ANALIZA IR 2018/2019 ĆWICZENIA 22 i 23 7 i 12.01.2019

ZADANIE 1 CAŁKI Z FUNKCJI WYMIERNYCH OD FUNKCJI TRYGONOMETRYCZNYCH

(a) $\int \frac{1+\sin x}{\sin x\,(1+\cos x)}\,dx$ podstawienie $t = \operatorname{tg}\frac{x}{2}$

(b) $\int \frac{dx}{\sin x \sin 2x}$ podstawienie $t=\cos x$ lub $t=\sin x$

(c) $\int \frac{2\operatorname{tg}x+3}{\sin^2 x + 2\cos^2 x}\,dx$ podstawienie $t=\operatorname{tg}x$

ZADANIE 2 PODSTAWIENIA HIPERBOLICZNE I TRYGONOMETRYCZNE DLA CAŁEK Z PIERWIASTKAMI

(a) $\int \frac{dx}{(x^2+a^2)\sqrt{x^2+b^2}}$ $x=b\operatorname{tg}t$ rozważyć $0<a<b$ i $0<b<c$

(b) $\int \frac{dx}{\sqrt{x^2+2x+2}}$ $x+1=\operatorname{tg}t$ lub $x+1=\sinh t$

(c) $\int_0^1 \frac{dx}{5+3\sqrt{1-x^2}}$ $x=\sin t$

Zadanie 7. Podstawienie Eulera służą do znajdowania funkcji pierwotnych do funkcji postaci

$$f(x) = R(x, \sqrt{ax^2+bx+c}),$$

gdzie R jest wymierną funkcją dwóch zmiennych. Chodzi o to, żeby funkcję zawierającą pierwiastek sprowadzić do funkcji wymiernej. Jeśli oznaczymy $y := \sqrt{ax^2+bx+c}$ sprowadza się to do sparametryzowania fragmentu krzywej

$$x \mapsto (x, \sqrt{ax^2+bx+c}) \in \mathbb{R}^2$$

dla $y>0$ parametrem t w taki sposób, aby $x(t)$ i $y(t)$ były wymierne.

(1) Pierwsze podstawienie Eulera działa gdy $a>0$, podstawiamy $y=\sqrt{a}x+t$. Wyznaczamy $x(t)$, $y(t)$ i $dx = x'(t)dt$;
(2) Drugie podstawienie Eulera działa gdy $c>0$, podstawiamy $y=tx+\sqrt{c}$;
(3) Trzecie podstawienie Eulera działa gdy łatwo jest wybrać punkt (x_0,y_0) na krzywej, $y_0=\sqrt{ax_0^2+bx_0+c}$, podstawiamy $y-y_0=t(x-x_0)$. Prowadząc rachunki warto zapisać y^2 w potęgach $x-x_0$ zamiast x.

Zadanie polega na obliczeniu trzema sposobami całki nieoznaczonej

$$\int \frac{dx}{x+\sqrt{x^2-x+1}} \quad \text{oraz jej wersji oznaczonej} \quad \int_0^1 \frac{dx}{x+\sqrt{x^2-x+1}}.$$

$$\int \frac{1+\sin x}{\sin x\,(1+\cos x)}\,dx = *$$

Uniwersalne podstawienie umożliwiające otrzymanie całki z funkcji wymiernych z całek z funkcji wymiernych od funkcji trygonometrycznych:

$$t = tg\frac{x}{2} \qquad dt = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\,dx = \frac{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}\,dx = \left(1+tg^2\frac{x}{2}\right)\cdot\frac{1}{2}\,dx = (1+t^2)\cdot\frac{1}{2}\,dx$$

$$dx = \frac{2dt}{1+t^2} \qquad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,\frac{\sin x/2\,\cos x/2}{\sin^2 x/2+\cos^2 x/2} = 2\,\frac{tg\,x/2}{1+tg^2 x/2} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2 x/2 - \sin^2 x/2}{\cos^2 x/2 + \sin^2 x/2} = \frac{1-tg^2 x/2}{1+tg^2 x/2} = \frac{1-t^2}{1+t^2}$$

