x}^2 = y^2, & \dot{y}^2 = x^1(1+x^1) + \frac{y^1\ y^2}{1+x^1} \end{array}$$

on TQ, where $Q = \{(x^1, x^2) \in \mathbb{R}^2 \ : \ x^1 > -1\}$. is transformed via the diffeomorphism Ψ

$$\begin{array}{rclrcl} \tilde{x}^1 & = & x^1, & \tilde{y}^1 & = & y^1, \\ \tilde{x}^2 & = & x^2 - \frac{1}{2}\left(\frac{y^2}{1+x^1}\right)^2, & \tilde{y}^2 & = & \frac{y^2}{1+x^1}, \end{array}$$

into the linear control system

$$(\mathcal{LMS})_1 : \begin{array}{rclrcl} \dot{\tilde{x}}^1 & = & \tilde{y}^1, & \dot{\tilde{y}}^1 & = & u, \\ \dot{\tilde{x}}^2 & = & \tilde{y}^2, & \dot{\tilde{y}}^2 & = & \tilde{x}^1. \end{array}$$

Notice that $(\mathcal{LMS})_1$ is a linear mechanical system but its mechanical structure is not MS-equivalent to that of $(\mathcal{MS})_1$. Indeed, Ψ does not map the vertical distribution $\mathfrak{V} = \text{span}\,\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\}$ of $(\mathcal{MS})_1$ onto the vertical distribution $\tilde{\mathfrak{V}} = \text{span}\,\{\frac{\partial}{\partial \tilde{y}^1}, \frac{\partial}{\partial \tilde{y}^2}\}$ of $(\mathcal{LMS})_1$. The question is thus whether we can bring $(\mathcal{MS})_1$ into a linear system that would be mechanically equivalent to $(\mathcal{MS})_1$? ◁

Linearization preserving the mechanical structure: main result

Theorem

The mechanical system $(\mathcal{MS})$ *is, locally around* $(x_0, y_0) \in TQ$, *MS-equivalent to a linear controllable mechanical system* $(\mathcal{LMS})$ *if and only if it satisfies, in a neighborhood of* (x_0, y_0), *the following conditions*

(LM1) $\dim \operatorname{span}\{\operatorname{ad}_F^q G_r,\ 0 \le q \le 2n-1, 1 \le r \le m\}(x, y) = 2n$,

(LM2) $\left[\operatorname{ad}_F^p G_r, \operatorname{ad}_F^q G_s\right] = 0$, *for* $1 \le r, s \le m,\ 0 \le p, q \le 2n$,

(LM3) *there exist* $d_{iq}^r \in \mathbb{R}$, *where* $1 \le i \le n$, $1 \le r \le m$, $0 \le q \le 2n-1$, *such that the vector fields*

$$V_i = \sum_{r,q} d_{iq}^r \operatorname{ad}_F^q G_r$$

span the vertical distribution $\mathfrak{V}$.

(LM3) is a compatibility condition

- It is well known that the conditions (LM1) and (LM2) are necessary and sufficient for a nonlinear control system of the form Σ : $\dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z)$ to be, locally, S-equivalent to a linear controllable system.
- In linearizing coordinates the vector fields $\mathrm{ad}_F^q G_r$ are constant
- The condition (LM3) is thus, clearly, a compatibility condition that assures that the mechanical and linear structure are conform: it implies that well chosen $\mathbb{R}$-linear combinations of the vector fields $\mathrm{ad}_F^q G_r$ span the vertical distribution $\mathfrak{V}$ that defines the tangent bundle structure of the mechanical system.

Example - cont.

For the system $(\mathcal{MS})_1$ of Example, we have

$$\mathfrak{V} = \text{span } \left\{ \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\}.$$

Simple Lie bracket calculations yield

$$\begin{array}{rcl} \mathrm{ad}_F\, G & = & -\frac{\partial}{\partial x^1} - \frac{y^2}{1+x^1}\frac{\partial}{\partial y^2}, \\ \mathrm{ad}_F^2\, G & = & \frac{y^2}{1+x^1}\frac{\partial}{\partial x^2} + (1+x^1)\frac{\partial}{\partial y^2}, \\ \mathrm{ad}_F^3\, G & = & -\frac{\partial}{\partial x^2}, \qquad \mathrm{ad}_F^4\, G = 0. \end{array}$$

