# Generalised Brègman relative entropies: a brief introduction 

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1 March 2024


#### Abstract

We present some basic elements of the theory of generalised Brègman relative entropies over nonreflexive Banach spaces. Using nonlinear embeddings of Banach spaces together with the EulerLegendre functions, this approach unifies two former approaches to Brègman relative entropy: one based on reflexive Banach spaces, another based on differential geometry. This construction allows to extend Brègman relative entropies, and related geometric and operator structures, to arbitrary-dimensional state spaces of probability, quantum, and postquantum theory. We give several examples, not considered previously in the literature.


## 1 Introduction

For any set $Z, D: Z \times Z \rightarrow[0, \infty]$ will be called an information on $Z$ (and $-D$ will be called a relative entropy on $Z)^{1}$ iff (cf. [8, p. 1019] [17, p. 794] [14, p. 161]) $D(x, y)=0 \Longleftrightarrow$ $x=y \forall x, y \in Z$. If $\varnothing \neq K \subseteq Z, x \in Z$, and $\arg \inf _{y \in K}\{D(y, x)\}\left(\right.$ resp., $\left.\arg ^{\inf }{ }_{y \in K}\{D(x, y)\}\right)$ is a singleton set, then we will denote the element of this set by $\overleftarrow{\mathfrak{P}}_{K}^{D}(x)$ (resp., $\overrightarrow{\mathfrak{P}}_{K}^{D}(x)$ ), while the map $x \mapsto \overleftarrow{\mathfrak{P}}_{K}^{D}(x)$ [51, p. 32] [33, Ch. 3.2] (resp., $x \mapsto \overrightarrow{\mathfrak{P}}_{K}^{D}(x)$ [13, Eqn. (16)]) will be called a left (resp., right) $D$-projection of $x$ onto $K$.

Let $M$ be a $C^{3}$-manifold with a tangent bundle $\mathbf{T} M$, a $\mathrm{C}^{3}$ riemannian metric tensor $\mathbf{g}$ on $\mathbf{T} M$, and a pair $(\nabla, \widetilde{\nabla})$ of $\mathbf{C}^{3}$ affine connections on $\mathbf{T} M$ (with an arbitrary torsion). Let $\mathbf{t}_{c}^{\nabla}$ denote a $\nabla$-parallel transport in $\mathbf{T} M$ along a curve $c$ in $M$. Then the Norden-Sen geometry is defined as a quadruple $(M, \mathbf{g}, \nabla, \widetilde{\nabla})$ satisfying any of the equivalent conditions [42, pp. 205-206, §2, §4] [52, p. 46]: ${ }^{2}$

$$
\begin{align*}
\mathbf{g}\left(\mathbf{t}_{c}^{\nabla}(\cdot), \mathbf{t}_{c}^{\widetilde{\nabla}}(\cdot)\right) & =\mathbf{g}  \tag{1}\\
\mathbf{g}\left(\nabla_{u} v, w\right)+\mathbf{g}\left(v, \widetilde{\nabla}_{u} w\right) & =u(\mathbf{g}(v, w)) \forall u, v, w \in \mathbf{T} M \tag{2}
\end{align*}
$$

If $Z$ is a finite dimensional $\mathrm{C}^{3}$-manifold and $D \in \mathrm{C}^{3}\left(Z \times Z ; \mathbb{R}^{+}\right)$has a positive definite hessian matrix, then a third order Taylor expansion of $D$ on $Z$ induces [17, pp. 795-796] [18, p. 357] a riemannian metric $\mathbf{g}^{D}$ on $\mathbf{T} Z$ and a pair $\left(\nabla^{D}, \widetilde{\nabla}^{D}\right)$ of torsion-free affine connections on $\mathbf{T} Z$, satisfying the characteristic property (2) of the Norden-Sen geometry. This way the global geometric properties of $D$ can be analysed in local terms of its torsion-free Norden-Sen differential geometry. ${ }^{3}$

[^0]
## $2 \quad D_{\Psi}$ : Brègman vs Brunk-Ewing-Utz

Given a strictly convex, differentiable function $\Psi: \mathbb{R}^{n} \rightarrow \mathbb{R}\left(\right.$ or $\Psi: M \rightarrow \mathbb{R}$ with convex $\left.M \subseteq \mathbb{R}^{n}\right)$, there are two approaches to construction of a functional encoding the first order Taylor expansion of $\Psi$ (together with its further use in optimisation problems): one going back to Brègman's [8, p. 1021]

$$
\begin{equation*}
D_{\Psi}(x, y):=\Psi(x)-\Psi(y)-\sum_{i=1}^{n}\left(x_{i}-y_{i}\right)\left(\operatorname{grad} \Psi\left(y_{i}\right)\right) \forall x, y \in \mathbb{R}^{n} \tag{3}
\end{equation*}
$$

(or $\forall x, y \in M$ ), another going back to the Brunk-Ewing-Utz [10, Eqn. (4.4)]

$$
\begin{equation*}
D_{\Psi}^{\mu}(x, y):=\int_{\mathcal{X} \subseteq \mathbb{R}^{m}} \mu(\chi) D_{\Psi}(x(x), y(x)), \tag{4}
\end{equation*}
$$

for $x, y: \mathcal{X} \rightarrow \mathbb{R}, n=1$, and a measure $\mu$ on the Borel subsets of $\mathbb{R}^{m}$.
The former approach has been generalised and widely developed for $\mathbb{R}^{n}$ replaced by a reflexive Banach space $\left(X,\|\cdot\|_{X}\right)$ (see Section 3). On the other hand, the latter approach was generalised and further developed for ( $\mathcal{X}, \mu$ ) given by any countably finite nonzero measure space (see [15] and references therein).

The passage from probabilistic to quantum theoretic setting corresponds to replacing $\left(L_{1}(\mathcal{X}, \mu),\|\cdot\|_{1}\right)$ by the Banach predual $\mathcal{N}_{\star}$ of a $\mathrm{W}^{*}$-algebra $\mathcal{N}$ (all of these spaces are nonreflexive). The noncommutative analogue $D_{\Psi}^{\operatorname{tr} \mathcal{H}}$ of $D_{\Psi}^{\mu}$ was introduced in $[56, \S 2.2]$ for finite dimensional real Hilbert spaces, and in [43, pp. $127-129]^{4}$ for type I W*-algebras (see also [23, §V] for type $\mathrm{I}_{n}$ JBW-algebras). However, due to nonreflexivity of $\mathcal{N}_{\star}$, this definition is incapable of utilising the vast body of reflexive Banach space theoretic results obtained for $D_{\Psi}$, and it is also unclear how to extend the definition of $D_{\Psi}^{\mathrm{tr}} \mathrm{H}_{\mathcal{H}}$ to arbitrary $\mathrm{W}^{*}$-algebras.

For a convex closed $C \subseteq M \subseteq \mathbb{R}^{n}$, $D_{\Psi}$ given by (3) exhibits [8, Lemm. 1],

$$
\begin{equation*}
D_{\Psi}\left(x, \overleftarrow{\mathfrak{P}}_{C}^{D_{\Psi}}(y)\right)+D_{\Psi}\left(\overleftarrow{\mathfrak{P}}_{C}^{D_{\Psi}}(y), y\right) \geq D_{\Psi}(x, y) \forall(x, y) \in C \times M \tag{5}
\end{equation*}
$$

(and analogously for $\overrightarrow{\mathfrak{P}}_{C} D_{\Psi}$ [37, Prop. 4.11]; cf. also [13, Thm. 1]), with $\geq$ replaced by $=$ for affine closed $C$. This property is a nonlinear generalisation of a pythagorean theorem, and is interpreted as an additive decomposition of an (information about) "data" into "signal" and "noise". It is a fundamental feature of $D_{\mathbb{\Psi}}$, characterising $\overleftarrow{\mathfrak{P}}_{C}^{D_{\Psi}}\left[6\right.$, Cor. 3.35] and $\overrightarrow{\mathfrak{P}}_{C}^{D_{\Psi}}$ [37, Prop. 4.11].

