Few remarks on Hamiltonian approach in classical theories

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Dual formulation

The system from the previous slide can be described by the set $C \subset \mathsf{T}^\star Q$ (constitutive set), defined by the variational principle:

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$\mathbb{T}$ is a one dimensional current in the space of parameters.
Two cases: $\mathbb{T} = [a, b]$, $\mathbb{T} = \frac{\partial}{\partial s}$ at $0 \in \mathbb{R}$.
Configurations $Q_{\mathbb{T}}$ are obtained by classification of curves in Q with respect to the action functional

$$L_{\mathbb{T}} :\ \gamma \mapsto \langle \mathbb{T}, L \circ \mathsf{t}\gamma \mathsf{d}s \rangle$$

As results we get:

For $\mathbb{T} = [a, b]$, $Q_{\mathbb{T}} = \{\gamma\colon [a, b] \to Q\}$, $L_{\mathbb{T}}([\gamma]) = \int_a^b L \circ \mathsf{t}\gamma$, elements of $\mathsf{T}Q_{\mathbb{T}}$ are curves in $\mathsf{T}Q$, $\hat{v}\colon [a, b] \to \mathsf{T}Q$.
A co-vector $\hat{a} \in \mathsf{T}^{\star}Q_{\mathbb{T}}$ can be represented by a pair of curves $f, p\colon [a, b] \to \mathsf{T}^{\star}Q$ with $\pi_Q \circ f = \pi_Q \circ p$. The evaluation between vectors and co-vectors is given by

$$\langle [(f, p)], \hat{v} \rangle = \langle p(b), \hat{v}(b) \rangle - \langle p(a), \hat{v}(a) \rangle - \int_a^b \langle f(s), \hat{v}(s) \rangle \mathsf{d}s$$

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For $\mathbb{T} = \frac{\partial}{\partial s}$, $Q_\mathbb{T} = \mathsf{T}Q$, $\mathsf{T}Q_\mathbb{T} = \mathsf{T}\mathsf{T}Q$, $L_\mathbb{T}(v) = L(v)$, and $\mathsf{T}^* Q_\mathbb{T} = \mathsf{T}^*\mathsf{T}Q$.

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One 2-dimensional case

In the dynamics of strings and statics of membranes, configurations are pieces of 2-dimensional submanifolds in Q equipped with a metric. Infinitesimal piece (a jet) we represent by a (simple) bi-vector on Q. Manifold of infinitesimal configurations is then $\wedge^2 \mathsf{T}Q$. The Nambu-Goto Lagrangian is given by $L(w) = \sqrt{(w|w)}$. Hamiltonian generating object is a Morse family

$$\begin{aligned} H\colon\ & \wedge^2 \mathsf{T}^*Q \times \mathbb{R}_+ \to \mathbb{R} \\ & : (p, r) \mapsto r(\sqrt{(p|p)} - 1) \end{aligned} \tag{1}$$

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The infinitesimal phase dynamics is a subspace $D \subset \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* Q$ given by $\mathrm{d}L,\ \mathrm{d}H$ via canonical mappings

$$\mathsf{T}^* \wedge^2 \mathsf{T}^* M \xleftarrow{\beta^2_M} \wedge^2 \mathsf{T} \wedge^2 \mathsf{T}^* M \xrightarrow{\alpha^2_M} \mathsf{T}^* \wedge^2 \mathsf{T} M$$

Analytical mechanics as a field theory

In field theories configurations are pieces of sections of a bundle. There is natural parametrization of a configuration by a domain of the base manifold. Let us see what happens in the case of the analytical mechanics of a point. The space-time is a manifold M fibred over time $T = \mathbb{R}$, $\tau\colon M \to T$. Motion of a point is a section of τ.

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$$T_1M = \{\mathsf{T}M \ni v : \mathsf{T}\tau(v) = \frac{\partial}{\partial t}\}$$

is an affine subbundle of $\mathsf{T}M$. The affine dual bundle can be identified with T^*M fibred by the pull-back of $\mathsf{d}t$, $\pi\colon \mathsf{T}^*M \to \mathsf{V}^*M$, where $\mathsf{V}M$ is the bundle of τ-vertical vectors.

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First order field theory

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Let us discuss the case of a scalar field over the space-time M with Lagrangian depending on first jets only. For simplicity:
$M = T \times Q$, $T = \mathbb{R}$ is the time, $Q = \mathbb{R}^3$ is the space. As a 4-dimensional current $\mathbb{T}$ on M we take the vector field $\frac{\partial}{\partial t}$ at $\Omega_t = \{t\} \times \Omega$.
Ω is a compact domain with smooth boundary $\partial\Omega$.

Evaluation between a 4-form α and $\mathbb{T}$:

$$\langle \mathbb{T}\,,\ \alpha \rangle = \int_{\Omega} \mathrm{i}_{\frac{\partial}{\partial t}} \alpha$$

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$\mathsf{T}^\star Q_\mathbb{T} = Q_\mathbb{T} \times Q_\mathbb{T}^\star$.
A co-vector $a \in Q_\mathbb{T}^\star$ can be represented by a pair (f, p),
$f\colon M \to \wedge^4 \mathsf{T}^* M$ and $p\colon M \to \wedge^3 \mathsf{T}^* M$.
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The evaluation between vectors and co-vectors is given by

$$\langle [(f, p)], [x] \rangle = \langle \mathbb{T}, \mathsf{d}(xp) - xf \rangle$$

.

More precisely

$p = \mathsf{d}t \wedge \bar{p} + p^0$ and

$$\mathrm{i}_{\frac{\partial}{\partial t}} \mathsf{d}(xp) = -\mathsf{d}^3(x\bar{p}) + \partial_t x p^0 + x \partial_t p^0$$

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Dual pairs: x and $\partial_t p^0 - \bar{f}$ on Ω; $\partial_t x$ and p^0 on Ω ; x and $\bar{p}$ on $\partial\Omega$.

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$x \in H^1(\Omega)$, $\partial_t p^0 - \bar{f} \in H^{-1}(\Omega)$, $\partial_t x \in H^0(\Omega)$, $p^0 \in H^0(\Omega)$,
$x|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$, $\bar{p}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$

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- For this we choose $x \in H^0(\Omega)$ and $\partial_t p^0 - \bar{f} \in H^0(\Omega)$
- This implies $\bar{f} = 0$ (no 'forces'), $\bar{p}|_{\partial\Omega} = 0$ (homogeneous boundary conditions)

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