Perimeter Institute for Theoretical Physics

Quantum measurements from entropic projections

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Abstract

In this essay I examine the potential of deriving quantum mechanical state update rules using the principle of constrained maximisation of quantum relative entropy. Two types of state update are surveyed: Lüders' rule, corresponding to a projective measurement, and partial trace, corresponding to restriction of a bipartite state to a state on one tensor factor. The original result contained in this work is a proof that partial trace between invertible matrices is a special case of constrained relative entropy maximisation.

The work on projective measurements builds upon existing work and I demonstrate that the conventional state update maximises relative entropy but I do not establish uniqueness.

Finally a strategy for a proof that would hold in infinite dimensional quantum mechanics is discussed.

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1 Introduction

Since the early days of quantum mechanics physicists have disagreed about the philosophical implications of the theory. At the essence of the theory are abstract mathematical objects. Some have argued that these abstract mathematical objects represent facts about the state of nature. Others have argued that they merely represent the available knowledge about outcomes of measurements. Since the two worldviews do not offer different predictions one might argue that the discussion itself is without consequence. However they may still provide an insight into how quantum theory could be founded mathematically.
Let us consider the latter view. Whatever information an experimenter has about a physical system is then supposed to be contained in a "state", and quantum theory is understood as a framework that provides a set of rules for how to update this state when manipulating the system in various ways. The updated state is then supposed to represent the new information available to the experimenter after performing the manipulations. But the only new information available to the experimenter is that he has performed the manipulation. Which mechanism is then specifically resonsible for incorporating this information into the new state? It may be possible to formulate quantum theory in such a way that the state update explicitly deals with the change of information. In this way a minimal amount of ontological baggage would reside in the theory itself since that theory wouldn't refer to external entities but only to knowledge about them. Information is however in general a vague concept that requires som specific formalisation before it can be incorporated into a theory. The concept of "relative entropy" is widely considered as providing a useful measure of difference in information between two states of knowledge.
Given two classical probability distributions $p(x)$ and $q(x)$ an often used relative entropy function is $S(p,q) = -\int dx p(x) \log \frac{p(x)}{q(x)}$ [21]. This is a stricty negatively valued function. Let $\mathcal{M}$ denote a space of probability distributions, let $p \in \mathcal{M}$ be given and let $\mathcal{Q} \subseteq \mathcal{M}$ be a convex subset. A constrained maximisation of $S(p,q)$ for $p \in \mathcal{Q}$ means picking the $q \in \mathcal{Q}$ such that $S(p,q)$ takes its biggest possible value. Bayes' rule is often taken as a primitive concept in classical probability theory for how one should infer from a probability distribution to a new one given some information. Caticha and Giffin [14] showed that the principle of constrained maximisation of relative entropy reproduces Bayes' rule in a special case. This means that one may take the principle of constrained maximisation of relative entropy as a fundamental general prescription for state update in classical probability theory. That something as fundamental in classical probability theory as Bayes' rule can be derived in this way leads to the question: is it possible to obtain an analogue of this result in quantum mechanics? It is the purpose of this essay to convince the reader of an answer in the positive.
First let us reacquaint ourselves with state update in conventional quantum mechanics. Suppose a projective measurement is being performed on a state described by a density matrix ρ on a Hilbert space $\mathcal{H}$. The measurement corresponds to a hermitian operator $O = \sum_{i\in I} \lambda_i P_i$, where I is a countable index set, $\{P_i\}_{i\in I}$ are projectors on $\mathcal{H}$ and $\{\lambda_i\}_{i\in I} \subset \mathbb{R}$. The new state is given by Lüders' rules. In the case where the outcome is known and corresponds to the projector

P_j, then the new state is $\rho_{\text{new}} = \frac{P_j \rho P_j}{\text{tr}[\rho P_j]}$. We will call this case **strong collapse**. When the outcome is not known the new state is given by $\rho_{\text{new}} = \sum_{i \in I} P_i \rho P_i$. This case will be called **weak collapse**.

When considering a state described by a density matrix ρ on a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ the conventional way to restrict to a state on $\mathcal{H}_A$ is by means of partial trace $\rho \mapsto \text{tr}_B \rho$. The state on the joint system $\mathcal{H}$ then is $\text{tr}_B \rho \otimes \mathbb{1}_B$.