## $3 D_{\Psi}$ : reflexive Banach space setting

$\left(X,\|\cdot\|_{X}\right)$ will denote a Banach space over $\mathbb{R}$. A Banach space $\left(X^{\star},\|\cdot\|_{X^{\star}}\right)$, consisting of elements given by continuous linear maps $X \rightarrow \mathbb{R}$, with a norm

$$
\begin{equation*}
\|y\|_{X^{\star}}:=\sup \left\{|y(x)| \mid x \in B\left(X,\|\cdot\|_{X}\right):=\left\{x \in X \mid\|x\|_{X} \leq 1\right\}\right\} \forall y \in X^{\star}, \tag{6}
\end{equation*}
$$

is called a Banach dual of $\left(X,\|\cdot\|_{X}\right)$, with respect to a bilinear duality

$$
\begin{equation*}
\llbracket x, y \rrbracket_{X \times X^{\star}}:=y(x) \in \mathbb{R} \forall(x, y) \in X \times X^{\star} . \tag{7}
\end{equation*}
$$

If there exists $\left(Y,\|\cdot\|_{Y}\right)$ with $\left(Y^{\star},\|\cdot\|_{Y^{\star}}\right)=\left(X,\|\cdot\|_{X}\right)$, then $Y=: X_{\star}$ is called a predual of $X$. Symbol $\operatorname{int}(W)$ (resp., $\mathrm{cl}(W)$ ) will denote an interior (resp., closure) of $W \subseteq X$ with respect to a topology of $\|\cdot\|_{X}$.

Given a Banach space $\left.\left.\left(X,\|\cdot\|_{X}\right), \Psi: X \rightarrow\right]-\infty, \infty\right]$ is called: proper iff

$$
\begin{equation*}
\operatorname{efd}(\Psi):=\{x \in X \mid \Psi(x) \neq \infty\} \neq \varnothing ; \tag{8}
\end{equation*}
$$

[^1]convex (resp., strictly convex) iff $\forall x, y \in \operatorname{efd}(\Psi) \forall \lambda \in] 0,1[$
\[

$$
\begin{equation*}
x \neq y \Rightarrow \Psi(\lambda x+(1-\lambda) y) \leq(\text { resp. },<) \lambda \Psi(x)+(1-\lambda) \Psi(y) \tag{9}
\end{equation*}
$$

\]

Let $\Gamma\left(X,\|\cdot\|_{X}\right)$ (resp., $\left.\Gamma^{\mathrm{G}}\left(X,\|\cdot\|_{X}\right)\right)$ be the set of all proper, convex, lower semicontinuous functions $\Psi: X \rightarrow]-\infty, \infty]$ (resp., that are also Gateaux differentiable on $\operatorname{int}(\operatorname{efd}(\Psi)) \neq \varnothing$, with $\mathfrak{D}^{\mathrm{G}} \Psi$ denoting a Gateaux derivative of $\Psi$ ).

For $\Psi \in \Gamma^{\mathrm{G}}\left(X,\|\cdot\|_{X}\right)$ the Brègman function reads [1, Eqn. (1)] $\forall x \in X$

$$
\begin{equation*}
D_{\Psi}(x, y):=\Psi(x)-\Psi(y)-\left[\left[x-y, \mathfrak{D}^{\mathrm{G}} \Psi(y)\right]\right]_{X \times X^{\star}} \forall y \in \operatorname{int}(\operatorname{efd}(\Psi)) \tag{10}
\end{equation*}
$$

and $D_{\Psi}(x, y):=\infty \forall y \in X \backslash \operatorname{int}(\operatorname{efd}(\Psi))$. $D_{\Psi}$ is an information on $X$ iff $\Psi$ is strictly convex on $\operatorname{int}(\operatorname{efd}(\Psi))$ [12, Prop. 1.1.9].

For a proper $\Psi: X \rightarrow]-\infty, \infty$, a Fenchel dual map [21, p. 75] [39, p. 8]

$$
\begin{equation*}
\left.\left.X^{\star} \ni y \mapsto \Psi^{\mathbf{F}}(y):=\sup _{x \in X}\left\{\llbracket x, y \rrbracket_{X \times X^{\star}}-\Psi(x)\right\} \in\right]-\infty, \infty\right], \tag{11}
\end{equation*}
$$

satisfies $\Psi^{\mathrm{F}} \in \Gamma\left(X^{\star},\|\cdot\|_{X^{\star}}\right)\left[9\right.$, Thm. 3.6]. If $\left(X,\|\cdot\|_{X}\right)$ is reflexive and $\Psi \in \Gamma^{\mathrm{G}}\left(X,\|\cdot\|_{X}\right)$, then $\Psi$ will be called Euler-Legendre ${ }^{5}$ iff [5, Def. 5.2.(iii), Thm. 5.4, Thm. 5.6] [47, §2.1] $\Psi^{\mathbf{F}} \in \Gamma^{\mathrm{G}}\left(X^{\star},\|\cdot\|_{X^{\star}}\right)$ and

$$
\left\{\begin{array}{l}
\operatorname{efd}\left(\mathfrak{D}^{\mathrm{G}} \Psi\right):=\left\{x \in \operatorname{efd}(\Psi) \mid \exists \mathfrak{D}^{\mathrm{G}} \Psi(x)\right\}=\operatorname{int}(\operatorname{efd}(\Psi))  \tag{12}\\
\operatorname{efd}\left(\mathfrak{D}^{\mathrm{G}} \Psi^{\mathbf{F}}\right)=\operatorname{int}\left(\operatorname{efd}\left(\Psi^{\mathbf{F}}\right)\right)
\end{array}\right.
$$

For $X=\mathbb{R}^{n}$, the definition of Euler-Legendre functions goes back to Rockafellar, who showed [49, Thm. C-K] [50, Thm. 1] that if $\varnothing \neq U \subseteq \mathbb{R}^{n}$ is open and convex, while $\left.\left.\Psi: U \rightarrow\right]-\infty, \infty\right]$ is strictly convex, differentiable on $U$, and

$$
\begin{equation*}
\lim _{t \rightarrow+0} \frac{\mathrm{~d}}{\mathrm{~d} t} \Psi(t x+(1-t) y)=-\infty \quad \forall(x, y) \in U \times(\mathrm{cl}(U) \backslash U) \tag{13}
\end{equation*}
$$

then $\operatorname{grad} \Psi$ is a bijection on $U, \operatorname{grad}\left(\Psi^{\mathbf{F}}\right)=(\operatorname{grad} \Psi)^{-1}$ on $(\operatorname{grad} \Psi)(U)$, and $\Psi^{\mathbf{F}}$ on $(\operatorname{grad} \Psi)(U)$ satisfies the same conditions as $\Psi$ on $U$.