We take $V_1 = G = \frac{\partial}{\partial y^1}$, that is, $d_{10} = 1$ and $d_{11} = d_{12} = d_{13} = 0$. In order to have $\mathfrak{V} = \text{span } \{V_1, V_2\}$, where $V_2 = d_{21}\, \mathrm{ad}_F\, G + d_{22}\, \mathrm{ad}_F^2\, G + d_{23}\, \mathrm{ad}_F^3\, G$, we need $d_{21} = 0$ and $d_{23} = \frac{y^2}{1+x^1} d_{22}$ so d_{22} and d_{23} cannot be taken as real constants, thus violating the condition (LM3) of Theorem 1. It follows that although the system $(\mathcal{MS})_1$ of Example 1 is S-equivalent to a linear mechanical system, it is not MS-equivalent to a linear mechanical system, that is, it cannot be linearized with simultaneous preservation of its mechanical structure. $\triangleleft$

Interpretation of linearizability conditions

- The linearizing diffeomorphism φ simultaneously rectifies the control vector fields, annihilates the Christoffel symbols, transforms the fiber-linear map $d(x)y$ into a linear one, and the vector field $e(x)$ into a linear vector field. Conditions that guarantee that all those normalizations take place and, moreover, that they can be effectuated simultaneously must be somehow encoded in the conditions (LM1)-(LM3). How?

- By (LM3), there exist $V_i = \sum_{r,q} d^r_{iq} \operatorname{ad}^q_F G_r,\ 1 \leq i \leq\ n$, that span the vertical distribution $\mathfrak{V}$ and are vertical lifts of vector fields v_i on Q due to

$$0 = [V_i, \operatorname{ad}_F V_j] \qquad 1 \leq i, j \leq n, \tag{1}$$

The commutativity conditions

$$0 = [\operatorname{ad}_F V_i, \operatorname{ad}_F V_j] = [v_i, v_j] \bmod \mathfrak{V}, \quad 1 \leq i, j \leq n, \tag{2}$$

imply that there exists a local diffeomorphism $\tilde{x} = \varphi(x)$ rectifying simultaneously all v_i, that is, $\varphi_* v_i = \frac{\partial}{\partial \tilde{x}_i}$. The extended point transformation $(\tilde{x}, \tilde{y})^T = \Phi(x, y) = (\varphi(x), \mathrm{D}\varphi \cdot y)^T$ maps V_i into $\tilde{V}_i = \Phi_* V_i = \frac{\partial}{\partial \tilde{y}_i}$.

- Now calculating in the $(\tilde{x}, \tilde{y})$-coordinates the commutativity relations

$$0 = [\tilde{V}_i, \mathrm{ad}_{\tilde{F}} \tilde{V}_j] = \Gamma^k_{ij} \frac{\partial}{\partial y^k}, \tag{3}$$

  we conclude that all Christoffel symbols vanish implying that the connection ∇ defining the mechanical system is locally Euclidean (its Riemannian tensor R vanishes) and that the local $\tilde{x}$-coordinates are flat and, simultaneously, rectifying coordinates for the v_i's.

- Finally, calculating the commutativity relations

$$0 = [\mathrm{ad}_F \tilde{V}_i, \mathrm{ad}^2_F \tilde{V}_j], \tag{4}$$

  we conclude that in the $\tilde{x}$-coordinates, the $(1,1)$-tensor d is constant and the vector field $e(x)$ is linear.

- All those informations are encoded in the commutativity conditions (LM2) but they are mixed up. Passing to the vector fields $V_i = \sum_{r,q} d^r_{iq} \mathrm{ad}^q_F G_r$ and using the conditions (LM2) in the form $0 = [\mathrm{ad}^p_F V_i, \mathrm{ad}^q_F V_j]$, for $0 \leq p, q \leq 2$ (equivalent to (1)-(4)), instead of applying directly to $\mathrm{ad}^q_F G_r$, allows to clearly identify the conditions responsable for the required form of, respectively, g_r's, the connection ∇, $d(x)$, and $e(x)$.

Linearization of Mechanizable Control Systems

- So far: is the linear structure compatible with a given mechanical structure?
- Now: we discuss general control-affine systems that admit both: a mechanical and a linear structure.
- If a system admits a unique mechanical structure, then the situation is that of the previous theorem
- When does a control system admit a mechanical structure and when is it unique?

