SERIA 2. 6.03.2020

DATA ZWROTU: 13.03.2020

ZADANIE 2.1

Korzystając z rozwinięć funkcji elementarnych sprawdzić, że

$$\lim_{x\to 0} \frac{\sqrt{1+x}\,\log(1+x) - \sin x}{\cos x - (1-x)e^x} = \frac{3}{8}$$

$$\lim_{x\to 0} \left[\sqrt[3]{1+3x+\frac{27}{8}x^3} - \frac{x}{1+x} \right]^{\frac{1}{x^3}} = \exp\left(\frac{43}{24}\right)$$

ZADANIE 2.2

Obliczyć sumy szeregów

$$\sum_{n=1}^{\infty} \frac{(n+2)}{n(n+1)2^n} \qquad \sum_{n=0}^{\infty} (16n^2-4n+1)a^n$$

ZADANIE 2.3

Znaleźć rozwinięcie funkcji $f(x)=\sqrt{(x-a)(b-x)}$ w szereg potęgowy wokół $x_0 = (a+b)/2$.
Znaleźć promień zbieżności otrzymanego szeregu.

ZADANIE 2.1

$$\lim_{x\to 0} \frac{\sqrt{1+x}\,\log(1+x) - \sin x}{\cos x - (1-x)e^{x}} = \frac{3}{8}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$(x+1)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \dots$$

$$\frac{\left[1+\frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 - \dots\right]\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(5)\right] - \left[x - \frac{x^3}{6} + \frac{x^5}{5!} - O(7)\right]}{1 - \frac{x^2}{2} + \frac{x^4}{4!} - O(6) - (1-x)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + O(4)\right)} =$$

$$\frac{\left[x + \frac{1}{2}x^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^3 - \frac{1}{8}x^3 + O(4)\right] - x + \frac{x^3}{6} + O(5)}{\cancel{1} - \cancel{\frac{x^2}{2}} + \frac{x^4}{4!} + O(6) - \left[\cancel{1} - x + x + \cancel{\frac{x^2}{2}} - \cancel{x^2} + \frac{x^3}{6} - \frac{x^3}{2} + O(4)\right]} =$$

$$= \frac{\left(\frac{1}{3} - \frac{1}{4} - \frac{1}{8} + \frac{1}{6}\right)x^3 + O(4)}{-\frac{1}{6}x^3 + O(4)} = \frac{x^3\,\frac{8-6-3+4}{24} + O(4)}{-\frac{1}{6}x^3 + O(4)} =$$

$$= \frac{-3/24 + O(1)}{-1/6 + O(1)} \xrightarrow{x\to 0} \frac{3}{24}\,6 = \frac{3}{8}$$

$$\lim_{x\to 0}\left[\sqrt[3]{1+3x+\frac{27}{8}x^3} - \frac{x}{1+x}\right]^{\frac{1}{x^3}} = \exp\left(\frac{43}{24}\right)$$

$$\frac{1}{x^3}\log\left[\boxed{\sqrt[3]{1+3x+\frac{27}{8}x^3}} - \frac{x}{1+x}\right]$$

$$(1+y)^{1/3} = \Big|_{y=0} 1. \qquad \frac{1}{3}\left(-\frac{2}{3}\right)(1+y)^{-5/3} = \Big|_{y=0} -\frac{2}{9}$$

$$\frac{1}{3}(1+y)^{-2/3}\Big|_{y=0} = \frac{1}{3} \qquad \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(1+y)^{-8/3} = \Big|_{y=0} \frac{10}{27}$$

$$(1+y)^{1/3} = \Big|_{y=0} 1. \qquad \frac{1}{3}\left(-\frac{2}{3}\right)(1+y)^{-5/3} = \Big|_{y=0} -\frac{2}{9}$$

$$\frac{1}{3}(1+y)^{-2/3}\Big|_{y=0} = \frac{1}{3} \qquad \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(1+y)^{-8/3} = \Big|_{y=0} \frac{10}{27}$$

$$(1+y)^{1/3} = 1 + \frac{1}{3}y - \frac{1}{2}\cdot\frac{2}{9}y^2 + \frac{1}{\cancel{6}_{3}}\cdot\frac{\cancel{10}^{5}}{27}\cdot y^3 + \dots = 1 + \frac{1}{3}y - \frac{1}{9}y^2 + \frac{5}{81}y^3 - \dots$$