## $4 D_{\Psi}$ : dually flat setting

The dually flat (a.k.a. hessian) geometry [53, Prop. (p. 213)] is characterised among all torsionfree Norden-Sen geometries by the flatness of $\nabla$ and $\nabla$. This is equivalent with existence of two coordinate systems, $\left\{\theta_{i} \mid i \in\{1, \ldots, n\}\right\}: M \rightarrow \mathbb{R}^{n}$ and $\left\{\eta_{i} \mid i \in\{1, \ldots, n\}\right\}: M \rightarrow \mathbb{R}^{n}$, such that, $\forall \rho \in M$,

$$
\left\{\begin{align*}
\eta_{i}(\rho) & =\frac{\partial \Psi(\theta(\rho))}{\partial \theta^{i}}, \theta_{i}(\rho)=\frac{\partial \Psi^{\mathbf{F}}(\eta(\rho))}{\partial \eta^{i}}  \tag{14}\\
\Psi^{\mathbf{F}}(y) & =\sup _{x \in \mathbb{R}^{n}}\left\{\sum_{i=1}^{n} x_{i} y_{i}-\Psi(x)\right\} \forall x \in \mathbb{R}^{n}
\end{align*}\right.
$$

and, for $D_{\theta, \Psi}(\rho, \sigma):=D_{\Psi}(\theta(\rho), \theta(\sigma))$ with $D_{\Psi}$ defined by (3),

$$
\left\{\begin{align*}
\Gamma_{i j k}^{\nabla^{D_{\theta, \Psi}}}(\theta(\rho)) & =0, \Gamma_{i j k}^{\tilde{\nabla}^{D_{\eta, \Psi}}}(\eta(\rho))=0  \tag{16}\\
\mathbf{g}_{i j}^{D_{\theta, \Psi}}(\theta(\rho)) & =\frac{\partial^{2} \Psi(\theta(\rho))}{\partial \theta^{i} \partial \theta^{j}}
\end{align*}\right.
$$

[^2]where $\Gamma^{\nabla}(u, v, w):=\mathbf{g}\left(\nabla_{u} v, w\right) \forall u, v, w \in \mathbf{T} M$, while the subscript ${ }_{i}$ denotes evaluation at the $i$-th component of a basis in $\mathbf{T} M$ given by coordinate system differentials (i.e., setting $u=\frac{\partial}{\partial \theta^{2}}$, etc., in (16)). (Also, $\mathbf{g}_{i j}^{D_{\eta}, \Psi}(\eta(\rho))=\frac{\partial^{2} \Psi^{\mathbf{F}}(\eta(\rho))}{\partial \eta^{2} \partial \eta^{j}}$.) When reconsidered in this setting, the left (resp., right) generalised pythagorean theorem is equivalent with: a projection of $y \in M$ onto $C$ along $\widetilde{\nabla}^{D_{\eta, \Psi}}$-(resp., $\nabla^{D_{\theta, \Psi}}$-) geodesics is $\mathbf{g}^{D_{\theta, \Psi}}$-orthogonal ( $=\mathbf{g}^{D_{\eta, \Psi}}$-orthogonal) to $C$ [3, Thm. 3.4].

Equation (15) is a special case of (11). Furthermore, (14) require only $\mathrm{C}^{1}$-differentiability. The approach presented in Section 5 is rooted in an observation that the correct generalisation of (14) requires two components: Euler-Legendre $\Psi$ on a reflexive Banach space $\left(X,\|\cdot\|_{X}\right)$, and nonlinear embeddings into $\left(X,\|\cdot\|_{X}\right)$ and $\left(X^{\star},\|\cdot\|_{X^{\star}}\right)$, replacing, respectively, $\theta$ and $\eta$.

## $5 D_{\ell, \Psi}$

In $[31, \S 3]$ we introduced a generalisation, $D_{\ell, \Psi}$, of a family of Brègman informations $D_{\Psi}$ on reflexive Banach spaces $\left(X,\|\cdot\|_{X}\right.$ ), applicable to a wide range of nonreflexive Banach spaces $\left(Y,\|\cdot\|_{Y}\right)$. (E.g., to postquantum state spaces, given by bases $Z \subseteq V^{+}$of positive cones $V^{+}$of radially compact base normed spaces in spectral duality, $\left(V,\|\cdot\|_{V}\right)=\left(Y,\|\cdot\|_{Y}\right)$.) The main idea is to pull back the properties exhibited by $D_{\Psi}$ with Euler-Legendre $\Psi$ acting on $\left(X,\|\cdot\|_{X}\right)$ into the properties exhibited by $D_{\ell, \Psi}(\cdot, \cdot):=D_{\Psi}(\ell(\cdot), \ell(\cdot))$, where $\ell: Z \rightarrow X$ and $Z \subseteq Y$.

Definition 5.1. [31, Def. 3.1] Let $\left(Y,\|\cdot\|_{Y}\right)$ be a Banach space, let $\left(X,\|\cdot\|_{X}\right)$ be a reflexive Banach space, let $\Psi \in \Gamma^{G}\left(X,\|\cdot\|_{X}\right)$ be strictly convex on $\operatorname{int}(\operatorname{efd}(\Psi))$ and Euler-Legendre, let $\varnothing \neq Z \subseteq Y$, and let $\ell: Z \rightarrow \ell(Z) \subseteq X$ be a bijection such that $\ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi)) \neq \varnothing$. Then:
(i) if $\varnothing \neq C \subseteq Y$, and $\ell(C)$ is convex (resp., closed; affine), then $C$ will be called $\ell$-convex (resp., $\ell$-closed; $\ell$-affine);
(ii) a triple $(Z, \ell, \Psi)$ will be called a generalised pythagorean geometry;
(iii) an $(\ell, \Psi)$-information (a generalised Brègman information) on $Z$ is

$$
\begin{equation*}
D_{\ell, \Psi}(\phi, \psi):=D_{\Psi}(\ell(\phi), \ell(\psi)) \quad \forall(\phi, \psi) \in Z \times \ell^{-1}(\ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi))) . \tag{18}
\end{equation*}
$$

Proposition 5.2. [31, Prop. 3.2] Under assumptions of Definition 5.1, let $\varnothing \neq C \subseteq Z$ be $\ell$-convex and $\ell$-closed, and let $\psi \in \ell^{-1}(\ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi)))$. Then:
(i) $D_{\ell, \Psi}$ is an information on $Z$;
(ii) $\arg \inf _{\phi \in C}\left\{D_{\ell, \Psi}(\phi, \psi)\right\}$ is a singleton set, denoted $\left\{\overleftarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}(\psi)\right\}$;
(iii) $\omega \in C$ is the unique solution of $D_{\ell, \Psi}(\phi, \omega)+D_{\ell, \Psi}(\omega, \psi) \leq D_{\ell, \Psi}(\phi, \psi) \forall \phi \in C$ iff $\omega=\overleftarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}(\psi)$ (in 'then' case, if $C$ is $\ell$-affine, then $=$ replaces $\leq$ );
(v) if $\ell$ is norm-to-norm continuous and $\overleftarrow{\mathfrak{P}}_{K}^{D_{\Psi}}$ is norm-to-norm continuous for any convex closed $\varnothing \neq K \subseteq \ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi))$, then $\overleftarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}$ is norm-to-norm continuous for any $\ell$-convex and closed $\varnothing \neq C \subseteq \ell^{-1}(\ell(Z) \cap \operatorname{int}(\operatorname{efd}(\Psi)))$.

An analogous result for $\overrightarrow{\mathfrak{P}}^{D_{\ell, \Psi}}$ also holds [32, Part I] (cf. also [13, Thm. 1]).
For $X=\mathbb{R}^{n}, D_{\ell, \Psi}$ recovers the setting of Brègman information $D_{\theta, \Psi}$ on an $n$-dimensional $\mathrm{C}^{1}$ manifold (hence, in particular, $\mathrm{C}^{\infty}$-manifold) $M$, with the map $\ell: M \rightarrow \mathbb{R}^{n}$ (resp., $\mathfrak{D}^{\mathrm{G}} \Psi \circ \ell: M \rightarrow$ $\mathbb{R}^{n}$ ) given by the coordinate system $\left\{\theta_{i}\right\}$ (resp., $\left\{\eta_{i}\right\}$ ). More specifically, a domain $M$ of a dually flat geometry is assumed to be a (suitably differentiable) manifold, covered by two global maps $\left\{\theta_{i}\right\}$ and $\left\{\eta_{i}\right\}$, without assuming $M \subseteq \mathbb{R}^{n}$, cf. [3, 54]. This is not addressed by (3), and is addressed (up to a weaker assumption on the order of differentiability) by (18).

