ANALIZA IR 2018/2019 ĆWICZENIA 21 21.12.2018

ROZGRZEWKA: OBLICZYĆ CAŁKI NIEOZNACZONE, TZN ZNALEŹĆ FUNKCJE PIERWOTNE

$$\int (x^2-2x+3)e^x\,dx$$

$$\int \sin^3 x\,dx$$

$$\int \sin^2 x \cos^4 x\,dx$$

$$\int \arcsin\left(\frac{x}{x+1}\right) dx$$

$$\int x^{-3/2}\log(1+\sqrt{x})\,dx$$

CAŁKOWANIE FUNKCJI WYMIERNYCH.

$$\int \frac{2x^4-x^2+1}{x^3-x}\,dx, \quad \int \frac{(x-1)^2}{(x+1)^3(x-4)}\,dx, \quad \int \frac{dx}{x^4-1}, \quad \int \frac{x^4+2x^2+4}{(1+x^2)^3}\,dx,$$

$$\int \frac{3x+1}{x(1+x^2)^2}\,dx$$

Rozwiązanie:

$$\underbrace{\int (x^2-2x+3)e^x\,dx}_{F(x)}$$

Obserwujemy że funkcje postaci $w(x)e^x$ są zamknięte ze względu na różniczkowanie, oraz zachowany jest stopień wielomianu

$$F(x) = (ax^2+bx+c)e^x \quad F'(x) = (2ax+b)e^x + (ax^2+bx+c)e^x = (ax^2+(2a+b)x+(b+c))e^x$$

$a=1 \quad 2a+b=-2 \quad b+c=3 \longrightarrow c = 3-b = 3+4 = 7$

$\quad \hookrightarrow b = -2-2a = -4$

$$F(x) = (x^2-4x+7)e^x + C$$

$$\int \sin^3 x\,dx = \left\{ \begin{array}{l} t = \cos x \\ dt = -\sin x\,dx \end{array} \right\} = \int \underset{(1-t^2)}{\sin^2 x}\, \underbrace{\sin x\,dx}_{-dt} = \int (t^2-1)\,dt = \frac{1}{3}t^3 - t + C = \frac{1}{3}\cos^3 t - \cos t + C$$

$$\int \sin^2 x \cos^4 x \, dx = \int (1-\cos^2 x)\cos^4 x \, dx = \int (\cos^4 x - \cos^6 x)\, dx$$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & &
\end{array}$$

$$\cos^4 x = \left[\frac{1}{2}\left(e^{ix}+e^{-ix}\right)\right]^4 = \frac{1}{2^4}\left[e^{4ix} + 4e^{3ix}e^{-ix} + 6e^{2ix}e^{-2ix} + 4e^{ix}e^{-3ix} + e^{-4ix}\right]$$

$$= \frac{1}{2^4}\left(\left[e^{4ix}+e^{-4ix}\right] + 4\left[e^{2ix}+e^{-2ix}\right] + 6\right) =$$

$$= \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}$$

$$\cos^6 x = \frac{1}{2^6}\left(e^{ix}+e^{-ix}\right)^6 = \frac{1}{2^6}\left(e^{6ix} + 6e^{5ix}e^{-ix} + 15e^{4ix}e^{-2ix} + 20e^{3ix}e^{-3ix} + 15e^{2ix}e^{-4ix} + 6e^{ix}e^{-5ix} + e^{-6ix}\right) =$$

$$= \frac{1}{32}\left(\cos 6x + 6\cos 4x + 15\cos 2x + 20\right) = \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x + \frac{15}{32}\cos 2x + \frac{5}{8}$$

$$\cos^4 x - \cos^6 x = -\frac{1}{32}\cos 6x + \left(\frac{1}{8} - \frac{3}{16}\right)\cos 4x + \left(\frac{1}{2} - \frac{15}{32}\right)\cos 2x + \frac{3}{8} - \frac{5}{8} =$$

$$= -\frac{1}{32}\cos 6x - \frac{1}{16}\cos 4x + \frac{1}{32}\cos 2x - \frac{1}{4}$$

$$\int \dots dx = -\frac{1}{32}\cdot\frac{1}{6}\sin 6x - \frac{1}{16}\cdot\frac{1}{4}\sin 4x + \frac{1}{32}\cdot\frac{1}{2}\sin 2x - \frac{1}{4}x + c =$$

$$= -\frac{1}{192}\sin 6x - \frac{1}{64}\sin 4x + \frac{1}{64}\sin 2x - \frac{1}{4}x + c$$

$$\int \arcsin\left(\frac{x}{1+x}\right) dx = \left\{\begin{array}{ll} f(x) = \arcsin\frac{x}{1+x} & g'(x) = 1 \\ f'(x) = \dfrac{1}{\sqrt{1-\frac{x^2}{(1+x)^2}}}\;\dfrac{1+x-x}{(1+x)^2} & g(x) = x \end{array}\right\} =$$