Mechanical Control Systems

A *mechanical control system* ($\mathcal{MS}$) as a 4-tuple $(Q, \nabla, \mathfrak{g}_0, d)$, in which

(i) Q is an n-dimensional manifold, called *configuration manifold*;

(ii) ∇ is a symmetric affine connection on Q;

(iii) $\mathfrak{g}_0 = (g_0, g_1, \ldots, g_m)$ is an $(m+1)$-tuple of vector fields on Q;

(iv) $d : TQ \to TQ$ is a map preserving each fiber and linear on fibers.

defining , in local coordinates (x, y) of TQ, the equations

$$\begin{aligned} \dot{x}^i &= y^i \\ \dot{y}^i &= -\Gamma^i_{jk}(x) y^j y^k + d^i_j(x) y^j + g^i_0(x) + \sum_{r=1}^{m} u_r g^i_r(x). \end{aligned}$$

- Γ^i_{jk} are the Christoffel symbols of the connection ∇.
- the terms $d^i_j(x) y^j$ correspond to dissipative-type (or gyroscopic-type) forces acting on the system,
- g_0 represents an uncontrolled force (which can be potential or not)
- $g_1, \ldots, g_m$ represent controlled forces.

Equivalence problem

When is the control system

$$\Sigma \ : \ \dot{z} = F(z) + \sum_{r=1}^{m} u_r Gr(z), \quad z \in M^{2n}, \ u \in \mathbb{R}^m,$$

mechanical? That is, when does there exist a (local) diffemorphism $\Phi : M \to TQ$ transforming Σ into a mechanical system $(\mathcal{MS})$?

In other words, a diffeomorphism $\Phi : M \to TQ$ such that

$$\begin{aligned} \Phi_* F &= y^i \frac{\partial}{\partial x^i} + \left(-\Gamma^i_{jk}(x) y^j y^k + d^i_j(x) y^j + g^i_0(x) \right) \frac{\partial}{\partial y^i} \\ \Phi_* G_r &= g^i_r(x) \frac{\partial}{\partial y^i}, \end{aligned}$$

Links with the inverse problem

When for the control system (differential equation)

$$\Sigma \ : \ \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z), \quad z \in M, \ u \in \mathbb{R}^m,$$

does there exist a (local) diffeomorphism $\Phi : M \to TQ$ such that

$$\begin{aligned} \Phi_* F &= y^i \frac{\partial}{\partial x^i} + \left(-\Gamma^i_{jk}(x) y^j y^k + d^i_j(x) y^j + g^i_0(x) \right) \frac{\partial}{\partial y^i} \\ \Phi_* G_r &= g^i_r(x) \frac{\partial}{\partial y^i}, \end{aligned}$$

- our problem is more specific: the right hand side is quadratic in velocities
- our problem is more general:
  - no á priori tangent bundle structure TQ
  - non potential forces g_0 are allowed
  - dissipative forces are allowed
- The vector fields G_r provide additional information encoded in the Lie algebra generated by them and F.

Symmetric product

- An affine connection ∇ defines the symmetric product:

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X \qquad X, Y \in \mathfrak{X}(Q).$$

- In coordinates given by

$$\langle X : Y \rangle = \left(\frac{\partial X^i}{\partial x^j} Y^j + \frac{\partial Y^i}{\partial x^j} X^j + \Gamma^i_{jk} X^j Y^k + \Gamma^i_{jk} Y^j X^k \right) \frac{\partial}{\partial x^i}.$$

- A distribution $\mathcal{D}$ on Q is called *geodesically invariant* with respect to an affine connection ∇ if every geodesic $\gamma : I \to Q$, such that $\gamma'(t_0) \in \mathcal{D}(\gamma(t_0))$ for some $t_0 \in I$, satisfies $\gamma'(t) \in \mathcal{D}(\gamma(t))$ for all $t \in I$.

- Geometric interpretation of the symmetric product (A. Lewis): a distribution $\mathcal{D}$ on a manifold Q, equipped with an affine connection ∇, is geodesically invariant if and only if

$$\langle X : Y \rangle \in \mathcal{D}, \quad \text{for every} \quad X, Y \in \mathcal{D}.$$

- So the symmetric products plays the same role for the geodesic invariance as the Lie brackets for integrability.

Geodesic accessibility

Consider the mechanical control system $(\mathcal{MS}) = (Q, \nabla, \mathfrak{g}_0, d)$. Let $\mathcal{SYM}(g_1, \ldots, g_m)$ be the smallest distribution on Q containing the input vector fields $g_1, \ldots, g_m$ and such that it is closed under the symmetric product defined by the connection ∇.

Definition

The system $(\mathcal{MS})$ is called geodesically accessible at $x_0 \in Q$ if

$$\mathcal{SYM}(g_1, \ldots, g_m)(x_0) = T_{x_0}Q,$$

and geodesically accessible if the above equality holds for all $x_0 \in Q$. A geodesically accessible mechanical system will be denoted by $(\mathcal{GAMS})$.

