# Zadania - liczby rzeczywiste, ciągi

MACIEJ GIZA

## Zadanie 7b) (str. 2)

Wykazać, że:

$$\lim_{n\to\infty} n\left(n+4\sqrt{n^2+n}-2\sqrt{n^2-n}-3\sqrt{n^2+2n}\right)=\frac{5}{4}$$

Korzystam z przekształconego wzoru skróconego mnożenia

$$(*)\quad \boxed{a-b=\frac{a^2-b^2}{a+b}}$$

$$\lim_{n\to\infty} n\left(n+4\sqrt{n^2+n}-2\sqrt{n^2-n}-3\sqrt{n^2+2n}\right)=\lim_{n\to\infty} n^2\left(1+4\sqrt{1+\tfrac{1}{n}}-2\sqrt{1-\tfrac{1}{n}}-3\sqrt{1+\tfrac{2}{n}}\right)=$$

$$\overset{*}{=}\lim_{n\to\infty} n^2\cdot\frac{\left(\overbrace{1+4\sqrt{1+\tfrac{1}{n}}}^{(a}\ \overbrace{-2\sqrt{1-\tfrac{1}{n}}-3\sqrt{1+\tfrac{2}{n}}}^{-b)}\right)\left(\overbrace{1+4\sqrt{1+\tfrac{1}{n}}}^{(a}\ \overbrace{+2\sqrt{1-\tfrac{1}{n}}+3\sqrt{1+\tfrac{2}{n}}}^{+b)}\right)}{\underbrace{1+4\sqrt{1+\tfrac{1}{n}}}_{(a}\ \underbrace{+2\sqrt{1-\tfrac{1}{n}}+3\sqrt{1+\tfrac{2}{n}}}_{+b)}}=$$

I raz korzystam ze wzoru (*)

$a=1+4\sqrt{1+\frac{1}{n}}$

$b=2\sqrt{1-\frac{1}{n}}+3\sqrt{1+\frac{2}{n}}$

Wyrażenie w mianowniku nazywam ciągiem $f_n$

$$f_n=1+4\sqrt{1+\tfrac{1}{n}}+2\sqrt{1-\tfrac{1}{n}}+3\sqrt{1+\tfrac{2}{n}}$$

$\lim\limits_{n\to\infty}\frac{1}{n}=0 \quad \lim\limits_{n\to\infty}\sqrt{1+\frac{1}{n}}=1 \quad \lim\limits_{n\to\infty}\sqrt{1-\frac{1}{n}}=1$

$\lim\limits_{n\to\infty}\sqrt{1+\frac{2}{n}}=1$

zatem $\lim\limits_{n\to\infty} f_n=1+4+2+3=10$

$$=\lim_{n\to\infty} n^2\cdot\frac{\overbrace{\left(1+4\sqrt{1+\tfrac{1}{n}}\right)^2-\left(2\sqrt{1-\tfrac{1}{n}}+3\sqrt{1+\tfrac{2}{n}}\right)^2}^{(a^2-b^2)}}{10}=$$

$$=\lim_{n\to\infty} n^2\cdot\frac{\left(1+8\sqrt{1+\tfrac{1}{n}}+16+\tfrac{16}{n}-4+\tfrac{4}{n}-12\sqrt{1-\tfrac{1}{n}}\cdot\sqrt{1+\tfrac{2}{n}}-9-\tfrac{18}{n}\right)}{10}=$$

$$=\lim_{n\to\infty} n^2\cdot\frac{\left(4+8\sqrt{1+\tfrac{1}{n}}-12\sqrt{1-\tfrac{1}{n}}\cdot\sqrt{1+\tfrac{2}{n}}+\tfrac{2}{n}\right)}{10}=$$

$$=\lim_{n\to\infty}\frac{n^2}{10}\cdot\frac{\left(\overbrace{4+8\sqrt{1+\tfrac{1}{n}}}^{(a}\ \overbrace{-12\sqrt{1-\tfrac{1}{n}}\cdot\sqrt{1+\tfrac{2}{n}}+\tfrac{2}{n}}^{-b)}\right)\left(\overbrace{4+8\sqrt{1+\tfrac{1}{n}}}^{(a}\ \overbrace{+12\sqrt{1-\tfrac{1}{n}}\cdot\sqrt{1+\tfrac{2}{n}}-\tfrac{2}{n}}^{+b)}\right)}{\cancel{10}\,1\cdot\left(\underbrace{4+8\sqrt{1+\tfrac{1}{n}}}_{(a}\ \underbrace{+12\sqrt{1-\tfrac{1}{n}}\cdot\sqrt{1+\tfrac{2}{n}}-\tfrac{2}{n}}_{+b)}\right)}=$$