This way the framework of generalised Brègman information $D_{\ell, \Psi}$ unifies reflexive Banach space theoretic and finite dimensional smooth information geometric approaches to Brègman information. If $\ell$ is a norm-to-norm continuous homeomorphism, then the $\ell$-closed sets in $Z$ are closed in terms of topology of $\|\cdot\|_{Y}$. This fragment of a theory provides a fusion of nonlinear convex analysis with nonlinear homeomorphic theory of Banach spaces. In particular, if $\ell$ is Hölder continuous, then it allows to pull back the conditions on Hölder continuity of $\overleftarrow{\mathfrak{P}}_{K}^{D_{\text {区 }}}$ and $\overrightarrow{\mathfrak{P}}_{K}^{D_{\text {区 }}}$ into results on Hölder continuity of $\overleftarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}$ and $\overrightarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}$. Generalised pythagorean geometry $(Z, \ell, \Psi)$ is a more general object than $D_{\ell, \Psi}$, and allows to suitably generalise also the affine connections (16) [32, Part IV].

In this context, our approach arises partially from an observation that the $\ell_{\gamma}$ (resp., $\ell_{\Upsilon}$ ) embeddings, cf. Example 6.1.(a) (resp., 6.1.(c)) below, used in [40, Eqn. (2.7)] (resp., [22, §7.2]), are finite dimensional Mazur (resp., Kaczmarz) maps [38, p. 83] (resp., [28, p.148]) on $\left(L_{1}(\mathcal{X}, \mu)\right)^{+}$. Drawing from an important example in $[27, \S 6-\S 8]$ (equal to Example 6.1.(a) with $\alpha=\gamma(1-\gamma)$ and $\beta=\gamma$ ), an abstract framework aiming at this unification was proposed in [30, Eqns. (24), (31)], while its implementation, based on the use of Euler-Legendre $\Psi$, was given in $[31, \S 3-\S 4]$. The resulting theory is developed in details in [32].

## 6 Examples of $(\ell, \Psi)$ with $Z \subseteq V^{+}$(for Proposition 5.2)

If $\left(Y,\|\cdot\|_{Y}\right)$ is partially ordered by $\geq$, then $Y^{+}:=\{x \in Y \mid x \geq 0\}$. All examples below feature $\left(Y,\|\cdot\|_{Y}\right)$ given by some kind of a radially compact base normed space $\left(V,\|\cdot\|_{V}\right)$. Such spaces provide the setting for the (linear) convex operational generalisation of quantum theory (a.k.a. "generalised probability theory" or "postquantum theory"), with state space given by $V_{1}^{+}:=\left\{\phi \in V^{+} \mid\|x\|_{V}=1\right\}$.

## Example 6.1.

(a). (=[31, Prop. 4.2].) If $\mathcal{N}$ is a $W^{*}$-algebra, $\left.\alpha \in\right] 0, \infty[, \beta, \gamma \in] 0,1\left[,\left(X,\|\cdot\|_{X}\right)=\left(L_{1 / \gamma}(\mathcal{N}),\|\cdot\|_{1 / \gamma}\right)\right.$, then the Mazur map

$$
\begin{equation*}
\ell=\ell_{\gamma}: Z=\mathcal{N}_{\star}^{+} \ni \phi \mapsto \phi^{\gamma} \in\left(L_{1 / \gamma}(\mathcal{N})\right)^{+} \tag{19}
\end{equation*}
$$

is Hölder continuous [48, Thm. (p. 37)]. If $\Psi=\Psi_{\alpha, \beta}:=\frac{\beta}{\alpha}\|\cdot\|_{X}^{1 / \beta}$, then

$$
\begin{equation*}
D_{\ell \gamma, \Psi_{\alpha, \beta}}(\phi, \psi)=\alpha^{-1}\left(\beta\|\phi\|_{1}^{\gamma / \beta}+(1-\beta)\|\psi\|_{1}^{\gamma / \beta}-\|\psi\|_{1}^{\gamma / \beta-1} \int\left(\phi^{\gamma} \psi^{1-\gamma}\right)\right) \tag{20}
\end{equation*}
$$

$\forall \phi, \psi \in \mathcal{N}_{\star}^{+}$, where $\int$ is understood as in [20, Eqn. (3.12')]; if $\mathcal{N}=\mathfrak{B}(\mathcal{H}):=\{$ bounded operators on a Hilbert space $\mathcal{H}\}$, then $\mathcal{N}_{\star}=\mathfrak{G}_{1}(\mathcal{H}) \equiv\{$ trace class operators on $\mathcal{H}\}, L_{1 / \gamma}(\mathcal{N})=$ : $\mathfrak{G}_{1 / \gamma}(\mathcal{H})$, and $\int \cdot=\operatorname{tr}_{\mathcal{H}}(\cdot)=\|\cdot\|_{1}$.
(b). (=[31, Prop. 4.7].) Let $A$ be a semifinite JBW-algebra with a Jordan product • a faithful normal semifinite trace $\tau, \alpha \in] 0, \infty[, \beta, \gamma \in] 0,1\left[,\left(X,\|\cdot\|_{X}\right)=\left(L_{1 / \gamma}(A, \tau),\|\cdot\|_{1 / \gamma}\right), \Psi=\Psi_{\alpha, \beta}\right.$. Then $\ell=\ell_{\gamma}: A_{\star}^{+} \ni \phi \mapsto \phi^{\gamma} \in\left(L_{1 / \gamma}(A, \tau)\right)^{+}$is Hölder continuous [31, Prop. 4.6], and $\forall \omega, \phi \in Z=A_{\star}^{+} D_{\ell \gamma, \Psi_{\alpha, \beta}}(\omega, \phi)=$

$$
\begin{equation*}
\alpha^{-1}\left(\beta(\tau(\omega))^{\gamma / \beta}+(1-\beta)(\tau(\phi))^{\gamma / \beta}-(\tau(\phi))^{\gamma / \beta-1} \tau\left(\omega^{\gamma} \bullet \phi^{1-\gamma}\right)\right) . \tag{21}
\end{equation*}
$$

(c). (=[31, Cor. 4.12].) If $(\mathcal{X}, \mu)$ is a nonatomic measure space, $\mu(\mathcal{X})<\infty, \Upsilon: \mathbb{R} \rightarrow \mathbb{R}^{+}$ is even, strictly convex, continuously differentiable, with $\Upsilon(1)=1, \Upsilon(u)=0$ iff $u=0$, $\lim \sup _{u \rightarrow \infty} \frac{\Upsilon(2 u)}{\Upsilon(u)}<\infty, \liminf _{u \rightarrow \infty} \frac{\Upsilon(2 u)}{\Upsilon(u)}>2, \lim _{u \rightarrow+0} \frac{\Upsilon(u)}{u}=0, \lim _{u \rightarrow \infty} \frac{\Upsilon(u)}{u}=\infty, t, s \in \mathbb{R}^{+}$, $t<s, u \mapsto \frac{\Upsilon^{-1}(u)}{u^{t}}$ is nondecreasing, and $u \mapsto \frac{\Upsilon^{-1}(u)}{u^{s}}$ is nonincreasing, then the Kaczmarz map