- For geodesically accessibile mechanical control systems, the smallest geodesically invariant distribution containing the control vector fields $g_1, \cdots, g_m$ is TQ.
- The planar rigid body is geodesically accessible but the robotic leg is NOT geodesically accessible (although accessible).

The basic object

- We will call a *zero-velocity point* for the mechanical control system $(\mathcal{MS})$ any point of the form $(x_0, \dot{x}_0) = (x_0, 0)$, that is, any point of the zero section of the tangent bundle TQ.

- For the control system

$$\Sigma \ : \ \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z),$$

let $\mathcal{V}$ denote the smallest vector space, over $\mathbb{R}$, containing the vector fields $G_1, \ldots, G_m$ and satisfying

$$[\mathcal{V}, \operatorname{ad}_F \mathcal{V}] \subset \mathcal{V},$$

where $[\mathcal{V}, \operatorname{ad}_F \mathcal{V}] = \{[V_i, \operatorname{ad}_F V_j] \mid V_i, V_j \in \mathcal{V}\}$.

Characterization of mechanical control systems

Theorem

Let M be a smooth $2n$-dimensional manifold. A system Σ is locally, at $z_0 \in M$, S-equivalent to a geodesically accessible mechanical system ($\mathcal{GAMS}$) around a zero-velocity point $(x_0, 0)$ if and only if

(MS0) $F(z_0) \in \mathcal{V}(z_0)$,

(MS1) $\dim \mathcal{V}(z) = n$ *and* $\dim \left(\mathcal{V} + [F, \mathcal{V}]\right)(z) = 2n$,

(MS2) $[\mathcal{V}, \mathcal{V}]\,(z) = 0$,

for any z in a neighborhood of z_0.

- The condition (MS0) implies that the diffeomorphism establishing the S-equivalence (if it exists) will map z_0 into a zero-velocity point.
- A mechanical system (more generally, a control system that is S-equivalent to a mechanical system ($\mathcal{MS}$)) is geodesically accessible around a zero-velocity point if and only if it satisfies (MS0) and (MS1).

- The condition (MS2) $[\mathcal{V}, \mathcal{V}] = 0$, is always necessary for S-equivalence to a mechanical system $(\mathcal{MS})$ and sufficient provided that (MS1) and (MS2) hold.

- It states that the Lie algebra $\mathcal{L} = \{F, G_1, \ldots, G_m\}_{LA}$ contains an abelian subalgebra $\mathcal{V}$ (that spans a distribution of rank n) which is the structural condition reflecting the existence of a mechanical structure of Σ.

- The conditions (MS0)-(MS2) are veryfiable: define

$$\begin{aligned} \mathcal{V}_1 &= \{G_r \mid 1 \leq r \leq m\} \\ \mathcal{V}_2 &= \{[G_r, \operatorname{ad}_F G_s] \mid 1 \leq r, s \leq m\} \end{aligned}$$

and, inductively,

$$\mathcal{V}_i = \bigcup_{p+l=i} \left[\mathcal{V}_p, \operatorname{ad}_F \mathcal{V}_l\right].$$

Put

$$\mathcal{V} := \operatorname{Vect}_{\mathbb{R}} \bigcup_{i=1}^{\infty} \mathcal{V}_i.$$

Constructing mechanical structure for Σ

- Objects defined for the system Σ in terms of F and G_i's (due to conditions (MS0), (MS1), (MS2)) will be denoted by Q^{Σ}, $g_r^{\Sigma}, \nabla^{\Sigma}$, etc.
- The configuration manifold

$$Q^{\Sigma} = \{z \in M \mid F(z) \in \operatorname{span}\mathcal{V}(z)\}.$$

- Define the surjective submersion $\pi : M \to Q^{\Sigma}$ by attaching to any $z \in M$ the point $q = \pi(z)$ defined as $q \in Q^{\Sigma} \cap L_z$, where L_z is the leaf passing through z.
- We identify the leaves of the distrib. $\operatorname{span}\mathcal{V}$ with the fibres T_xQ.
- Due to $[\operatorname{ad}_F V, \mathcal{V}] \subset \mathcal{V}$, any vector field $V \in \mathcal{V}$ gives rise to a vector field v on Q^{Σ}. by

$$v := -\pi_*(\operatorname{ad}_F V)$$

- In particular, since $G_r \in \mathcal{V}, 1 \leq r \leq m$, we define the control vector fields by:

$$g_r^{\Sigma} := -\pi_*(\operatorname{ad}_F G_r).$$

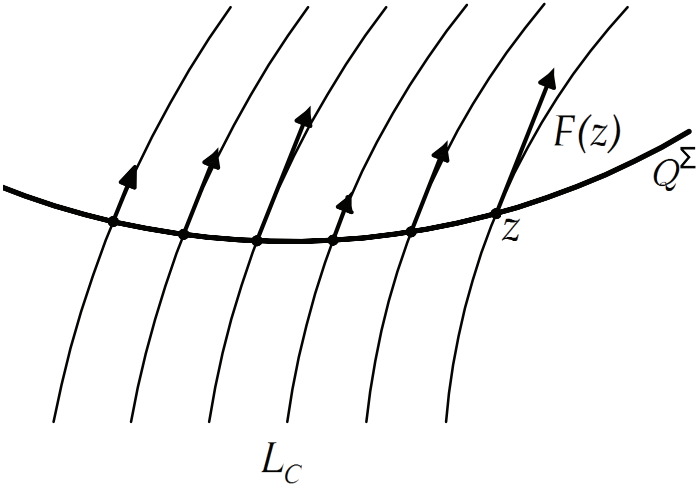

Figure: The configuration manifold Q^Σ consists of points where F is tangent to the leaves L_C of the foliation defined by $\mathcal{V}$

Constructing mechanical structure for Σ (cont.)

- Similarly, we define the drift g_0 and the fibre-linear map d.
- Conversely, given a frame $v_1, \ldots, v_n$ on Q^Σ, there exists a unique collection of independent vector fields $V_1, \ldots, V_n \in \mathcal{V}$ satisfying $v_i := -\pi_*(\operatorname{ad}_F V_i)$ whose vertical lifts are, actually, V_i, i.e., the of $V_i = (v_i)^{\text{vlift}}$.
- We define an affine connection $\nabla^\Sigma : \mathfrak{X}(Q^\Sigma) \times \mathfrak{X}(Q^\Sigma) \to \mathfrak{X}(Q^\Sigma)$ by

$$\nabla^\Sigma_{v_j} v_i := \frac{1}{2}\pi_* \left(\left[\operatorname{ad}_F^2 V_i, V_j \right] \right),$$

with v_i, v_j arbitrary vector fields on Q^Σ and V_i, V_j their corresponding vertical lifts.

Uniqueness of the $(\mathcal{GAMS})$ structure

Any system Σ that is already $(\mathcal{GAMS}) = (Q, \nabla, \mathfrak{g}_0, d)$, satisfies conditions (MS0),(MS1) and (MS2) (in particular $[\mathcal{V}, \mathcal{V}] = 0$) and thus we can construct for it the $\mathcal{V}$-canonical structure $(Q^\Sigma, \nabla^\Sigma, \mathfrak{g}_0^\Sigma, d^\Sigma)$.

Theorem

(i) *For any Σ that is already $(\mathcal{GAMS}) = (Q, \nabla, \mathfrak{g}_0, d)$ the original mechanical structure coincides with the canonical one, that is,*

$$(Q, \nabla, \mathfrak{g}_0, d) = (Q^\Sigma, \nabla^\Sigma, \mathfrak{g}_0^\Sigma, d^\Sigma).$$

(ii) *If a control system Σ admits a $(\mathcal{GAMS})$-structure, then it is unique (among $\mathcal{GAMS}$-structures). More precisely, if Σ is S-equivalent to two mechanical structures $(\mathcal{GAMS})_1$ and $(\mathcal{GAMS})_2$, then they are $\mathcal{MS}$-equivalent (by an extended point transformation).*

Uniqueness of the $(\mathcal{MS})$ structures

- If a control system Σ admits a $(\mathcal{GAMS})$-structure, then it is unique.
- Can Σ (accessible, but not geodesically accessible) admit multiple mechanical structure?

Bi-mechanical control systems exist!!!

The system

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{x}_2 &= y_2 \\ \dot{y}_1 &= u \\ \dot{y}_2 &= x_1 \end{aligned}$$

is bi-mechanical: it is S-equivalent (but not mechanically S-equivalent) to the mechanical system

$$\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{y}_1 \\ \dot{\tilde{x}}_2 &= \tilde{y}_2 \\ \dot{\tilde{y}}_1 &= u \\ \dot{\tilde{y}}_2 &= \tilde{x}_1(1+\tilde{x}_1) + \frac{\tilde{y}_1\tilde{y}_2}{1+\tilde{x}_1} \end{aligned}$$

- Bi-mechanical (accessible) control systems exist!!!
- Geodesic accessibility is therefore not just a technical (rank) assumption but is responsable for for the uniqueness of a mechanical structure.