II raz korzystam ze wzoru (*)

$a=4+8\sqrt{1+\frac{1}{n}}$

$b=12\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}-\frac{2}{n}$

Wyrażenie w mianowniku drugiego ułamka → ciąg $g_n$.

$$g_n=4+8\sqrt{1+\tfrac{1}{n}}+12\sqrt{1-\tfrac{1}{n}}\cdot\sqrt{1+\tfrac{2}{n}}-\tfrac{2}{n}$$

Korzystam z poprzednich ~~przykładów~~ granic oraz tego, że $\lim\limits_{n\to\infty}\left(-\frac{2}{n}\right)=-2\cdot\lim\limits_{n\to\infty}\frac{1}{n}=0$

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty}\left(4+8\sqrt{1+\frac{1}{n}}+12\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}-\frac{2}{n}\right)=4+8+12-0=24$$

$$a^2 \qquad -b^2$$

$$=\lim_{n\to\infty}\frac{n^2}{10\cdot 24}\cdot\left(\left(4+8\sqrt{1+\frac{1}{n}}\right)^2-\left(12\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}-\frac{2}{n}\right)^2\right)=$$

$$=\lim_{n\to\infty}\frac{n^2}{240}\cdot\left(16+64\sqrt{1+\frac{1}{n}}+64+\frac{64}{n}-144-\frac{144}{n}+\frac{288}{n^2}+\frac{48}{n}\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}-\frac{4}{n^2}\right)=$$

$$\left\{\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}=\sqrt{1-\frac{1}{n}+\frac{2}{n}-\frac{2}{n^2}}=\sqrt{1+\frac{1}{n}-\frac{2}{n^2}}\right\}$$

$$=\lim_{n\to\infty}\frac{n^2}{240}\cdot\left(\underbrace{-64}_{-b}+\underbrace{64\sqrt{1+\frac{1}{n}}}_{a}\underbrace{-\frac{80}{n}}_{-b}+\underbrace{\frac{284}{n^2}+\frac{48}{n}\cdot\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}}_{a}\right)=$$

Stosuję II raz wzór ⊛

$a=64\sqrt{1+\frac{1}{n}}+\frac{284}{n^2}+\frac{48}{n}\cdot\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}$

$b=64+\frac{80}{n}$

Wyrażenie w mianowniku drugiego ułamka nazywam ciągiem $h_n$

$h_n=64\sqrt{1+\frac{1}{n}}+\frac{284}{n^2}+\frac{48}{n}\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}+64+\frac{80}{n}$

Wiedząc, że $\lim_{n\to\infty}\frac{1}{n^2}=\lim_{n\to\infty}\frac{1}{n}=0$ oraz korzystając z poprzednich granic: $\lim_{n\to\infty}h_n=64+0+0+64+0=128$.

$$=\lim_{n\to\infty}\frac{n^2}{240}\cdot\frac{\left(\overbrace{64\sqrt{1+\frac{1}{n}}+\frac{284}{n^2}+\frac{48}{n}\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}}^{a}\overbrace{-64-\frac{80}{n}}^{-b}\right)\cdot\left(\overbrace{64\sqrt{1+\frac{1}{n}}+\frac{284}{n^2}+\frac{48}{n}\sqrt{1-\frac{1}{n}}\sqrt{1+\frac{2}{n}}}^{a}\overbrace{+64+\frac{80}{n}}^{-b}\right)}{\underbrace{\underbrace{64\sqrt{1+\frac{1}{n}}+\frac{284}{n^2}+\frac{48}{n}\sqrt{1-\frac{1}{n}}\sqrt{1+\frac{2}{n}}}_{a}\underbrace{+64+\frac{80}{n}}_{+b}}_{h_n}}=$$

$$=\lim_{n\to\infty}\frac{n^2}{240\cdot128}\left(\left(64\sqrt{1+\frac{1}{n}}+\frac{284}{n^2}+\frac{48}{n}\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}\right)^2-\left(64+\frac{80}{n}\right)^2\right)=$$

$$=\lim_{n\to\infty}\frac{n^2}{240\cdot128}\left(4096+\frac{4096}{n}+\frac{80656}{n^4}+\frac{2304}{n^2}\left(1+\frac{1}{n}-\frac{2}{n^2}\right)+\frac{36352}{n^2}\cdot\sqrt{1+\frac{1}{n}}+\right.$$

$$\left.+\frac{6144}{n}\sqrt{1+\frac{1}{n}}\cdot\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}+2\cdot\frac{284\cdot48}{n^3}\cdot\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}}-4096-\frac{10240}{n}-\frac{6400}{n^2}\right)=$$