$$
\begin{equation*}
\ell=\ell_{\Upsilon}: Z=\left(L_{1}(\mathcal{X}, \mu)\right)_{1}^{+} \ni \phi \mapsto \Upsilon^{-1}(\phi) \in\left(L_{\Upsilon}(\mathcal{X}, \mu)\right)_{1}^{+} \tag{22}
\end{equation*}
$$

is Hölder continuous for the Morse-Transue-Nakano-Luxemburg norm $\|\cdot\|_{\Upsilon}$ on Orlicz space $L_{\Upsilon}(\mathcal{X}, \mu)$ [16, Cor. 2.5]. For $\left.\Psi=\Psi_{\beta, \beta}, \beta \in\right] 0,1[$, this gives

$$
\begin{equation*}
D_{\ell_{\Upsilon}, \Psi_{\beta, \beta}}(\omega, \phi)=\beta^{-1}(1-\bar{\Upsilon}(\omega, \phi) / \bar{\Upsilon}(\phi, \phi)), \tag{23}
\end{equation*}
$$

where $\bar{\Upsilon}(\omega, \phi):=\int \mu \Upsilon^{-1}(\omega) \Upsilon^{\prime}\left(\Upsilon^{-1}(\phi)\right)$, and $(\cdot)^{\prime}$ denotes a derivative.
All these cases have norm-to-norm continuous $\overleftarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}$. In [32] we prove this also for $\overrightarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}$, and establish conditions for Hölder continuity of $\overleftarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}$ and $\overrightarrow{\mathfrak{P}}_{C}^{D_{\ell, \Psi}}$.

## Example 6.2.

$\left(=\left[31\right.\right.$, Prop. 4.14] for $\varphi(t)=\varphi_{\alpha, \beta}(t)=\frac{1}{\alpha} t^{1 / \beta-1}$, i.e. $\Psi=\Psi_{\alpha, \beta}=\Psi_{\varphi_{\alpha, \beta}} ;\left[32\right.$, Part I] for $\left.\Psi=\Psi_{\varphi}\right)$. Let $\left(V,\|\cdot\|_{V}\right)$ be a generalised spin factor [7, Def. 4], i.e. $V=\mathbb{R} \oplus X$, where $\left(X,\|\cdot\|_{X}\right)$ is a reflexive Banach space, and

$$
\forall v=(\lambda, x) \in V\left\{\begin{array}{l}
v \geq 0: \Longleftrightarrow \lambda \geq\|x\|_{X}  \tag{24}\\
\|v\|_{V}:=\max \left\{|\lambda|,\|x\|_{X}\right\} .
\end{array}\right.
$$

Let $\Psi(x)=\Psi_{\varphi}(x):=\int_{0}^{\|x\|_{X}} \mathrm{~d} t \varphi(t)$, where $\varphi: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$is positive, strictly increasing, continuous, $\varphi(0)=0$, and $\lim _{t \rightarrow \infty} \varphi(t)=\infty .{ }^{6}$ Then $\Psi_{\varphi}$ (and, in particular, $\Psi_{\alpha, \beta}$ ) is Euler-Legendre iff $\left(V,\|\cdot\|_{V}\right)$ satisfies spectral duality condition [2, Def. (p. 55)]. This gives a family $D_{\ell_{X}, \Psi_{\varphi}}$ on $Z=\left\{w \in V^{+} \mid\right.$ $\left.\|w\|_{V}=1\right\}$, where $\quad \ell=\ell_{X}: Z \ni v=:(1, x) \mapsto x \in B\left(X,\|\cdot\|_{X}\right)$.

## Example 6.3.

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ with $n:=(\operatorname{dim} \mathcal{H})^{2} \in \mathbb{N}$ (hence, $\left.\mathfrak{G}_{1 / \widetilde{\gamma}}(\mathcal{H})=\mathfrak{G}_{1 / \gamma}(\mathcal{H}) \forall \gamma, \widetilde{\gamma} \in\right] 0,1[)$. Let $(\cdot)^{\text {sa }}:=$ self-adjoint part of $(\cdot)$. Let $\boldsymbol{\lambda}(x)$, with

$$
\begin{equation*}
\mathcal{K}:=\left(\mathfrak{G}_{2}(\mathcal{H})\right)^{\text {sa }}=\{\text { hermitean } n \times n \text { matrices }\} \ni x \mapsto \boldsymbol{\lambda}(x) \in \mathbb{R}^{n}, \tag{26}
\end{equation*}
$$

be a vector of eigenvalues of $x$ ordered nonincreasingly. For $\left.\left.\Phi: \mathbb{R}^{n} \rightarrow\right]-\infty, \infty\right]$, let $\Phi(s(x))=\Phi(x)$ $\forall$ permutation matrices $s: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$. Then $\Psi=\Phi \circ \boldsymbol{\lambda}$ is Euler-Legendre iff $\Phi$ is Euler-Legendre [36, Cor. 3.2, Cor. 3.3]. E.g., if: $\Phi(x)=$
(a). [4, Ex. 6.5, Cor. 5.13] $\sum_{i=1}^{n}\left(x_{i} \log \left(x_{i}\right)-x_{i}\right)$ if $x \geq 0$, and $\infty$ otherwise;
(b). [11] [4, Ex. 6.7, Cor. 5.13] $-\sum_{i=1}^{n} \log \left(x_{i}\right)$ on $] 0, \infty{ }^{n}$, and $\infty$ otherwise; ${ }^{7}$
(c). [29, Eqn. (60)] [4, Ex. 6.6, Cor. 5.13] $\sum_{i=1}^{n}\left(x_{i} \log \left(x_{i}\right)+\left(1-x_{i}\right) \log \left(1-x_{i}\right)\right)$ on $[0,1]^{n}$, and $\infty$ otherwise;
(d). [4, Ex. 6.1, Cor. 5.13] $\sum_{i=1}^{n} \gamma\left|x_{i}\right|^{1 / \gamma}$ on $\mathbb{R}^{n}$ with $\left.\gamma \in\right] 0,1[$;
(e). [46, Eqn. (37)] [46, §7.2] $\Phi_{\alpha}(x):=\frac{1}{\alpha-1} \sum_{i=1}^{n}\left(x_{i}^{\alpha}-1\right)$ for $(x, \alpha) \in\left[0, \infty\left[{ }^{n} \times\right] 0,1\left[,-\Phi_{\alpha}(x)\right.\right.$ for $(x, \alpha) \in] 0, \infty\left[{ }^{n} \times\right]-\infty, 0\left[\right.$, and $\infty$ otherwise; ${ }^{8}$
and $\mathcal{K}_{0}^{+}:=\left(\mathfrak{G}_{2}(\mathcal{H})\right)_{0}^{+}=\{$strictly positive definite $n \times n$ matrices $\}$, then: $D_{\Phi \circ \lambda}(\xi, \zeta)=$
(a). [57, Def.1] $\operatorname{tr}_{\mathcal{H}}(\xi(\log \xi-\log \zeta)-\xi-\zeta) \forall(\xi, \zeta) \in \mathcal{K}^{+} \times \mathcal{K}_{0}^{+}$;