$$* = \int \frac{1+\frac{2t}{1+t^2}}{\frac{2t}{1+t^2}\left(1+\frac{1-t^2}{1+t^2}\right)}\,\frac{2dt}{1+t^2} = \int \frac{(t^2+2t+1)\cancel{(1+t^2)^2}}{\cancel{(1+t^2)}\,\cancel{2}t\,(1+t^2+1-t^2)\cancel{(1+t^2)}}\,\cancel{2}dt = \int \frac{t^2+2t+1}{2t}\,dt = \frac{1}{2}\int\left(t+2+\frac{1}{t}\right)dt =$$

$$= \frac{1}{2}\left[\frac{1}{2}t^2+2t+\log|t|\right]+C = \frac{1}{4}t^2+t+\frac{1}{2}\log|t|+C = \boxed{\frac{1}{4}tg^2\frac{x}{2}+tg\frac{x}{2}+\frac{1}{2}\log\left|tg\frac{x}{2}\right|+C}$$

Jeśli pod całką są jedynie parzyste potęgi sin i cos, tzn $\sin^{2k}x$, $\cos^{2l}x$, $\sin^k x\cos^k x$, $tg^l x$ można podstawić $t = tg\,x$

$$t = tg\,x \quad dt = \frac{1}{\cos^2 x}\,dx = (1+t^2)\,dx \quad dx = \frac{dt}{1+t^2} \quad \sin^2 x = \frac{t^2}{1+t^2} \quad \cos^2 x = \frac{1}{1+t^2} \quad \sin x\cos x = \frac{t}{1+t^2}$$

$$\int \frac{2tg\,x+3}{\sin^2 x+2\cos^2 x}\,dx = \int \frac{2t+3}{\frac{t^2}{1+t^2}+\frac{2}{1+t^2}}\,\frac{dt}{1+t^2} = \int\frac{2t+3}{t^2+2}\,dt = \log(t^2+2)+\int\frac{3dt}{t^2+2} =$$

$$\log(t^2+2)+\frac{3}{2}\int\frac{dt}{1+(t/\sqrt{2})^2} = \log(t^2+2)+\frac{3}{2}\sqrt{2}\,arctg\left(\frac{t}{\sqrt{2}}\right)+C = \log(tg^2x+2)+\frac{3\sqrt{2}}{2}\,arctg\left(\frac{tg\,x}{\sqrt{2}}\right)+C$$

Czasami nadają się też inne podstawienia:

$$\int\frac{1}{\sin x\sin 2x}\,dx = \int\frac{1}{2\sin^2 x\cos x}\,dx = \int\frac{\cos x\,dx}{2\sin^2 x\cos^2 x} = \int\frac{-d(\sin x)}{2\sin^2 x(1-\sin^2 x)} = -\int\frac{dy}{2y^2(1-y^2)} = *$$

$$\frac{A}{y^2}+\frac{B}{y}+\frac{C}{1-y}+\frac{D}{1+y} = \frac{A(1-y^2)+By(1-y^2)+Cy^2(1+y)+Dy^2(1-y)}{\dots} = \frac{1}{\dots}$$

$y=0 \quad A=1$

$y=1 \quad 2C=1 \quad C=1/2$

$y=-1 \quad 2D=1 \quad D=1/2$

pochodne w $y=0$: $B=0$

$$* = \int\left(\frac{1}{y^2}+\frac{1/2}{1-y}+\frac{1/2}{1+y}\right)dy = -\frac{1}{y}-\frac{1}{2}\log(1-y)+\frac{1}{2}\log(1+y)+C =$$

$$= -\frac{1}{y}+\frac{1}{2}\log\frac{(1+y)}{1-y}+C = -\frac{1}{\sin x}+\frac{1}{2}\log\frac{1+\sin x}{1-\sin x}+C$$