Wyrażenia zaznaczone w niebieskie kółko są nieistotne w dalszych obliczeniach, gdyż ich iloczyn z "$n^2$" stojącym przed nawiasem da składnik rzędu $\frac{1}{n}$ lub mniejszy. Granica tego czynnika będzie równa zero.

③

c.d. Zadanie 7 b) $\left\{ \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\left(1+\frac{2}{n}\right) = \left(1+\frac{1}{n}-\frac{2}{n^2}\right)\left(1+\frac{2}{n}\right) = 1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3} \right\}$

$$= \lim_{n\to\infty} \frac{n^2}{240\cdot 128}\left(\frac{6144}{n}\sqrt{1+\frac{1}{n}}\sqrt{1-\frac{1}{n}}\cdot\sqrt{1+\frac{2}{n}} - \frac{6144}{n} + \frac{29952}{n^2} + \frac{2304}{n^2} + \frac{2304}{n^3} - \frac{4608}{n^4}\right) =$$

$$= \lim_{n\to\infty} \frac{n^2}{240\cdot 128}\left(\frac{6144}{n}\cdot\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}} - \frac{6144}{n} + \frac{32\,256}{n^2}\right) =$$

Stosuję ostatni raz wzór (*)

$a = \frac{6144}{n}\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}$

$b = \frac{6144}{n} - \frac{32256}{n^2}$

Wyr. w mianowniku II ułamka nazywam $i_n$

$i_n = \cancel{6144}\ 6144\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}} + 6144 - \frac{32256}{n}$

$\lim_{n\to\infty}\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}} = 1 \qquad \lim_{n\to\infty} -\frac{32256}{n} = 0$

$\lim_{n\to\infty} i_n = 6144\cdot 1 + 6144 - 0 = 12\,288$

Korzystam też z poprzednich własności.

$$= \lim_{n\to\infty}\frac{n^2}{240\cdot 128}\cdot\frac{\left(\overset{a}{\frac{6144}{n}\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}} \overset{-b}{- \frac{6144}{n} + \frac{32256}{n^2}}\right)\cdot\left(\overset{a}{\frac{6144}{n}\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}} \overset{+b}{+ \frac{6144}{n} - \frac{32256}{n^2}}\right)}{\underset{a}{\frac{6144}{n}\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}} \underset{+b}{+ \frac{6144}{n} - \frac{32256}{n^2}}} =$$

$$= \lim_{n\to\infty}\frac{n^2}{240\cdot 128}\cdot\frac{\left(\frac{6144}{n}\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}}\right)^2 - \left(\cancel{\frac{32256}{n^2}}\ \frac{6144}{n} - \frac{32256}{n^2}\right)^2}{\frac{1}{n}\cdot\underbrace{\left(6144\sqrt{1+\frac{2}{n}-\frac{1}{n^2}-\frac{2}{n^3}} + 6144 - \frac{32256}{n}\right)}_{\to i_n}} =$$

$$= \lim_{n\to\infty}\frac{n^3}{240\cdot 128\cdot 12288}\cdot\left(\frac{6144^2}{n^2}\cdot 1 + \frac{6144^2}{n^3}\cdot 2 - \frac{6144^2}{n^4} - \frac{6144^2}{n^5}\cdot 2 - \frac{6144^2}{n^2} +\right.$$

$$\left. + 2\cdot\frac{6144\cdot 32256}{n^3} - \frac{32256^2}{n^4}\right) =$$

$$= \lim_{n\to\infty}\frac{n^3}{240\cdot 128\cdot 12288}\cdot\frac{1}{n^3}\left(2\cdot 6144^2 + 2\cdot 6144\cdot 32256\right) =$$

$$\cancel{\lim_{n\to\infty}} = \frac{6144\cdot 2\left(6144+32256\right)}{240\cdot 128\cdot 12288} = \frac{38400}{240\cdot 128} = \frac{5}{4}$$

$\square$.