Linear mechanical control systems

An important class of mechanical control systems are linear systems

$$\Lambda : \quad \begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & Dy + Ex + Bu \end{array}$$

corresponding to

- a Euclidean metric
- the control vector fields B_i are constant in flat coordinates
- the dissipative terms $d(y) = Dy$ and the uncontrolled vector field $g_0(x) = Ex$ are linear with respect to linear coordinates.

They are never $(\mathcal{GA})$ (geodesically accessible) unless $m = n$ (i..e, unless we control all degrees of freedom) so $\mathcal{V}$ does not contain enough information to define the bundle structure TQ.

Theorem

The following conditions are equivalent for a nonlinear control system of the form $\Sigma : F(z) + \sum_{r=1}^{m} u_r G_r(z)$ *on a* $2n$*-dimensional manifold:*

(i) the system Σ *is S-equivalent, locally at* z_0*, to a controllable linear mechanical system* ($\mathcal{LMS}$)*;*

(ii) Σ *satisfies, in a neighborhood of* z_0*, the following conditions*

(LM1) $\dim \operatorname{span}\{\operatorname{ad}_F^q G_r,\ 1 \le r \le m, 0 \le q \le 2n-1\}(z) = 2n,$

(LM2) $\left[\operatorname{ad}_F^p G_r, \operatorname{ad}_F^q G_s\right] = 0,$ *for* $1 \le r, s \le m,\ 0 \le p, q \le 2n,$

(LM3)' *there exist* $d_{iq}^r \in \mathbb{R}$*, where* $1 \le i \le n$*,* $1 \le r \le m$*,* $0 \le q \le 2n-1$*, such that the distribution*

$$\mathfrak{V} = \operatorname{span}\ \{\sum_{r,q} d_{iq}^r \operatorname{ad}_F^q G_r,\ 1 \le i \le n\}$$

is of rank n*, contains* G_r*, for* $1 \le r \le m$*, and satisfies*

$$\mathfrak{V} + [F, \mathfrak{V}] = TM.$$

(iii) Σ *satisfies (LM1), (LM2) and*

(LM3)" $\dim \operatorname{span}\{G_r, \operatorname{ad}_F G_r,\ 1 \le r \le m\}(z) = 2m.$

Interpretation of the conditions

- The difference between the condition (LM3) and (LM3)' (or (LM3)") explains very clearly the difference between the problems considered in this and the previous theorem.
- If a mechanical system is given (the case of the former theorem), then n vector fields of the form $V_i = \sum_{r,q} d_{iq}^r \operatorname{ad}_F^q G_r$ have to span its vertical distribution $\mathfrak{V}$.
- If a mechanical structure is not given (the case of the last theorem), it is the distribution $\mathfrak{V} = \operatorname{span}\{V_1, \ldots, V_n\}$ which will be the vertical distribution of the mechanical structure to be constructed, provided that $\mathfrak{V}$ satisfies (LM3)' (or, equivalently, (LM3)").

Example-cont.

Clearly, the system $(\mathcal{MS})$ of Example, satisfies the conditions (LM1) and (LM2) (actually, we have given a linearizing diffeomorphism Ψ explicitly). To analyze the condition (LM3)', we take $V_1 = G = \frac{\partial}{\partial y^1}$, that is, $d_{10} = 1$ and $d_{11} = d_{12} = d_{13} = 0$, and $V_2 = d_{21} \operatorname{ad}_F G + d_{22} \operatorname{ad}_F^2 G + d_{23} \operatorname{ad}_F^3 G$. We look for reals d_{21}, d_{22}, d_{23} such that the distribution $\mathfrak{V} = \operatorname{span} \{V_1, V_2\}$ satisfies $\mathfrak{V} + [F, \mathfrak{V}] = TM$. A direct calculation shows that this is the case if and only if

$$d_{21} d_{23} - d_{22}^2 \neq 0.$$

Therefore the system $(\mathcal{MS})_1$ of Example admits infinitely many non-equivalent linear mechanical structures whose vertical distribution can be any distribution $\operatorname{span} \{G, d_{21} \operatorname{ad}_F G + d_{22} \operatorname{ad}_F^2 G + d_{23} \operatorname{ad}_F^3 G\}$, where the real coefficients d_{2q} satisfy the above condition. $\triangleleft$

Reducing the problem to the case of linear systems

- The conditions (LM1) and (LM2) are necessary and sufficient for S-equivalence of Σ to a linear controllable system.
- Therefore the problem becomes that of when a linear control system admits a linear mechanical structure.
- Therefore the last Theorem reduces actually to the following one, which is of independent interest.