$$= x\arcsin\frac{x}{1+x} - \int \frac{x\,dx}{(1+2x+x^2-x^2)^{1/2}}\,\frac{\cancel{(1+x)}}{(1+x)^{\cancel{2}}} = x\arcsin\frac{x}{1+x} - \int \frac{x\,dx}{(1+x)\sqrt{1+2x}} =$$

$$u = \sqrt{1+2x} \qquad du = \frac{1}{2\sqrt{1+2x}}\cdot 2\,dx = \frac{dx}{\sqrt{1+2x}}$$

$$\int \frac{u^2-1}{u^2+1}\,du = \int 1 + \frac{-2}{u^2+1}\,du =$$

$$u - 2\,\text{arctg}\, u + c$$

$$u^2 = 1+2x$$

$$x = \frac{u^2-1}{2} \qquad \frac{x}{1+x} = \frac{\frac{u^2-1}{2}}{\frac{u^2-1}{2}+1} = \frac{u^2-1}{u^2+1}$$

$$= x\arcsin\frac{x}{x+1} - \sqrt{1+2x} - 2\,\text{arctg}\sqrt{1-2x} + C$$

$$\int x^{-3/2}\log\left(1+\sqrt{x}\right)dx = \int \frac{2}{t^2}\log(1+t)\,dt = \left\{\begin{array}{ll} f(t) = \log(1+t) & g'(t) = \frac{2}{t^2} \\ f'(t) = \frac{1}{1+t} & g(t) = -\frac{2}{t} \end{array}\right\} =$$

$$t = \sqrt{x}$$

$$dt = \frac{1}{2\sqrt{x}}\,dx \qquad \frac{dx}{\sqrt{x}} = 2\,dt \qquad = -\frac{2\log(1+t)}{t} + \int \frac{2}{t(1+t)}\,dt = -\frac{2\log(1+t)}{t} +$$

$$2\int\left(\frac{1}{t} - \frac{1}{1+t}\right)dt = -\frac{2\log(1+t)}{t} + 2\log t - 2\log(1+t) + C =$$

$$= -2\log\left(1+\sqrt{x}\right)\left(1/\sqrt{x} + 1\right) + \log x + C$$

$$\int \frac{2x^4 - x^2 + 1}{x^3 - x}\,dx = \int \left(2x + \frac{x^2+1}{x^3-x}\right) dx = x^2 + \int \frac{x^2+1}{x(x^2-1)} = *$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{Ax^2 - A + Bx^2 + Bx + Cx^2 - Cx}{\dots} = \frac{(A+C)x^2 + (B-C)x + (-A)}{\dots}$$

$$A+C=1 \qquad B-C=0 \qquad -A=1$$

$$A=-1 \qquad C=2 \qquad B=2$$

$$* = x^2 + \int\left(-\frac{1}{x} + \frac{2}{x-1} + \frac{2}{x+1}\right) dx = x^2 - \log|x| + 2\log|x-1| + 2\log|x+1| + c = x^2 + \log\frac{(x^2-1)^2}{|x|} + c$$

$$\int \frac{(x-1)^2}{(x+1)^3(x-4)}\,dx$$

$$\frac{A}{(x+1)^3} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)} + \frac{D}{(x-4)} = \frac{A(x-4) + B(x+1)(x-4) + C(x-1)^2(x-4) + D(x+1)^3}{\dots}$$

wartość w $x=4$: $D \cdot 5^3 = (4-1)^2 = 3^2$ $\boxed{D = 9/125}$

wartość w $x=-1$ $A\cdot(-1-4) = (-2)^2$ $-5A = 4$ $\boxed{A = -4/5}$

pochodna w $x=-1$ $B(-1-4) + A = 2(-1-1)$ $-5B + A = -2$ $-5B = -2 - A = -2 + \frac{4}{5} = -\frac{6}{5}$ $\boxed{B = \frac{6}{25}}$

druga pochodna w $x=-1$ $2B + 2C(-1-4) = 2$ $B - 5C = 1$ $-5C = 1 - \frac{6}{25} = \frac{19}{25}$ $C = -\frac{19}{125}$

$$\int\left(\frac{-4/5}{(x+1)^3} + \frac{6/25}{(x+1)^2} + \frac{-19/125}{(x+1)} + \frac{9/125}{x-4}\right) dx = \frac{4/15}{(x+1)^2} - \frac{3/25}{(x+1)} - \frac{19}{125}\log|x+1| + \frac{9}{125}\log|x-4| + C$$

UWAGA: Szukając funkcji pierwotnej dla (na przykład) $f(x) = \frac{1}{(x-1)(x+1)}$ piszemy zazwyczaj:

$$\frac{1}{(x+1)(x-1)} = \frac{1/2}{x-1} - \frac{1/2}{x+1} \quad \text{zatem} \quad \int \frac{dx}{(x+1)(x-1)} = \int\left(\frac{1/2}{x-1} - \frac{1/2}{x+1}\right) dx = \frac{1}{2}\log|x-1| - \frac{1}{2}\log|x+1| + C$$

