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ZADANIE DO SAMODZIELNEGO ROZWIĄZANIA: Obliczyć granice ciągów

$$x_n = \frac{3n-2}{2n+1} \qquad y_n = \frac{3n^3+4n+5}{5n^3+6n^2+7} \qquad z_n = \sqrt{2n^2+3n+1} - \sqrt{2n^2-3n+1}$$

WSPÓLNIE WYKAŻEMY $\sqrt[n]{a} \xrightarrow{n\to\infty} 1 \qquad \sqrt[n]{n} \xrightarrow{n\to\infty} 1$

PO JEDNYM CIĄGU DLA KAŻDEGO: Zbadać zbieżność i ewentualnie obliczyć granice następujących ciągów

(a) $\sqrt[100]{n^{100}+n^{99}} - n$

(b) $n\left(n+4\sqrt{n^2+n} - 2\sqrt{n^2-n} - 3\sqrt{n^2+2n}\right)$

(c) $\sqrt{n^2+\sqrt{n^3+\sqrt{n^5}}} - \sqrt{n^2+\sqrt{n^3}}$

(d) $n^2\left[\left(1+\frac{p}{n}\right)^q - \left(1+\frac{q}{n}\right)^p\right] \qquad p,q\in\mathbb{N}$

(e) $\sqrt[n]{5a^{2n}+4a^n+3} \qquad a>0$

(f) $\sqrt[n]{(3+x)^n+(1-x)^n} \qquad x\in\mathbb{R}$

(g) $\sin\left(\pi\sqrt{n^2+1}\right)$

(h) $\cos\left(\frac{\pi n^2}{n+3}\right)$

(i) $\frac{n^2}{2^n}$

(j) $\frac{n^n}{n!}$

(k) $\frac{n\,3^n+2n^2-5}{n!+1}$

(l) $\sqrt[3]{n^3+2n^2+3} - \sqrt{n^2+2n+3}$

(m) $\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}$ (bez granicy)

Nad fioletowymi pomyśleć do następnych ćwiczeń

Zielone rozwiązane

$$\sqrt{n^2+\sqrt{n^3+\sqrt{n^5}}} - \sqrt{n^2+\sqrt{n^3}} = \frac{\cancel{n^2}+\sqrt{n^3+\sqrt{n^5}}-\cancel{n^2}-\sqrt{n^3}}{\sqrt{\quad}+\sqrt{\quad}} =$$

$$= \frac{\sqrt{n^3+\sqrt{n^5}}-\sqrt{n^3}}{\sqrt{\quad}+\sqrt{\quad}} = \frac{\cancel{n^3}+\sqrt{n^5}-\cancel{n^3}}{\left(\sqrt{\quad}+\sqrt{\quad}\right)\left(\sqrt{n^3+\sqrt{n^5}}-\sqrt{n^3}\right)} = \frac{n^{5/2}}{\left(\sqrt{\quad}+\sqrt{\quad}\right)\left(\sqrt{\quad}+\sqrt{\quad}\right)}$$

$$= \frac{1}{\frac{1}{n}\left(\sqrt{n^2+\sqrt{n^3+\sqrt{n^5}}}+\sqrt{n^2+\sqrt{n^3}}\right)\frac{1}{n^{3/2}}\left(\sqrt{n^3+\sqrt{n^5}}+\sqrt{n^3}\right)} =$$

$$= \frac{1}{\left(\sqrt{1+\sqrt{\frac{1}{n}+\sqrt{\frac{1}{n^3}}}}+\sqrt{1+\sqrt{\frac{1}{n}}}\right)\left(\sqrt{1+\sqrt{\frac{1}{n}}}+1\right)} \xrightarrow{n\to\infty} \frac{1}{4}$$