[^3]（b）．$[26, \S 5]\left\langle\xi, \zeta^{-1}\right\rangle_{\mathcal{K}}-\log \operatorname{det}\left(\xi \zeta^{-1}\right)-n=h\left(\zeta^{-1 / 2} \xi \zeta^{-1 / 2}\right)-n \forall(\xi, \zeta) \in \mathcal{K}_{0}^{+} \times \mathcal{K}_{0}^{+}$，for $h(\xi):=$ $\operatorname{tr}_{\mathcal{K}}(\xi)-\log \operatorname{det}(\xi) ;$
（c）．［41，p．376］ $\operatorname{tr}_{\mathcal{H}}(\xi(\log \xi-\log \zeta)+(\mathbb{I}-\xi)(\log (\mathbb{I}-\xi)-\log (\mathbb{I}-\zeta))) \forall(\xi, \zeta) \in B^{+} \times \operatorname{int}\left(B^{+}\right)$，where $B^{+}:=\mathcal{K}^{+} \cap B\left(\mathcal{K},\|\cdot\|_{2}\right) ;$
（d）．［31，Cor．4．18．（ii）］ $\operatorname{tr}_{\mathcal{H}}\left(\gamma|\xi|^{1 / \gamma}+(1-\gamma) \zeta^{1 / \gamma}-\xi \zeta^{1 / \gamma-1}\right) \forall(\xi, \zeta) \in \mathcal{K} \times \mathcal{K}_{0}^{+}$（under restriction of a domain of $\zeta$ to $\mathcal{K}_{0}^{+}$）；
（e）．［31，Cor．4．18．（iii）］$\left.D_{\alpha}(\xi, \zeta):=\operatorname{tr}_{\mathcal{H}}\left(\zeta^{\alpha}-\frac{1}{1-\alpha} \xi^{\alpha}+\frac{\alpha}{1-\alpha} \zeta^{\alpha-1} \xi\right) \forall(\xi, \zeta, \alpha) \in \mathcal{K}^{+} \times \mathcal{K}_{0}^{+} \times\right] 0,1[$ ， $\left.-D_{\alpha}(\xi, \zeta) \forall(\xi, \zeta, \alpha) \in \mathcal{K}_{0}^{+} \times \mathcal{K}_{0}^{+} \times\right]-\infty, 0[;$
with＂$D_{\Phi \circ \lambda}(\xi, \zeta):=\infty$ otherwise＂in all cases，and $\langle\xi, \zeta\rangle_{\mathcal{K}}:=\operatorname{tr}_{\left(\mathfrak{G}_{2}(\mathcal{H})\right)^{\text {sa }}}(\xi \zeta)$ ．All cases（a）－（e）of $D_{\Phi \circ \lambda}$ are also the special cases of $D_{\Psi}^{\operatorname{tr} \mathcal{H}}$ ，with a range of good optimisation theoretic properties implied by the fact that $\Phi \circ \boldsymbol{\lambda}$ is Euler－Legendre．$\ell$ can be set to be any automorphism of $\left(\mathfrak{G}_{2}(\mathcal{H})\right)^{\text {sa }}$ preserving $\operatorname{int}(\operatorname{efd}(\Phi \circ \boldsymbol{\lambda}))$ ，e．g．a restriction of $\ell_{1 / 2}$ to a subset of $\left(\mathfrak{G}_{1}(\mathcal{H})\right)^{\text {sa }}$ ，corresponding to $\operatorname{int}(\operatorname{efd}(\Phi \circ \boldsymbol{\lambda}))$ ．

## Acknowledgements

I thank：Lucien Hardy，Ravi Kunjwal，Jerzy Lewandowski，and Marcin Marciniak for hosting me as a visitor；Francesco Buscemi，Paolo Gibilisco，and Anna Jenčová for hospitality and discussions；Michał Eckstein，Jan Głowacki，and Karol Horodecki for help；Perimeter Institute for Theoretical Physics and Polish National Science Centre（grants 2015／18／E／ST2／00327 and 2021／42／A／ST2／00356）for support．Research at Perimeter Institute is supported by the Government of Canada through In－ dustry Canada and by the Province of Ontario through the Ministry of Research and Innovation．

Remark．Cyrillic names and titles were bijectively transliterated from the original，using the system：ц $=\mathbf{c}$ ，
 ь＝＇，and analogously for capitalised letters．Symbol＊in front of a bibliographic item indicates that I have not seen this work．