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{dx}{\sqrt{(x+1)^2+1}} \overset{*}{=} \int \frac{\cosh\alpha \, d\alpha}{\sqrt{sh^2\alpha+1}} = \int d\alpha = \alpha + C = arsh(x+1) + C =$$

$$= \log\left(x+1+\sqrt{x^2+2x+1}\right) + c$$

$x+1 = \sinh\alpha \qquad dx = \cosh\alpha \, d\alpha$

$x+1 = tg\varphi \quad (1+x)^2+1 = tg^2\varphi + 1 = \frac{1}{\cos^2\varphi} \qquad dx = \frac{1}{\cos^2\varphi} d\varphi \qquad \varphi \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\qquad \frac{\varphi}{2} \in \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$

$t \in]-1, 1[$

$$* = \int \frac{d\varphi}{\cos^2\varphi \cdot \frac{1}{\cos\varphi}} = \int \frac{d\varphi}{\cos\varphi} = \int \frac{\cos\varphi}{\cos^2\varphi} d\varphi = \int \frac{d(\sin\varphi)}{1-\sin^2\varphi} = \int \frac{dy}{1-y^2} = \frac{1}{2}\int\left(\frac{1}{1-y} + \frac{1}{1+y}\right) dy$$

$$= \frac{1}{2}\left[\log(1-y)(-1) + \log(1+y)\right] + C = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) + c = \frac{1}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) + c =$$

$$= \log \frac{1+\sin\varphi}{\cos\varphi} + c = \log\left(tg\varphi + \frac{1}{\cos\varphi}\right) + c =$$

$$= \log\left(x+1+\sqrt{x^2+2x+2}\right) + c$$

$$\boxed{\int_0^1 \frac{dx}{x+\sqrt{x^2-x+1}}}$$

EULER I

$y = \sqrt{a}\,x + t \qquad y = x+t \qquad \cancel{x^2} - x + 1 = \cancel{x^2} + 2xt + t^2 \qquad -x+1 = 2xt + t^2$

$2xt + x = 1 - t^2 \qquad x(2t+1) = 1 - t^2 \qquad x = \frac{1-t^2}{2t+1}$

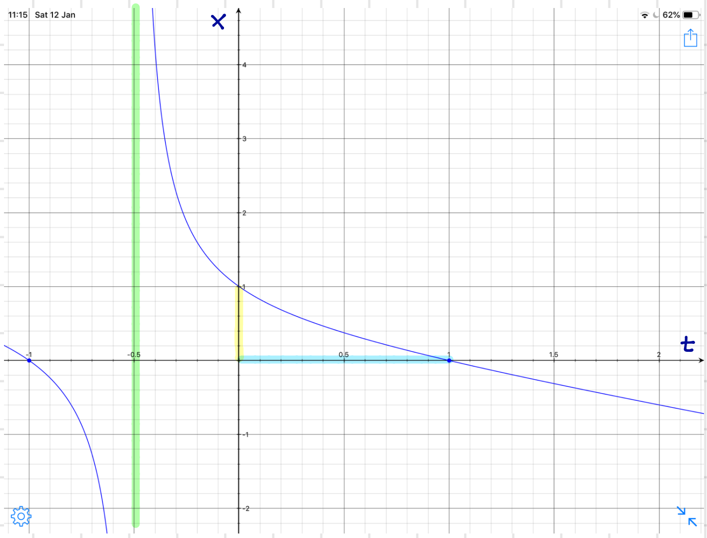
$$y = x + t = \frac{1-t^2+2t^2+t}{2t+1} = \frac{t^2+t+1}{2t+1}$$

$$dx = \frac{-2t(2t+1) - 2(1-t^2)}{(2t+1)^2} =$$

$$= \frac{-4t^2-2t-2+2t^2}{(2t+1)^2} dt = \frac{-2t^2-2t-2}{(2t+1)^2} dt = (-2)\frac{t^2+t+1}{(2t+1)^2} dt$$