Proposition

Consider a linear controllable system of the form
$\Lambda: \ \dot{z} = Az + \sum_{r=1}^{m} u_r b_r$, *where* $z \in \mathbb{R}^{2n}$. *The following conditions are equivalent:*

(i) the system Λ *is S-equivalent, via a linear transformation, to a linear mechanical system* $(\mathcal{LMS})$;

(ii) there exists an n*-dimensional linear subspace* $\mathsf{V} \subset \mathbb{R}^{2n}$ *containing the vectors* b_r, *for* $1 \leq r \leq m$, *and satisfying*

$$\mathsf{V} + A\mathsf{V} = \mathbb{R}^{2n}.$$

(iii) all controllability indices of Λ *equal at least two.*

The above proposition explains that all linear controllable systems (excepts for those possessing a controllability index equal to one) admit a linear mechanical structure. Moreover, such a structure is, in general, highly non unique: any n-dimensional linear subspace V satisfying (ii) of the above proposition leads to such a structure.

Affine connection control systems

Mechanical control systems subject

- neither to dissipative-type (or gyroscopic-type) forces, i.e., $d = 0$
- nor to uncontrolled forces, i.e., $g_0 = 0$

are called affine connection control systems and are thus defined as a 3-tuple $(\mathcal{ACS}) = (Q, \nabla, \mathfrak{g})$, with Q and ∇ as before and $\mathfrak{g} = (g_1, \ldots, g_m)$ an m-tuple of input vector fields on Q. For an $(\mathcal{ACS})$, we have

$$\begin{aligned} \dot{x}^i &= y^i, \\ \dot{y}^i &= -\Gamma^i_{jk}(x) y^j y^k + \sum_{r=1}^{m} u_r g^i_r(x), \end{aligned}$$

- Let $\mathrm{Sym}(\mathfrak{g})$ denote the smallest family of vector fields on Q containing $g_1, \ldots, g_m$ and closed under the symmetric product defined by the connection ∇. Elements of $\mathrm{Sym}(\mathfrak{g})$ are thus iterative symmetric products of vector fields $g_1, \ldots, g_m$.

- Let Let $\mathcal{SYM}(\mathfrak{g})$ be the distribution on Q spanned by $\mathrm{Sym}(\mathfrak{g})$.

- Recall that the system $(\mathcal{MS})$ is called geodesically accessible at $x_0 \in Q$ if

$$\mathcal{SYM}(\mathfrak{g})(x_0) = T_{x_0}Q,$$

  and geodesically accessible if the above equality holds for all $x_0 \in Q$.

- Geodesically accessible mechanical control systems are denoted by $(\mathcal{GAMS})$. If additionally, the system is affine connection then it will be called geodesically accessible affine connection system and it will be denoted shortly by $(\mathcal{GACS})$.

Conform frames

- The geodesic accessibility property guarantees the existence of n independent vector fields $v_1, \ldots, v_n \in \mathrm{Sym}(\mathfrak{g})$ and $\tilde{v}_1, \ldots, \tilde{v}_n \in \mathrm{Sym}(\tilde{\mathfrak{g}})$.
- Two frames $(v_1, \ldots, v_n)$ and $(\tilde{v}_1, \ldots, \tilde{v}_n)$, for two systems, are conform if each $\tilde{v}_j,\ 1 \le j \le n$, is constructed as an analogous iterative symmetric product as that defining v_j

Fundamental relations

- Fix a frame $(v_1, \ldots, v_n)$ and consider the fundamental equalities

$$\text{(LAR)} \qquad \Big[v_{i_q}, \ldots, \big[v_{i_3}, [v_{i_2}, v_{i_1}]\big] \ldots\Big] = \alpha^s_{i_1 \ldots i_q}\, v_s, \text{ and}$$

$$\text{(SAR)} \qquad \langle v_{i_q} : \ldots \langle v_{i_3} : \langle v_{i_2} : v_{i_1} \rangle\rangle \ldots \rangle = \beta^s_{i_1 \ldots i_q}\, v_s,$$

  defining the *structure functions* $\alpha^s_{i_1 \ldots i_q}$ and $\beta^s_{i_1 \ldots i_q}$, where $q \geq 2$ and $1 \leq i_1, \ldots, i_q \leq n$.

- Equalities (LAR) and (SAR) give, respectively, information about the Lie algebraic relations and the symmetric algebraic relations of the system.