## References

［1］Al＇ber Ya．I．，Butnariu D．，1997，Convergence of Bregman projection methods for solving consistent convex feasibility problems in reflexive Banach spaces，J．Optim．Theor．Appl．92，33－61．$\uparrow 3$.
［2］Alfsen E．M．，Shultz F．W．，1976，Non－commutative spectral theory for affine function spaces on convex sets，Mem．Amer．Math．Soc．172，American Mathematical Society，Providence．$\uparrow 6$ ．
［3］Amari S．（甘利俊一），Nagaoka H．（長岡浩司），1993，Jōhō kika no hōhō（情報幾何の方法），Iwanami Shoten，Tōkyō（Engl．transl．rev．ed．：2000，Methods of information geometry，Transl．Math．Monogr． 191，American Mathematical Society，Providence）．$\uparrow 4$.
［4］Bauschke H．H．，Borwein J．M．，1997，Legendre functions and the method of random Bregman projections，J．Conv．Anal．4，27－67．people．ok．ubc．ca／bauschke／Research／07．pdf．$\uparrow 6$.
［5］Bauschke H．H．，Borwein J．M．，Combettes P．L．，2001，Essential smoothness，essential strict convexity， and Legendre functions in Banach spaces，Commun．Contemp．Math．3，615－647． people．ok．ubc．ca／bauschke／Research／18．pdf．$\uparrow 3$.
［6］Bauschke H．H．，Borwein J．M．，Combettes P．L．，2003，Bregman monotone optimization algorithms，Soc． Industr．Appl．Math．J．Contr．Optim．42，596－636．people．ok．ubc．ca／bauschke／Research／28．pdf．$\uparrow 2$.
［7］Berdikulov M．A．，Odilov S．T．，1995，Obobšennye spin－faktory，Uzbek．mat．zhurn．1995：1，10－15． www．fuw．edu．pl／～kostecki／scans／berdikulovodilov1995．pdf．$\uparrow 6$ ．
［8］Brègman L．M．，1966，Relaksacionnyı̆ metod nahozhdeniya obšě̆ tochki vypuklykh mnozhestv i ego primenenie dlya zadach optimizacii，Dokl．Akad．nauk SSSR 171，1019－1022．mathnet．ru：dan32741 （Engl．transl．：1966，A relaxation method of finding a common point of convex sets and its application to problems of optimization，Soviet Math．Dokl．7，1578－1581）．$\uparrow 1,2$.
［9］Brøndsted A．，1964，Conjugate convex functions in topological vector spaces，Kong．Danske Vidensk． Selsk．Mat．－fys．Medd．34，1－26．$\uparrow 3$.
［10］Brunk H．D．，Ewing G．M．，Utz W．R．，1957，Minimizing integrals in certain classes of monotone functions，Pacific J．Math．7，833－847．euclid：pjm／1103043663．$\uparrow 2$.
［11］＊Burg J．P．，1967，Maximum entropy spectral analysis，Texas Instruments，Dallas（repr．in：Childers D．G．（ed．），1978，Modern spectrum analysis，IEEE Press，New York，pp．34－41）．$\uparrow 6$.
［12］Butnariu D．，Iusem A．N．，2000，Totally convex functions for fixed point computation and infinite dimensional optimization，Kluwer，Dordrecht．$\uparrow 3$.
［13］Chencov N．N．，1968，Nesimmetrichnoe rasstoyanie mezhdu raspredeleniyami veroyatnosteŭ，entropiya $i$ teorema Pifagora，Mat．zametki 4，323－332．mathnet．ru：mz9452（Engl．transl．：1968，Nonsymmetrical distance between probability distributions，entropy and the theorem of Pythagoras，Math．Notes Acad． Sci．USSR 4，686－691）．$\uparrow 1,2,4$.
［14］Csiszár I．，1995，Generalized projections for non－negative functions，Acta Math．Hung．68，161－185． $\uparrow 1$.
［15］Csiszár I．，Matúš F．，2012，Generalized minimizers of convex integral functions，Bregman distance， pythagorean identities，Kybernetika 48，637－689．arXiv：1202．0666．$\uparrow 2$.
［16］Delpech S．，2005，Modulus of continuity of the Mazur map between unit balls of Orlicz spaces and approximation by Hölder mappings，Illinois J．Math．49，195－216．$\uparrow 6$.
［17］Eguchi S．，1983，Second order efficiency of minimum contrast estimators in a curved exponential family，Ann．Statist．11，793－803．euclid：aos／1176346246．$\uparrow 1$.
［18］Eguchi S．，1985，A differential geometric approach to statistical inference on the basis of contrast functionals，Hiroshima Math．J．15，341－391．euclid：hmj／1206130775．$\uparrow 1$.
［19］Euler L．，1770，Institutionum calculi integralis，Vol．3，Academia Scientiarum Imperialis， Sankt－Peterburg．pbc．gda．pl／dlibra／publication／20282／edition／16413（Russ．transl．：1958，Integral＇noe ischislenie，Vol．3，Gosudarstvennoe izdatel＇stvo fiziko－matematicheskô̆ literatury，Moskva；Engl． transl．：2010，Foundations of integral calculus，Vol．3， www．17centurymaths．com／contents／integralcalculusvol3．htm）．$\uparrow 3$ ．
［20］Falcone A．J．，Takesaki M．（竹崎正道），2001，The non－commutative flow of weights on a von Neumann algebra，J．Funct．Anal．182，170－206．www．math．ucla．edu／～mt／papers／QFlow－Final．tex．pdf．$\uparrow 5$.
［21］Fenchel W．，1949，On conjugate convex functions，Canadian J．Math．1，73－77． www．cs．cmu．edu／～suvrit／teach／papers／1949＿fenchel＿conjugate＿convex＿functions．pdf．$\uparrow 3$.
［22］Gibilisco P．，Pistone G．，1998，Connections on non－parametric statistical manifolds by Orlicz space geometry，Inf．Dim．Anal．Quant．Prob．Relat．Top．1，325－347． art．torvergata．it／retrieve／handle／2108／49737／18230／IDAQP1998．pdf．$\uparrow 5$.
［23］Harremoës P．，2017，Quantum information on spectral sets，in：Bossert M．，Hanly S．，ten Brink S．， Ulukus S．（eds．）， 2017 IEEE International Symposium on Information Theory，IEEE，Piscataway，pp． 1549－1553．arXiv：1701．06688．$\uparrow 2$.
［24］Havrda J．，Chárvat F．，1967，Quantification method of classification processes：concept of structural a－entropy，Kybernetika 3，30－35．www．kybernetika．cz／content／1967／1／30／paper．pdf．$\uparrow 6$.
［25］Itakura F．（板倉文忠），Saito S．（齐藤收三），1968，Analysis synthesis telephony based on the maximum likelihood method，in：Kohasi Y．（小橋豊）（ed．），Reports of the 6th international congress on acoustics， Vol．2，Maruzen，Tōkyō，pp．C17－C20．$\uparrow 6$.
［26］James W．，Stein C．，1961，Estimation with quadratic loss，in：Neyman J．（ed．），Proceedings of the fourth Berkeley symposium on mathematical statistics and probability，Vol．1，University of California Press，Berkeley，pp．361－379．$\uparrow 7$ ．
［27］Jenčová A．，2005，Quantum information geometry and non－commutative $L_{p}$ spaces，Inf．Dim．Anal． Quant．Prob．Relat．Top．8，215－233．www．mat．savba．sk／～jencova／pdf／lpspaces．pdf（early version： arXiv：math－ph／0311004）．$\uparrow 5$ ．
［28］Kaczmarz S．，1933，O homeomorfji pewnych przestrzeni．－The homeomorphy of certain spaces，Bull． Internat．Acad．Polon．Sci．Lett．，Class．Sci．Math．Natur．：Sér．A，Sci．Math．1933：2，145－148． www．fuw．edu．pl／～kostecki／scans／kaczmarz1933．pdf．$\uparrow 5$.
［29］Kapur J．N．，1972，Measures of uncertainty，mathematical programming and physics，J．Indian Soc． Agric．Stat．24，47－66．$\uparrow 6$.
［30］Kostecki R．P．，2011，The general form of $\gamma$－family of quantum relative entropies，Open Sys．Inf．Dyn． 18，191－221．arXiv：1106．2225．$\uparrow 5$.
［31］Kostecki R．P．，2017，Postquantum Brègman relative entropies，arXiv：1710．01837．$\uparrow 4,5,6,7$.
［32］Kostecki R．P．，2023，Generalised Brègman relative entropies and quasi－nonexpansive operators．I． Banach space setting；II．Families induced by geometry of $L_{p}$ spaces over $W^{*}$－and JBW－algebras；III．