$$\int \frac{dx}{x+\sqrt{}} = \int \frac{(-2)\frac{t^2+t+1}{(2t+1)^2} dt}{\frac{1-t^2}{2t+1} + \frac{t^2+t+1}{2t+1}} =$$

$$= -2\int \frac{t^2+t+1}{(2t+1)(t+2)} dt =$$



$$\frac{t^2+t+1}{2t^2+5t+2} = \frac{1}{2} + \frac{-\frac{3}{2}t}{2t^2+5t+2}$$

$$= -t + \int \frac{3t}{(2t+1)(t+2)} =$$

$$= -t + \int\left(\frac{-1}{2t+1} + \frac{2}{t+2}\right) dt = -t - \frac{1}{2}\log|2t+1| + 2\log|t+2| + C$$

$$\frac{A}{2t+1} + \frac{B}{t+2} = \frac{At+2A+2Bt+B}{\dots} \qquad \begin{array}{l} A+2B = 3 \\ 2A+B = 0 \quad /\cdot 2 \end{array} \qquad \begin{array}{ll} -3A = 3 & A = -1 \\ B = -2A & B = 2 \end{array}$$

Całka oznaczona (w zmiennej t od 1 do 0)

$$-\frac{1}{2}\log(1) + 2\log 2 + 1 + \frac{1}{2}\log 3 - 2\log 3 = 1 + 2\log 2 - \frac{3}{2}\log 3 \approx 0{,}7384$$

$$-t-\frac{1}{2}\log|2t+1|+2\log|t+2|+C \qquad t=\sqrt{\quad}-x$$

$$x-\sqrt{\quad}-\frac{1}{2}\log\left(2\sqrt{\quad}-2x+1\right)+2\log\left(\sqrt{\quad}-x+1\right)$$

WolframAlpha

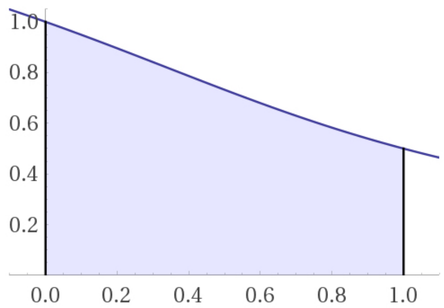
integral 1/(x+√(x^2-x+1)) for x from 0 to 1

Definite integral

$$\int_0^1 \frac{1}{x+\sqrt{x^2-x+1}}\,dx = 1+\log\left(\frac{4}{3}\right)-\sinh^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 0.7384$$

More digits | Step-by-step solution

Visual representation of the integral



Indefinite integral

$$\int \frac{1}{x+\sqrt{x^2-x+1}}\,dx = -\sqrt{x^2-x+1}+\log\left(2\sqrt{x^2-x+1}+x+1\right)+x-\frac{1}{2}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+\text{constant}$$

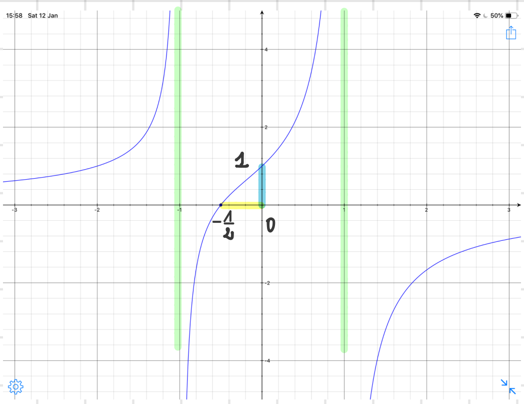
EULER II

$y=tx+\sqrt{c}$ $\quad y=tx+1$ $\quad x^2-x+\cancel{1}=t^2x^2+2tx+\cancel{1}$ $\quad x^2-x=tx^2+2tx$

