Rozwiązania do zad. 8 a) b) c) d) – seria z całkami

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a) $F_n(x) = \int \left(\frac{x^2}{x^2+1}\right)^n dx$ przez części:

$f'(x) = x^{2n}$

$f(x) = \frac{1}{1+2n} x^{2n+1}$

$g(x) = \frac{1}{(x^2+1)^n}$

$g'(x) = \frac{0(\ldots) - 1 \cdot n(x^2+1)^{n-1} \cdot 2x}{(x^2+1)^{2n}}$

$g'(x) = \frac{-2nx}{(x^2+1)^{n+1}}$

$$\int \left(\frac{x^2}{x^2+1}\right)^n dx = \frac{x^{2n+1}}{(2n+1)(x^2+1)^n} - \int \frac{x^{2n+1}(-2nx)}{(2n+1)(x^2+1)^{n+1}} dx$$

$$\int \left(\frac{x^2}{x^2+1}\right)^n dx = \frac{x^{2n+1}}{(2n+1)(x^2+1)^n} + \frac{2n}{(2n+1)} \int \frac{(x^2)^{n+1}}{(x^2+1)^{n+1}} dx$$

$$\frac{2n}{2n+1} \int \left(\frac{x^2}{x^2+1}\right)^{n+1} dx = \int \left(\frac{x^2}{x^2+1}\right)^n dx - \frac{x^{2n+1}}{(2n+1)(x^2+1)^n} \quad // \cdot \frac{2n+1}{2n}$$

$$\int \left(\frac{x^2}{x^2+1}\right)^{n+1} dx = \frac{2n+1}{2n} \cdot \int \left(\frac{x^2}{x^2+1}\right)^n dx - \frac{x^{2n+1}}{(2n+1)(x^2+1)^n}$$

$$\boxed{F_{n+1}(x) = \frac{2n+1}{2n} \cdot F_n(x) - \frac{x^{2n+1}}{(2n+1)(x^2+1)^n}}$$

b)

$$F_n(x) = \int \frac{1}{x(x^2+1)^n}\,dx = \int \frac{x^2+1}{x(x^2+1)^{n+1}}\,dx = \int \frac{x^2}{x(x^2+1)^{n+1}}\,dx + \int \frac{1\,dx}{x(x^2+1)^{n+1}}$$

$$\int \frac{x}{(x^2+1)^{n+1}}\,dx = \left| \begin{array}{ll} t = x^2+1 & \\ \frac{dt}{dx} = 2x & dx = \frac{dt}{2x} \end{array} \right| = \int \frac{x}{t^{n+1}} \cdot \frac{dt}{2x} = \frac{1}{2}\int t^{-n-1}\,dt =$$

$$\frac{1}{2} \cdot \frac{1}{1+(-n-1)}\, t^{1+(-n-1)} = \frac{1}{2(-n)} \cdot t^{-n}$$

$$\int \frac{x}{(x^2+1)^{n+1}}\,dx = \frac{-1}{2n(x^2+1)^n}$$

$$\int \frac{1}{x(x^2+1)^n}\,dx = \frac{-1}{2n(x^2+1)^n} + \int \frac{1}{x(x^2+1)^{n+1}}\,dx$$

$$\int \frac{1}{x(x^2+1)^{n+1}}\,dx = \int \frac{1}{x(x^2+1)^n}\,dx + \frac{1}{2n(x^2+1)^n}$$

$$\boxed{F_{n+1}(x) = F_n(x) + \frac{1}{2n(x^2+1)^n}}$$

$$\int \frac{1}{x(x^2+1)}\,dx = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$Ax^2 + A + Bx^2 + Cx$$

$$\begin{cases} A+B=0 \\ C=0 \qquad B=-1 \\ A=1 \end{cases}$$

$$\int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx$$

$$\ln x - \frac{1}{2}\ln|x^2+1| =$$

$$\ln x - \ln\sqrt{x^2+1} =$$

$$\boxed{\ln \frac{x}{\sqrt{x^2+1}}}$$

$$F_4(x) = F_3(x) + \frac{1}{6(x^2+1)^3}$$

$$F_4(x) = \left[F_2(x) + \frac{1}{4(x^2+1)^2} \right] + \frac{1}{6(x^2+1)^3} = F_1(x) + \frac{1}{2(x^2+1)} + \frac{1}{4(x^2+1)^2} + \frac{1}{6(x^2+1)^3}$$

$$F_4(x) = \boxed{\int \frac{1}{x(x^2+1)}\,dx} + \frac{6(x^4+2x^2+1)}{12(x^2+1)^3} + \frac{3(x^2+1)}{12(x^2+1)^3} + \frac{2}{12(x^2+1)^3}$$

$$F_4(x) = \ln \frac{x}{\sqrt{x^2+1}} + \frac{6x^4+15x^2+11}{12(x^2+1)^3} + C$$

$$\int \frac{1}{x(x^2+1)^4}\,dx =: F_4(x)$$