Families induced by geometry of noncommutative Orlicz spaces；IV．Affine connections，to be submitted．$\uparrow 4,5,6$ ．
［33］Kullback S．，1959，Information theory and statistics，Wiley，New York（2nd rev．ed．：1968，Dover，New York）．$\uparrow 1$.
［34］Lauritzen S．L．，1987，Statistical manifolds，in：Amari S．（甘利俊一），Barndorff－Nielsen O．E．，Kass R．E．，Lauritzen S．L．，Rao C．R．，Differential geometry in statistical inference，Institute of Mathematical Statistics，Hayward，pp．163－216．$\uparrow 1$.
［35］Legendre A．－M．，1787，Mémoire sur l＇intégration de quelques équations aux différences partielles，Mém． Académ．Royale Sci．1787，309－351．thibaut．horel．org／convex／legendre－1787．pdf．$\uparrow 3$.
［36］Lewis A．S．，1996，Convex analysis on the hermitian matrices，Soc．Industr．Appl．Math．J．Optim．6， 164－177．$\uparrow 6$.
［37］Martín－Márquez V．，Reich S．，Sabach S．，2012，Right Bregman nonexpansive operators in Banach spaces，Nonlin．Anal．Theor．Meth．Appl．75，5448－5465． ssabach．net．technion．ac．il／files／2015／12／MRS2012－1．pdf．$\uparrow 2$.
［38］Mazur S．M．，1929，Une remarque sur l＇homéomorphie des champs fonctionnels，Stud．Math．1，83－85． matwbn．icm．edu．pl／ksiazki／sm／sm1／sm114．pdf．$\uparrow 5$.
［39］Moreau J．－J．，1962，Fonctions convexes en dualité，Séminaires de mathématiques，Faculté des sciences de Montpellier，Montpellier．thibaut．horel．org／convex／moreau－62．pdf．$\uparrow 3$.
［40］Nagaoka H．（長岡浩司），Amari S．（甘利俊一），1982，Differential geometry of smooth families of probability distributions，Technical report METR 82－7，University of Tōkyō，Tōkyō． www．fuw．edu．pl／～kostecki／scans／nagaokaamari1982．pdf．$\uparrow 5$.
［41］Nock R．，Magdalou B．，Briys E．，Nielsen F．，2013，Mining matrix data with Bregman matrix divergences for portfolio selection，in：Nielsen F．，Bhatia R．（eds．），Matrix information geometry， Springer，Berlin，pp．373－402．$\uparrow 7$ ．
［42］Norden A．P．，1937，Über Paare konjugierter Parallerübertragungen，Trudy semin．vekt．tenzorn．anal． 4，205－255．www．fuw．edu．pl／～kostecki／scans／norden1937．pdf．$\uparrow 1$.
［43］Petz D．，2007，Bregman divergence as relative operator entropy，Acta Math．Hungar．116，127－131． web．archive．org／web／20170705131857／http：／／www．renyi．hu／～petz／pdf／112bregman．pdf．$\uparrow 2$.
［44］Pinsker M．S．，1960，Èntropiya，skorost＇sozdaniya èntropii i èntropiŭnaya ustoŭchivost＇gaussovskikh sluchaŭnykh velichin i processov，Dokl．Akad．nauk SSSR 133，531－534．mathnet．ru：dan28087（Engl． transl．：1960，The entropy，the rate of establishment of entropy and entropic stability of gaussian random variables and processes，Soviet Math．Dokl．1，886－889）．$\uparrow 6$.
［45］Pinsker M．S．，1960，Informaciya i informacionnaya ustoŭchivost＇sluchaĭnykh velichin i processov， Izdatel＇stvo Akademii nauk SSSR，Moskva（Engl．transl．：1964，Information and information stability of random variables and processes，Holden－Day，San Francisco）．$\uparrow 6$.
［46］Reem D．，Reich S．，De Pierro A．，2019，Re－examination of Bregman functions and new properties of their divergences，Optimization 68，279－348．arXiv：1803．00641．$\uparrow 6$.
［47］Reich S．，Sabach S．，2009，A strong convergence theorem for a proximal－type algorithm in reflexive Banach spaces，J．Nonlin．Conv．Anal．10，471－485． ssabach．net．technion．ac．il／files／2015／12／RS2009．pdf．$\uparrow 3$.
［48］Ricard É．，2015，Hölder estimates for the noncommutative Mazur maps，Arch．Math．104，37－45． arXiv：1407．8334．$\uparrow 5$.
［49］Rockafellar R．T．，1963，Convex functions and dual extremum problems，Ph．D．thesis，Harvard University，Cambridge．sites．math．washington．edu／～rtr／papers／rtr001－PhDThesis．pdf．$\uparrow 3$.
［50］Rockafellar R．T．，1967，Conjugates and Legendre transforms of convex functions，Canad．J．Math．19， 200－205．sites．math．washington．edu／～rtr／papers／rtr014－LegendreTransform．pdf．$\uparrow 3$ ．
［51］Sanov I．N．，1957，O veroyatnosti bol＇shikh otkloneniŭ sluchaĭnykh velichin，Matem．sb． 84 （nov．ser． 42），11－42．mathnet．ru：msb5043（Engl．transl．：1961，On the probability of large deviations of random variables，Sel．Transl．Math．Statist．Probab．1，213－244）．$\uparrow 1$.
［52］Sen R．N．，1948，Parallel displacement and scalar product of vectors，Proc．Nat．Inst．Sci．India 14， 45－52．个 1.
［53］Shima H．（志磨裕彦），1976，On certain locally flat homogeneous manifolds of solvable Lie groups， Osaka J．Math．13，213－229．个 3.
［54］Shima H．（志磨裕彦），2007，The geometry of hessian structures，World Scientific，Singapore．$\uparrow 4$.
［55］Tsallis C．，1988，Possible generalization of Boltzmann－Gibbs statistics，J．Stat．Phys．52，479－487．$\uparrow 6$.
［56］Tsuda K．，Rätsch G．，Warmuth M．K．，2005，Matrix exponentiated gradient updates for on－line learning and Bregman projection，J．Mach．Learn．Res．6，995－1018．$\uparrow 2$.
［57］Umegaki H．（梅垣寿春），1961，On information in operator algebras，Proc．Jap．Acad．37，459－461．
euclid:pja/1195523632. $\uparrow 6$.
[58] Wiener N., 1948, Cybernetics or control and communication in the animal and the machine,
Hermann/Technology Press/Wiley, Paris/Cambridge/New York (2nd rev. ed.: 1961). $\uparrow 1$.
[59] Woo H.K. (우현균), 2017, A characterization of the domain of beta-divergence and its connection to Bregman variational model, Entropy 19:482, 1-27. $\uparrow 6$.


[^0]:    ${ }^{1}$ Cf. «information is the negative of the quantity (...) defined as entropy» [58, p. 76].
    ${ }^{2}$ In comparison, given $(M, \mathbf{g})$, the Levi-Civita affine connection $\nabla^{\mathbf{g}}$ is characterised among all torsion-free affine connections on $\mathbf{T} M$ by $\mathbf{g}\left(\mathbf{t}_{c}^{\nabla \mathbf{g}}(\cdot), \mathbf{t}_{c}^{\nabla \mathbf{g}}(\cdot)\right)=\mathbf{g}$. Each torsion-free Norden-Sen geometry determines $\nabla^{\mathbf{g}}$ by $\nabla^{\mathbf{g}}=$ $\frac{1}{2}(\nabla+\widetilde{\nabla})[42$, p. 211].
    ${ }^{3}$ Following [34, §4], the torsion-free Norden-Sen geometries are sometimes called "statistical manifolds". Apart from not crediting the original authors, this terminology is misleading, since these geometries are independent of any notion of statistics.

[^1]:    ${ }^{4}$ More precisely, $D_{\Psi}^{\operatorname{tr} \mathcal{H}}(x, y):=\operatorname{tr}_{\mathcal{H}}\left(D_{\Psi}(x, y)\right)$ for a convex and Gateaux differentiable $\Psi: W \rightarrow \mathfrak{B}(\mathcal{H})$, where $W$ is a convex subset of a Banach space, e.g. $W=(\mathfrak{B}(\mathcal{H}))_{\star}^{+}$. The evaluation of $D_{\Psi}^{\operatorname{tr} \mathcal{H}}(x, y)$ is thus defined by spectral calculus applied to $\Psi$.

[^2]:    ${ }^{5}$ These functions are usually called "Legendre" (for $X=\mathbb{R}^{n}$ they were introduced namelessly in [49, Thm. C-K]). Yet, the transformation $\mathrm{d}(z(x, y)-p x-q y)=-x \mathrm{~d} p-y \mathrm{~d} q$, with $p=\frac{\partial z(x, y)}{\partial x}$ and $q=\frac{\partial z(x, y)}{\partial y}$, was introduced first by Euler [19, Part I, Probl. 11], and only 17 years later by Legendre [35, p. 347].

[^3]:    ${ }^{6}$ Cf. [31, Rem. 4.15]. In [32] we also extend Example 6.1 to $\Psi=\Psi_{\varphi}$.
    ${ }^{7} D_{\Phi}(x, y)=\sum_{i=1}^{n}\left(-\log \frac{x_{i}}{y_{i}}+\frac{x_{i}}{y_{i}}-1\right) \forall(x, y) \in\left(\mathbb{R}^{n}\right)_{0}^{+} \times\left(\mathbb{R}^{n}\right)_{0}^{+}$, corresponding to $\Phi$ in (b), was introduced by Pinsker in [44, Eqn. (4)] [45, Eqn. (10.5.4)]. The result by Itakura-Saito [25, Eqn. (7)], usually cited as a reference for this $D_{\Phi}$, has appeared 8 years later, and contains only a formula $2 \log (2 \pi)+\frac{1}{2 \pi} \int_{-\pi}^{\pi} \mathrm{d} t\left(\log (y(t))+\frac{x(t)}{y(t)}\right)$.
    ${ }^{8}$ Cf.: $-\frac{2^{\alpha-1}(\alpha-1)}{2^{\alpha-1}-1}\left(\Phi_{\alpha}+\frac{n-1}{\alpha-1}\right) \forall \alpha>0$ in [24, Thm. 1]; $-\Phi_{\alpha}-\frac{n-1}{\alpha-1} \forall \alpha \in \mathbb{R}$ in [55, Eqn. (1)]; a detailed analysis when $\frac{1}{\alpha}\left(-\Phi_{\alpha}-\frac{n}{\alpha-1}\right)$ is Euler-Legendre in [59, Thm. 5].