0

$$\sqrt[3]{n^3+2n^2+3} - \sqrt{n^2+2n+3} = \sqrt[3]{\quad} - n + n - \sqrt{\quad} = \frac{\cancel{n^2}-\cancel{n^2}-2n+3}{n+\sqrt{n^2+2n+3}} +$$

$$+ \frac{\cancel{n^3}+2n^2+3-\cancel{n^3}}{(n^3+2n^2+3)^{2/3}+n(n^3+2n^2+3)^{1/3}+n^2} = \frac{-2+3/n}{1+\sqrt{1+\frac{2}{n}+\frac{3}{n^2}}} + \frac{2+3/n^2}{\left(1+\frac{2}{n}+\frac{3}{n^3}\right)^{2/3}+\left(1+\dots\right)^{1/3}+1}$$

$$\xrightarrow{n\to\infty} -1+\frac{2}{3} = -\frac{1}{3}$$

$$n\left(n+4\sqrt{n^2+n}-2\sqrt{n^2-n}-3\sqrt{n^2+2n}\right) =$$

$$= n\left[2\left(\sqrt{n^2+n}-\sqrt{n^2-n}\right)+2\left(\sqrt{n^2+n}-\sqrt{n^2+2n}\right)+\left(n-\sqrt{n^2+2n}\right)\right] =$$

$$= n\left[2\,\frac{\cancel{n^2}+n-\cancel{n^2}+n}{\sqrt{n^2+n}+\sqrt{n^2-n}} + 2\,\frac{\cancel{n^2}+n-\cancel{n^2}-2n}{\sqrt{n^2+n}-\sqrt{n^2+2n}} + \frac{\cancel{n^2}-\cancel{n^2}-2n}{n+\sqrt{n^2+2n}}\right] =$$

$$= n\left[\frac{4n}{\sqrt{n^2+n}+\sqrt{n^2-n}} + \frac{-2n}{\sqrt{n^2+n}-\sqrt{n^2+2n}} + \frac{-2n}{n+\sqrt{n^2+2n}}\right] =$$

$$= n\left[\frac{4n}{\sqrt{n^2+n}+\sqrt{n^2-n}} + \frac{-2n}{\sqrt{n^2+n}+\sqrt{n^2+2n}} + \frac{-2n}{n+\sqrt{n^2+2n}}\right] =$$

$$= 2n^2\left[\frac{1}{\sqrt{n^2+n}+\sqrt{n^2-n}} - \frac{1}{\sqrt{n^2+n}+\sqrt{n^2+2n}} + \frac{1}{\sqrt{n^2+n}+\sqrt{n^2-n}} - \frac{1}{n+\sqrt{n^2+2n}}\right] =$$

$$= 2n^2\left[\frac{\cancel{\sqrt{n^2+n}}+\sqrt{n^2+2n}-\cancel{\sqrt{n^2+n}}-\sqrt{n^2-n}}{(\quad)\cdot(\quad)} + \frac{n+\sqrt{n^2+2n}-\sqrt{n^2+n}-\sqrt{n^2-n}}{(\quad)(\quad)}\right]$$

$$2n^2\left[\frac{\dfrac{\cancel{n^2}+2n-\cancel{n^2}+n}{\sqrt{n^2-2n}+\sqrt{n^2-n}}}{(\sqrt{\quad}+\sqrt{\quad})(\sqrt{\quad}+\sqrt{\quad})} + \frac{\dfrac{\cancel{n^2}-\cancel{n^2}+n}{\sqrt{\quad}+\sqrt{\quad}} + \dfrac{\cancel{n^2}+2n-\cancel{n^2}-n}{\sqrt{\quad}+\sqrt{\quad}}}{(\sqrt{\quad}+\sqrt{\quad})(\sqrt{\quad}+\sqrt{\quad})}\right] =$$

$$= \frac{6n^3}{(\quad)(\quad)(\quad)} + 2n^2\,\frac{\overbrace{\frac{n}{(\quad)}}^{\to\frac12}+\overbrace{\frac{n}{(\quad)}}^{\to\frac12}}{(\quad)(\quad)} \xrightarrow[n\to\infty]{} \frac{3}{4}+\frac{2}{4}=\frac{5}{4}$$

$$\frac{6}{8}=\frac{3}{4} \qquad\qquad 2\cdot 2=4$$

$$\underbrace{\sqrt[100]{n^{100}+n^{99}}}_{a} - n = \frac{\cancel{n^{100}}+n^{99}-\cancel{n^{100}}}{\underbrace{a^{99}+a^{98}n+a^{97}n^2+\dots+an^{98}+n^{99}}_{100}} \xrightarrow[n\to\infty]{} \frac{1}{100}$$