$x-1=t^2x+2t$ $\quad x-t^2x=2t+1$ $\quad x(1-t^2)=2t+1$ $\quad x=\dfrac{2t+1}{1-t^2}$ $\quad y=\dfrac{2t^2+t}{1-t^2}+1=\dfrac{2t^2+t+1-t^2}{1-t^2}$

$=\dfrac{1+t+t^2}{1-t^2}$ $\quad dx=\dfrac{2(1-t^2)+2t(2t+1)}{(1-t^2)^2}\,dt=$

$=\dfrac{2-2t^2+4t^2+2t}{(1-t^2)^2}\,dt=2\dfrac{1+t+t^2}{(1\ t^2)^2}\,dt$



$$\int \frac{dx}{x+\sqrt{x^2-x+1}} = 2\int \frac{\frac{1+t+t^2}{(1-t^2)^2}\,dt}{\frac{2t+1}{1-t^2}+\frac{1+t+t^2}{1-t^2}} = 2\int \frac{1+t+t^2}{(1-t^2)(t^2+3t+2)}\,dt = 2\int \frac{1+t+t^2}{(1-t)(1+t)(t+1)(t+2)}\,dt =$$

$$= 2\int \frac{1+t+t^2}{(1+t)^2(1-t)(t+2)}\,dt =$$

$$\frac{A}{(1+t)^2}+\frac{B}{(1+t)}+\frac{C}{(1-t)}+\frac{D}{t+2} \qquad A(1-t)(t+2)+B(1+t)(1-t)(t+2)+C(1+t)^2(t+2)+D(1+t)^2(1-t) = 1+t+t^2$$

$t=1 \qquad C\cdot 4\cdot 3 = 3 \qquad C=\frac{1}{4}$

$t=-1 \qquad A\cdot 2\cdot 1 = 1 \qquad A=\frac{1}{2}$

$t=0 \qquad 2A+2B+2C+D=1$

$1+2B+\frac{1}{2}+1=1$

$2B+3/2=0$

$B=-\frac{3}{4}$

$t=-2 \qquad D\cdot 1\cdot 3 = 3 \qquad D=1$

$$= 2\int\left(\frac{1/2}{(1+t)^2}-\frac{3/4}{(1+t)}+\frac{1/4}{(1-t)}+\frac{1}{t+2}\right)dt = 2\left[-\frac{1/2}{1+t}-\frac{3}{4}\log|1+t|-\frac{1}{4}\log|1-t|+\log|t+2|\right]+C =$$

$$= -\frac{1}{1+t}-\frac{3}{2}\log|1+t|-\frac{1}{2}\log|1-t|+2\log|t+2|+C$$

całka oznaczona: t od $-\frac{1}{2}$ do 0

$$-1+2\log 2+\frac{1}{1-\frac{1}{2}}+\frac{3}{2}\log\frac{1}{2}+\frac{1}{2}\log\frac{3}{2}-2\log\frac{3}{2} = -2+4+2\log 2-\frac{3}{2}\log 2+\frac{1}{2}\log 3-\frac{1}{2}\log 2-2\log 3+2\log 2$$

$= 1+2\log 2-\frac{3}{2}\log 3$ WYNIK JAK POPRZEDNIO.

EULER III

Bierzemy $y_0=1$ $x_0=1$ $\qquad (y-y_0)=t(x-x_0) \qquad y=t(x-1)+1 \qquad x^2-x+1 = t^2(x-1)^2+2t(x-1)+1$