$$y = 3x + \frac{27}{8}x^3$$

$$(1+y)^{1/3} = 1 + \frac{1}{3}\left(3x + \frac{27}{8}x^3\right) - \frac{1}{9}\left(3x + \frac{27}{8}x^3\right)^2 + \frac{5}{81}\left(3x + \frac{27}{8}x^3\right)^3 - \dots =$$

$$= 1 + x + \frac{9}{8}x^3 - \frac{1}{9}\left(9x^2 + O(4)\right) + \frac{5}{81}27x^3 + O(5) =$$

$$= 1 + x - x^2 + x^3\left(\frac{9}{8} + \frac{5}{3}\right) + O(4) =$$

$$= 1 + x - x^2 + \frac{67}{24}x^3 + O(4)$$

$$\frac{x}{1+x} = \frac{x+1-1}{1+x} = 1 - \frac{1}{1+x} = 1 - \left(1 - x + x^2 - x^3 + O(4)\right) = x - x^2 + x^3 + O(4)$$

$$\sqrt[3]{1 + 3x + \frac{27}{8}x^3} - \frac{x}{1+x} = 1 + \cancel{x} - \cancel{x^2} + \frac{67}{24}x^3 - \cancel{x} + \cancel{x^2} - x^3 + O(4) =$$

$$= 1 + \frac{43}{24}x^3 + O(5)$$

$$\log\left(\sqrt[3]{1 + 3x + \frac{27}{8}x^3} - \frac{x}{1+x}\right) = \log\left(1 + \frac{43}{24}x^3 + O(4)\right) = \frac{43}{24}x^3 + O(4)$$

$$\frac{1}{x^3}\log\left(\sqrt[3]{1 + 3x + \frac{27}{8}x^3} - \frac{x}{1+x}\right) = \frac{43}{24} + O(1) \xrightarrow{x \to 0} \frac{43}{24}.$$

ZADANIE 2.2

$\sum_{n=1}^{\infty} \frac{(n+2)}{n(n+1)2^n}$: Rozważamy szereg $f(x) = \sum_{n=1}^{\infty} \frac{n+2}{n(n+1)}x^n$. Jest to szereg o promieniu zbieżności równym 1. $f(1/2) = \sum_{n=1}^{\infty} \frac{n+2}{n(n+1)2^n}$.

Szukamy $f(x)$:

$$\frac{n+2}{n(n+1)} = \frac{2}{n} - \frac{1}{n+1} \qquad f(x) = \sum_{n=1}^{\infty} \frac{n+2}{n(n+1)}x^n = \sum_{n=1}^{\infty} \frac{2}{n}x^n - \sum_{n=1}^{\infty} \frac{1}{n+1}x^n$$

$$f(x)=\sum_{n=1}^{\infty}\frac{n+2}{n(n+1)}x^n=\sum_{n=1}^{\infty}\frac{2}{n}x^n-\sum_{n=1}^{\infty}\frac{1}{n+1}x^n=-2\log(1-x)+\frac{1}{x}\log(1-x)+1=$$

$$=\boxed{\left(\frac{1}{x}-2\right)\log(1-x)+1}$$

$$2\sum_{n=1}^{\infty}\frac{x^n}{n}=-2\log(1-x)$$

$$\sum_{n=1}^{\infty}\frac{x^n}{n+1}=\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}=\frac{1}{x}\sum_{n=2}^{\infty}\frac{x^n}{n}=$$

$$=\frac{1}{x}\left(-\log(1-x)-x\right)$$

$$\sum_{n=1}^{\infty}\frac{n+2}{n(n+1)2^n}=f(1/2)=\left(\frac{1}{1/2}-2\right)\log(1/2)+1=1$$

$$\sum_{n=0}^{\infty}(16n^2-4n+1)a^n$$

szereg ma promień zbieżności 1.