$$\sqrt[3]{n^3+2n^2+3}-\sqrt{n^2+2n+3} = \sqrt[3]{n^3+2n^2+3}-\sqrt[3]{n^3}+\sqrt{n^2}-\sqrt{n^2+2n+3} =$$

$$= \frac{\cancel{n^3}+2n+3-\cancel{n^3}}{(\quad)^{2/3}+(\quad)^{1/3}n+n^{2/3}} + \frac{\cancel{n^2}-\cancel{n^2}-2n-3}{n+\sqrt{n^2+2n+3}} \longrightarrow \frac{2}{3}-1=-\frac{1}{3}$$

$$\cos\left(\frac{\pi n^2}{n+3}\right) = \cos\left(\pi(n-3)+\frac{9\pi}{n+3}\right) = \cos(\pi(n-3))\cos\left(\frac{9\pi}{n+3}\right) - \cancel{\underbrace{\sin(\pi(n-3))}_{0}\underbrace{\sin\left(\frac{9\pi}{n+3}\right)}_{\text{ograniczone}}}$$

$$\frac{n^2}{n+3} = n + \frac{-3n}{n+3} = n-3+\frac{9}{n+3} \qquad = (-1)^{n-3}\cos\left(\frac{9\pi}{n+3}\right)$$

$$\cos\left(\frac{9\pi}{n+3}\right) \to 1$$

ciąg nie ma granicy

$$\sin\left(\pi\sqrt{n^2+1}\right) = \sin\left(\pi\left(\sqrt{n^2+1}-n+n\right)\right) = \sin\Big(\underbrace{\pi\left(\sqrt{n^2+1}-n\right)}_{\alpha}+\pi n\Big) =$$

$$= \sin\alpha\cos(n\pi) + \cos\alpha\,\underbrace{\sin(n\pi)}_{=0} = (-1)^n\sin\left(\pi\frac{\cancel{n^2}+1-\cancel{n^2}}{n+\sqrt{n^2+1}}\right) = \underbrace{(-1)^n}_{\text{ograniczone}}\underbrace{\sin\left(\pi\frac{1}{n+\sqrt{n^2+1}}\right)}_{\to 0}$$

$$\xrightarrow{n\to\infty} 0$$

$$x_n = \frac{n^2}{2^n} \qquad \frac{x_{n+1}}{x_n} = \frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right)^2 \xrightarrow{n\to\infty} \frac{1}{2}$$

dla dostatecznie dużych n mamy więc $\frac{a_{n+1}}{a_n} < \frac{3}{4}$ zatem $a_{n+1} < \frac{3}{4}a_n$

$$a_{n+1} < \frac{3}{4}a_n < \left(\frac{3}{4}\right)^2 a_{n-1} < \dots < \left(\frac{3}{4}\right)^{n+1-N} a_N = \underbrace{\left(\frac{3}{4}\right)^n}_{\to 0}\underbrace{\left(\frac{3}{4}\right)^{1-N} a_N}_{\text{const}} \xrightarrow{n\to\infty} 0$$

Wiadomo więc, że $0 \leq a_{n+1} \leq z_n$ gdzie $z_n \xrightarrow{n\to\infty} 0$

z tw. o trzech ciągach $a_n \longrightarrow 0$

$$\frac{n^n}{n!}$$

Sposób I $\quad n! = 1\cdot 2\cdot 3\cdot\ldots\cdot n \leq 1\cdot n\cdot n\cdot\ldots\cdot n = n^{n-1}$

$$\frac{1}{n!} \geq \frac{1}{n^{n-1}}$$

$$\frac{n^n}{n!} \geq \frac{n^n}{n^{n-1}} = n \longrightarrow \infty$$

Ciąg $\frac{n^n}{n!}$ jest rozbieżny do ∞ przez porównanie z ciągiem rozbieżnym

Sposób II

$$x_n = \frac{n^n}{n!} \qquad \frac{x_{n+1}}{x_n} = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \frac{(n+1)^n\cancel{n!}}{\cancel{n!}\;n^n} = \left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty} e = 2{,}7\ldots > 2$$

d.d.d. n $\quad \frac{x_{n+1}}{x_n} > 2 \quad$ tzn $\quad x_{n+1} > 2x_n > 4x_{n-1} > \dots > 2^{n+1-N}x_N \xrightarrow{n\to\infty} \infty$