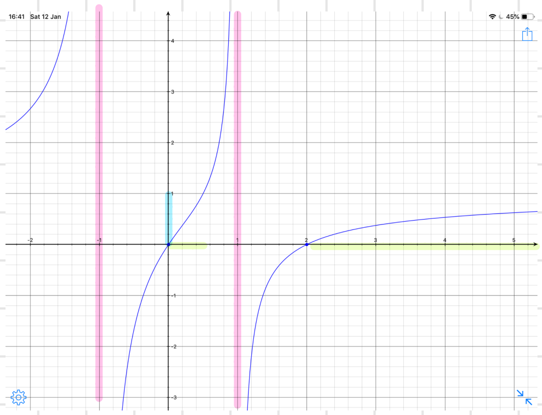
$$x^2-x+1 = (x-1)^2+2x-1+x+1 = (x-1)^2+(x-1)+1$$

$$(x-1)^2+(x-1)+\cancel{1} = t^2(x-1)^2+2t(x-1)+\cancel{1} \qquad (x-1)+1 = t^2(x-1)+2t$$

$$x = t^2x-t^2+2t \qquad x(1-t^2) = 2t-t^2 \qquad x = \frac{2t-t^2}{1-t^2}$$

$$y = t(x-1)+1 = t\left(\frac{2t-\cancel{t^2}-1+\cancel{t^2}}{1-t^2}\right)+1 = \frac{2t^2-t+1-t^2}{1-t^2} = \frac{t^2-t+1}{1-t^2}$$

$$dx = \frac{(2-2t)(1-t^2)+2t(2t-t^2)}{(1-t^2)^2}\,dt = \frac{2-2t-2t^2+\cancel{2t^3}+4t^2-\cancel{2t^3}}{(1-t^2)^2}\,dt = 2\,\frac{1-t+t^2}{(1-t^2)^2}\,dt$$



$t\in[0,\frac{1}{2}]$ lub $t\in[2,+\infty[$

w tym przedziale $y<0$, więc to nie jest dobry przedział.

$$\int \frac{dx}{x+\sqrt{x^2-x+1}} = 2\int \frac{\frac{t^2-t+1}{(1-t^2)^2}}{\frac{2t-t^2}{1-t^2}+\frac{t^2-t+1}{1-t^2}}\,dt = 2\int \frac{t^2-t+1}{(1-t^2)(t+1)}\,dt = 2\int \frac{t^2-t+1}{(t+1)^2(1-t)}\,dt$$

$$\frac{A}{(t+1)^2}+\frac{B}{(t+1)}+\frac{C}{1-t} \qquad A(1-t)+B(1+t)(1-t)+C(1+t)^2 = t^2-t+1$$

$t=1 \qquad 4C=1 \qquad C=1/4$

$t=-1 \qquad 2A=3 \qquad A=3/2$

$t=0 \qquad A+B+C=1 \qquad \frac{3}{2}+\frac{1}{4}+B=1 \quad B=1-\frac{6}{4}-\frac{1}{4}=-\frac{3}{4}$

$$=2\int\left(\frac{3/2}{(1+t)^2}-\frac{3/4}{(t+1)}+\frac{1/4}{1-t}\right)dt = 2\left[\frac{-3/2}{1+t}-\frac{3}{4}\log|1+t|-\frac{1}{4}\log|1-t|\right]+C=$$

$$=\frac{-3}{1+t}-\frac{3}{2}\log|1+t|-\frac{1}{2}\log|1-t|+C$$

Całka oznaczona dla t od 0 do $1/2$

$$\frac{-3}{1+1/2}-\frac{3}{2}\log\frac{3}{2}-\frac{1}{2}\log\frac{1}{2}+3=-\frac{\cancel{3}\cdot 2}{\cancel{3}}+3-\frac{3}{2}\log 3+\frac{3}{2}\log 2+\frac{1}{2}\log 2=$$

$$=1-\frac{3}{2}\log 3+2\log 2$$ OK.

$$\int \frac{dx}{(x^2+a^2)\sqrt{x^2+b^2}} \qquad x=b\,tg\,t \quad \text{rozważyć} \quad 0<a<b \ \text{ i } \ 0<b<c$$