Dla $|a|<1$:

$$\sum a^n=\frac{1}{1-a}\qquad \sum_{n=0}^{\infty}na^n=\sum_{n=1}^{\infty}na^n=a\sum_{n=1}^{\infty}na^{n-1}=a\left(\sum_{n=1}^{\infty}a^n\right)'=$$

$(a^n)'$

$$=a\left[\frac{1}{1-a}-1\right]'=\frac{a}{(1-a)^2}$$

$$\sum_{n=0}^{\infty}n^2a^n=\sum_{n=1}^{\infty}n^2a^n=a\sum_{n=1}^{\infty}n^2a^{n-1}=a\sum_{n=1}^{\infty}(na^n)'=a\left(\sum_{n=1}^{\infty}na^n\right)'=a\left[\frac{a}{(1-a)^2}\right]'=$$

$$=a\left[\frac{1}{(1-a)^2}+a\cdot(-2)\frac{1}{(1-a)^3}(-1)\right]=\frac{a}{(1-a)^3}\left[1-a+2a\right]=\frac{a(1+a)}{(1-a)^3}$$

$$\sum_{n=0}^{\infty}(16n^2-4n+1)a^n=16\frac{a(1+a)}{(1-a)^3}-4\frac{a}{(1-a)^2}+\frac{1}{(1-a)}=$$

$$=\frac{1}{(1-a)^3}\left[16a(1+a)-4a(1-a)+(1-a)^2\right]=\boxed{\frac{21a^2+10a+1}{(1-a)^3}}$$

$$16a+16a^2-4a+4a^2+1-2a+a^2$$

$$21a^2+10a+1$$

ZADANIE 3.1

$$f(x)=\sqrt{(x-a)(b-x)}$$

$$x_0 = (a+b)/2$$

$$x = x_0 + u \qquad x-a = \frac{a+b}{2} - a + u = \frac{b-a}{2} + u$$

$$b-x = b - \frac{a+b}{2} - u = \frac{b-a}{2} - u$$

$$(x-a)(b-x) = \left(\frac{b-a}{2}\right)^2 - u^2$$

$$\sqrt{\left(\frac{b-a}{2}\right)^2 - u^2} = \left(\frac{b-a}{2}\right)\sqrt{1 - \left(\frac{2u}{b-a}\right)^2} \qquad t = \frac{2u}{(b-a)}$$

$$\parallel$$

$$\left(\frac{b-a}{2}\right)\sqrt{1-t^2}$$

Rozwijamy $g(x) = (1-y)^{1/2}$

$$g(0) = 1$$

$$g'(y) = \frac{1}{2}(1-y)^{-1/2}(-1) = -\frac{1}{2}(1-y)^{-1/2}\Big|_{y=0} = -\frac{1}{2}$$

$$g''(y) = \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)(1-y)^{-3/2}(-1) = -\frac{1}{4}(1-y)^{-3/2}\Big|_{y=0} = -\frac{1}{4}$$

$$g^{(3)}(y) = \left(-\frac{1}{4}\right)\left(-\frac{3}{2}\right)(1-y)^{-5/2}(-1)\Big|_{y=0} = -\frac{3}{8}$$

$$g^{(4)}(y)\Big|_{y=0} = \left(-\frac{3}{8}\right)\left(-\frac{5}{2}\right)(-1) = -\frac{5!!}{16}$$

$$g^{(k)}(y)\Big|_{y=0} = -\frac{1}{2^k}(2k-3)!! \qquad k \geq 2$$

$$\sqrt{1-y} = 1 - \frac{1}{2}y - \sum_{k=2}^{\infty} \frac{1}{2^k}\frac{(2k-3)!!}{k!}y^k$$

$$y = t^2 = \frac{4u^2}{(b-a)^2} = \frac{4(x-x_0)^2}{(b-a)^2}$$

$$f(x) = 1 - \frac{2}{(b-a)^2}(x-x_0)^2 - \sum_{k=2}^{\infty}\frac{(2k-3)!!\,2^k}{k!\,(b-a)^{2k}}(x-x_0)^{2k}$$

Promień zbieżności

$$\sqrt[2k]{\frac{(2k-3)!!\;2^{k}}{k!\,(b-a)^{2k}}} = \frac{1}{(b-a)}\sqrt{2}\left\{\left(\frac{(2k-3)!!}{k!}\right)^{\frac{1}{k}}\right\}^{1/2} \longrightarrow \frac{2}{(b-a)}$$

$$\lim_{k\to\infty}\sqrt[k]{x_k} = \lim_{k\to\infty}\frac{x_{k+1}}{x_k}$$

$$\frac{(2k-1)!!}{(k+1)!}\cdot\frac{k!}{(2k-3)!!} = \frac{2k-1}{k} \xrightarrow{k\to\infty} 2$$

$$\boxed{R = \frac{b-a}{2}}$$