Co jednak należy pod tym napisem rozumieć? f jest określone na $\mathbb{R}\setminus\{-1,1\}$, zatem F także ma taką dziedzinę. Stałe na każdym spójnym obszarze mogą być inne. Mamy więc trójparametrową rodzinę funkcji pierwotnych

$$F_{C_1,C_2,C_3}(x) = \begin{cases} \frac{1}{2}\log(1-x) - \frac{1}{2}\log(-x-1) + C_1 & \text{dla } x < -1 \\ \frac{1}{2}\log(1-x) - \frac{1}{2}\log(x+1) + C_2 & \text{dla } -1 < x < 1 \\ \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1) + C_3 & \text{dla } x > 1 \end{cases}$$

$$\int \frac{dx}{x^4-1} = \int \frac{dx}{(x^2+1)(x^2-1)} = \int \frac{dx}{(x^2+1)(x-1)(x+1)}$$

$$\frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{x+1} = \frac{(Ax+B)(x^2-1) + C(x+1)(x^2+1) + D(x-1)(x^2+1)}{\dots}$$

$$Ax^3+Bx^2-Ax-B+Cx^3+Cx^2+Cx+C+Dx^3-Dx^2+Dx-D = 1$$

$$x^3(A+C+D)+x^2(B+C-D)+x(-A+C+D)-B+C-D=1$$

$$\begin{array}{l} A+C+D=0 \\ B+C-D=0 \\ -A+C+D=0 \\ -B+C-D=1 \\ \hline 4C=1 \quad C=1/4 \end{array}$$

$2C+2D=0 \quad D=-C=-1/4$

$A=C+D=0$

$B=D-C=-1/2$

$$\frac{-1/2}{x^2+1} + \frac{1/4}{x-1} - \frac{1/4}{x+1} \qquad F(x) = -\frac{1}{2}\operatorname{arctg} x + \frac{1}{4}\log\left|\frac{x-1}{x+1}\right| + C$$

$$\frac{Ax+B}{(1+x^2)^3} + \frac{Cx+D}{(1+x^2)^2} + \frac{Ex+F}{1+x^2} :$$

$$\int \frac{x^4+2x^2+4}{(1+x^2)^3}\,dx = * \qquad Ax+B+\underbrace{(Cx+D)(1+x^2)}_{Cx^3+Dx^2+Cx+D}+\underbrace{(Ex+F)(1+x^2)^2}_{(Ex+F)(x^4+2x^2+1)} = x^4+2x^2+4$$

$$(Ex+F)(x^4+2x^2+1) = Ex^5+Fx^4+2Ex^3+2Fx^2+Ex+F$$

$$Ex^5+Fx^4+x^3(C+2E)+x^2(D+2F)+x(A+C+E)+(B+D+F)=x^4+2x^2+4$$

$E=0 \quad F=1 \quad C=0 \qquad D+2F=2 \qquad A+C+E=0 \qquad B+D+F=4$

$D+2=2 \qquad A=0 \qquad B+1=4 \quad B=3$

$D=0$

$$\stackrel{*}{=} \int\left(\frac{3}{(1+x^2)^3} + \frac{1}{1+x^2}\right)dx = \operatorname{arctg} x + \underbrace{3\int\frac{dx}{(1+x^2)^3}}$$ liczymy oddzielnie

$$\int\frac{1}{(1+x^2)^3}dx = \int\frac{1+x^2-x^2}{(1+x^2)^3} = \int\frac{dx}{(1+x^2)^2} - \int\frac{1}{2}x\,\frac{2x}{(1+x^2)^3} = \int\frac{dx}{(1+x^2)^2} - \left(\frac{1}{2}x\right)\left(-\frac{1}{2}\right)\frac{1}{(1+x^2)^2} +$$

$$\int\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\frac{dx}{(1+x^2)^2} = \frac{1/4\,x}{(1+x^2)^2} + \left(1-\frac{1}{4}\right)\int\frac{dx}{(1+x^2)^2} = \frac{\frac{1}{4}x}{(1+x^2)^2} + \frac{3}{4}\int\frac{dx}{(1+x^2)^2} =$$

$$= \frac{x}{4(1+x^2)^2} + \frac{3}{4}\int\left(\frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2}\right)dx = \frac{x}{4(1+x^2)^2} + \frac{3}{4}\operatorname{arctg} x - \frac{3}{4}\cdot\frac{1}{2}\int x\,\frac{2x}{(x^2+1)^2}\,dx =$$

$$= \frac{x}{4(1+x^2)^2} + \frac{3}{4}\operatorname{arctg} x - \frac{3}{8}\left[\frac{-x}{2(x^2+1)} + \int\frac{dx}{2(x^2+1)}\right] =$$

$$\frac{x}{4(1+x^2)^2} + \frac{3x}{16(x^2+1)} - \frac{3}{16}\operatorname{arctg} x + \frac{3}{4}\operatorname{arctg} x + C = \frac{x}{4(1+x^2)^2} + \frac{3x}{16(x^2+1)} + \frac{9}{16}\operatorname{arctg} x + C$$

raczej nie wierzę, że tu nie ma błędów...