$x=b\,tg\,t \qquad x^2+b^2=b^2tg^2t+b^2=b^2(tg^2t+1)=\frac{b^2}{\cos^2t} \qquad t\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[\qquad dx=\frac{1}{\cos^2t}\,dt$

$$\int\frac{dx}{(x+a^2)\sqrt{b^2+x^2}}=\int\frac{dt/\cos^2t}{(b^2tg^2t+a^2)\,1/\cos t}=\int\frac{\cos t\,dt}{(b^2\sin^2t+a^2\cos^2t)\cos^2t}=\int\frac{dy}{[b^2y^2+a^2(1-y^2)](1-y^2)}=$$

$y=\sin t \quad dy=\cos t\,dt$

$$=\int\frac{dy}{[a^2+(b^2-a^2)y^2][1-y^2]}=\int\left(\frac{1-a^2/b^2}{a^2+(b^2-a^2)y^2}+\frac{1/b^2}{1-y^2}\right)dy$$

$$\frac{A}{a^2+(b^2-a^2)y^2}+\frac{B}{1-y^2} \qquad A(1-y^2)+B(a^2+(b^2-a^2)y^2)=(A+a^2B)+y^2\left(B(b^2-a^2)-A\right)=1$$

$$\begin{aligned} A+a^2B&=1\\ -A+(b^2-a^2)B&=0\\ \hline b^2B&=1 \qquad B=1/b^2\end{aligned} \qquad A=B(b^2-a^2)=\frac{(b^2-a^2)}{b^2}=1-a^2/b^2$$

$$\frac{1}{b^2}\int\frac{dy}{1-y^2}=\frac{1}{2b^2}\int\left(\frac{1}{1-y}+\frac{1}{1+y}\right)dy=\frac{1}{2b^2}\left(-\log|1-y|+\log|1+y|\right)+C=\frac{1}{2b^2}\log\left|\frac{1+y}{1-y}\right|=$$

$$=\frac{1}{2b^2}\log\left(\frac{1+\sin t}{1-\sin t}\right)=\frac{1}{2b^2}\log\frac{(1+\sin t)^2}{\cos^2t}=\frac{1}{b^2}\log\left(tg\,t+\frac{1}{\cos t}\right)=\frac{1}{b^2}\log\left(\frac{x}{b}+\frac{1}{b}\sqrt{x^2+b^2}\right)$$

$$\int \frac{1-\frac{a^2}{b^2}}{a^2+(b^2-a^2)y^2}\,dy$$

$b>a$

$$= \left(1-\frac{a^2}{b^2}\right)\frac{1}{a^2}\int \frac{dy}{1+\left(\frac{b^2}{a^2}-1\right)y^2} = \left(\frac{1}{a^2}-\frac{1}{b^2}\right)\operatorname{arctg}\left(\sqrt{b^2-a^2}\,\frac{y}{a}\right)\cdot\frac{\sqrt{b^2-a^2}}{a} + c$$

$$= \frac{(b^2-a^2)^{3/2}}{a^3b^2}\operatorname{arctg}\left(\sqrt{b^2-a^2}\,\frac{\sin t}{a}\right) = \operatorname{sgn} x\,\frac{(b^2-a^2)^{3/2}}{a^3b^2}\operatorname{arctg}\left(\frac{\sqrt{b^2-a^2}}{a}\sqrt{\frac{\sqrt{b^2+x^2}-b}{\sqrt{b^2+x^2}}}\right) + C$$

$$\frac{b}{\cos^2 t} = \sqrt{x^2+b^2} \qquad \frac{b}{\sqrt{\;\;}} = \cos^2 t \qquad \sin^2 t = 1-\frac{b}{\sqrt{\;\;}} = \frac{\sqrt{\;\;}-b}{\sqrt{\;\;}}$$

$$\sin t = \pm\sqrt{\frac{\sqrt{\;\;}-b}{\sqrt{\;\;}}} = \pm\frac{\sqrt{x^2+b^2-b\sqrt{\;\;}}}{\sqrt{x^2+b^2}}$$

$b<a$

... nie chce mi się liczyć - rozkładamy na ułamki proste.