- Analogously, we can derive the structure functions $\tilde{\alpha}^s_{i_1 \ldots i_q}$ and $\tilde{\beta}^s_{i_1 \ldots i_q}$ for $(\widetilde{\mathcal{GACS}})$. We consider the families of structure functions

$$\begin{aligned} \mathfrak{s} &= \{\alpha^s_{i_1 \ldots i_q},\ \beta^s_{i_1 \ldots i_q} \mid q \geq 2\} \quad \text{and} \\ \tilde{\mathfrak{s}} &= \{\tilde{\alpha}^s_{i_1 \ldots i_q},\ \tilde{\beta}^s_{i_1 \ldots i_q} \mid q \geq 2\} \end{aligned}$$

  defined by the Lie algebraic relations (LAR) and the symmetric algebraic relations (SAR).

Rank and order of a family of functions

- A family of smooth functions $\{\gamma^s_{i_1 \dots i_q} \mid q \geq 2\}$ is of a constant rank r, in an open neighborhood U of $x_0 \in Q$, if $\left\{ d\gamma^s_{i_1 \dots i_q}(x) \mid q \geq 2 \right\}$ span an r-dimensional space at any $x \in U$.
- We call the order of a family of constant rank r to be the minimal number ρ such that

$$\dim \operatorname{span} \left\{ d\gamma^s_{i_1 \dots i_q} \mid 2 \leq q \leq \rho \right\} (x_0) = r.$$

Eqivariants of mechanical control systems

Theorem

Two geodesically accessible affine connection systems $(\mathcal{GACS}) = (Q, \nabla, \mathfrak{g})$ *and* $(\widetilde{\mathcal{GACS}}) = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathfrak{g}})$, *whose families of structure functions* $\mathfrak{s}$ *and* $\tilde{\mathfrak{s}}$ *are of constant rank in neighborhoods of* $x_0 \in Q$ *and* $\tilde{x}_0 \in \tilde{Q}$, *are MS-equivalent around* x_0 *and* $\tilde{x}_0$, *respectively, if and only if there exists a diffeomorphism* $\varphi : W_{x_0} \to \tilde{W}_{\tilde{x}_0}$, *where* W_{x_0} *and* $\tilde{W}_{\tilde{x}_0}$ *are neighborhoods of* x_0 *and* $\tilde{x}_0$ *in* Q *and* $\tilde{Q}$, *respectively, such that*

$$\begin{aligned} (\mathrm{LAC}) \quad \alpha^s_{i_1 \ldots i_q} &= \tilde{\alpha}^s_{i_1 \ldots i_q} \circ \varphi, \\ (\mathrm{SAC}) \quad \beta^s_{i_1 \ldots i_q} &= \tilde{\beta}^s_{i_1 \ldots i_q} \circ \varphi, \end{aligned}$$

for $q \leq \rho + 1$, *with* ρ *being the common order of families* $\mathfrak{s}$ *and* $\tilde{\mathfrak{s}}$.

- (LAC) says that the Lie modules, generated by the symmetric vector fields $\mathrm{Sym}(g_1, \ldots, g_m)$ of $(\mathcal{GACS})$ and and $\mathrm{Sym}(\tilde{g}_1, \ldots, \tilde{g}_m)$ of $(\widetilde{\mathcal{GACS}})$, coincide (up to the conjugation by a diffeomorphism of the configuration manifolds Q and $\tilde{Q}$); (SAC) states that the symmetric modules, generated by all symmetric vector fields of $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$, coincide (up to the conjugation by the same diffeomorphism).

- If a diffeomorphism ϕ establishing the equivalence of $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$ exists then it is unique (since it transforms the frame $(v_1, \ldots, v_n)$ onto the frame $(\tilde{v}_1, \ldots, \tilde{v}_n)$ and $\phi(x_0) = \tilde{x}_0$). On the other hand, the diffeomorphism φ conjugating the structure functions may or may not be unique: we can distinguish three cases:

  (i) If $r = n$, that is, the families $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are of maximal possible rank, then the diffeomorphism φ conjugating them is unique and φ and ϕ coincide;

  (ii) If $r = 0$, which correspond to $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ consisting of constant functions only (homogenous case), then (LAC) and (SAC) imply that the structure functions have to be the same and, if this is the case, any diffeomorphism φ conjugates them;

  (iii) If $0 < r < n$, then only a "part" of the diffeomorphism ϕ is determined by the diffeomorphism φ.

Conclusions

- We described mechanical systems that admit a mechanical structure
- We constructed canonical mechanical structure
- Equivariants of mechanical control systems
- Linearization of mechanical and mechanizable control systems
- Perspectives : systems with particular structure (hamiltonian, on Lie groups etc), systems with nonholonomic constraints, reductions of mechanical control systems