In Section 2 the main work of this essay is done. It is shown here that in finite dimensional Hilbert spaces the above rules for state update are obtained as unique maximisers of relative entropy under right assumptions. For weak collapse choose the constraint set $\{\omega \in \mathcal{D} \,|\, [\omega, P_i] = 0 \,\forall i \in I\}$. For a discussion on why the constraint set takes the form it does see Herbut [16]. The strong collapse is obtained as a limit of the quantum Jeffreys rule $\rho \mapsto \rho_{\text{new}} = \sum_{i \in I} p_i \frac{P_i \rho P_i}{\text{tr}[\rho P_i]}$ (that applies when the outcome related to P_i is known with a certain probability p_i for each outcome $i \in I$). For quantum Jeffreys the constraint set is $\{\omega \in \mathcal{D} \,|\, [\omega, P_i] = 0, \, \text{tr}[\omega P_i] = p_i \forall i \in I\}$. Strong collapse is then achieved from quantum Jeffreys in the limit $p_i \to 0 \,\forall i \neq j$ and $p_j \to 1$. The derivations of the collapse rules follow Kostecki [10]. The work on partial trace is entirely original and here the constraint set is $\{\omega \in \mathcal{D} \,|\, \omega = \alpha \omega_A \otimes \mathbb{1}_B, \omega_A$ is a density matrix on $\mathcal{H}_A, \ \alpha \in \mathbb{C}\}$, representing states of maximal uncertainty on system $\mathcal{H}_B$.

Section 3 discusses a strategy for proving the above results for infinite dimensional systems. The mathematical framework for which this strategy is intended is that of W^*-algebras. This Section contains a review of their structure and the formula for relative entropy between states on a W^*-algebra proposed by Araki [7][6]. It ends by discussing the concept of conditional expectations and states a result that provides the first step of the proof.

2 Finite dimensions

In this section I will derive weak and strong collapse as well as partial trace from minimising entropic distance under appropriate constraint sets for finite dimensional Hilbert spaces.

The strategy of the two proofs are the same and before going at the individual proofs I will give the general results that will be employed. In short we will see that if any distance satisfies a property called triangle equality for an arbitrary point and a point in some subset then the distance between those two points is the minimal distance from the arbitrary point and any point in the subset. The strategy in the proofs therefore is to establish this triangle equality for the collapse rules and partial trace.

2.1 The general proof strategy of this essay

Let $D : \mathcal{M} \times \mathcal{M} \to [0, \infty]$ be a distance on a set $\mathcal{M}$, meaning that $D(\phi, \psi) \geq 0 \,\forall \phi, \psi \in \mathcal{M}$ and $D(\phi, \psi) = 0 \Leftrightarrow \phi = \psi$. Notice that a distance is more general than a metric since it is not required that it satisfies triangle inequality and it is not required to be symmetric.

Definition 2.1. Let $\mathcal{M}$ be an arbitrary set with $\mathcal{Q} \subseteq \mathcal{M}$ and $\psi \in \mathcal{M}$. If there exists a $\rho \in \mathcal{Q}$

such that $\forall \phi \in \mathcal{Q} : D(\phi, \psi) = D(\phi, \rho) + D(\rho, \psi)$ then $\mathcal{Q}$ is said to satisfy the triangle equality for ψ at ρ with respect to D.

As a matter of notation I will write $\underset{x \in \mathrm{dom} f}{\operatorname{arginf}}\{f(x)\}$ to mean the argument for which the function $f : \mathrm{dom}(f) \to \mathbb{R}$ takes its minimal value. To be more precise $\underset{x \in \mathrm{dom}(f)}{\operatorname{arginf}}\{f(x)\} \in \mathrm{Powerset}(\mathrm{dom}(f))$. With the notation $\rho = \underset{x \in \mathrm{dom}(f)}{\operatorname{arginf}}\{f(x)\}$ is meant the singleton $\{\rho\} = \underset{x \in \mathrm{dom}(f)}{\operatorname{arginf}}\{f(x)\}$. Note that it is possible to have the case $\underset{x \in \mathrm{dom}(f)}{\operatorname{arginf}}\{f(x)\} = \emptyset$.
When I use the word relative entropy I mean a function $S : \mathcal{M} \times \mathcal{M} \to [-\infty, 0]$ such that $-S := D$ is a distance on $\mathcal{M}$.
The following Lemma provides the main insight behinds the proofs:

Lemma 2.2. *If $\mathcal{Q} \subseteq \mathcal{M}$ satifies triangle equality for ψ at $\rho \in \mathcal{Q}$ with respect to D and $D(\rho, \psi) < \infty$, then*

$$\rho = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \psi)\}$$

Proof. Since the distance D is non-negative and $D\{\phi, \rho\} = 0$ only at $\phi = \rho$ we have that $\rho = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho)\}$. We can rewrite $\underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho)\}$ as $\underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho) + D(\rho, \psi)\}$ since $D(\rho, \psi)$ is a fixed positive and finite number for all $\phi \in \mathcal{Q}$ by assumption. Hence, using the triangle equality,

$$\rho = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho)\} = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho) + D(\rho, \psi)\} = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\rho, \psi)\}$$

□

Remark 2.3. Finiteness of $D(\rho, \psi)$ is used explicitly in order to achieve the Lemma. If $D(\rho, \psi) = \infty$ then the expression $\underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho) + D(\rho, \psi)\}$ certainly is not the same as $\underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho)\}$ since the latter is a singleton and the former is equal to $\mathcal{Q}$. Showing the finiteness of $D(\rho, \psi)$ is therefore crucial if we want to obtain a minimiser of distance.

Corollary 2.4. *Assume that $\mathcal{Q} \subseteq \mathcal{M}$ satisfies triangle equality for $\psi \in \mathcal{M}$ at $\rho \in \mathcal{Q}$ with respect to D and $D(\rho, \psi) < \infty$. Then $D(\phi, \psi)$ for $\phi \in \mathcal{Q}$ is uniquely minimised at ρ.*

Proof. It is immediate from Lemma 2.2. To be more specific assume that triangle equality for $\mathcal{Q}$ is also satisfied at $\sigma \in \mathcal{Q}$ with respect to D and that $D(\sigma, \psi) < \infty$. Making again use of the fact that D is a distance, we have

$$\rho = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho)\} = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \rho) + D(\rho, \psi)\} = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \psi)\}$$

$$= \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \sigma) + D(\sigma, \psi)\} = \underset{\phi \in \mathcal{Q}}{\operatorname{arginf}}\{D(\phi, \sigma)\} = \sigma.$$

□

Lemma 2.5. *Let $\mathcal{D}$ be the space of density matrices of a fixed finite dimension. Then*

$$D(\rho, \psi) := tr[\psi(\log \psi - \log \rho)]$$

is a distance on $\mathcal{D}$.

Proof. This was shown in more generality by Umegaki [5]. □

In the following two sections where we will show that collapse rules and partial trace minimise entropic distance we will work with $\mathcal{D}$ and the distance $D(\rho, \psi) := \mathrm{tr}[\psi(\log \psi - \log \rho)]$. The proofs for the collapse rules follow Kostecki [10] and the proof of partial trace is original work for this essay.

2.2 Derivation of collapse rules

We will start by deriving weak collapse. Let $\mathcal{D}$ denote the space of density matricies on a Hilbert space $\mathcal{H}$ of a fixed finite dimension and consider the set of projectors $\{P_i\}_{i\in I}$ where $P_i P_j = \delta_{ij}\, \forall i, j \in I$ and I is a finite index set. We would then like to show that $\underset{\phi\in\mathcal{Q}_W}{\mathrm{arginf}}\{D(\phi, \psi)\} = \sum_{i\in I} P_i \psi P_i$ where the constraint set $\mathcal{Q}_W = \{\omega \in \mathcal{D} \,|\, [P_i, \omega] = 0\, \forall i \in I\}$.
Let us approach this problem by first considering case where there is only the projectors P and $\mathbb{1} - P$ and the constraint set therefore is $\mathcal{Q} = \{\omega \in \mathcal{D} | [\omega, P] = 0\}$. Let us first address the finiteness of the alleged minimiser $\rho = P\psi P + (\mathbb{1} - P)\psi(\mathbb{1} - P)$. The proof of this is incomplete but I will provide a sketch for how I think one should be able to show this.

Conjecture 2.6. *If $\psi \in \mathcal{D}$ is arbitrary and $\rho = P\psi P + (\mathbb{1} - P)\psi(\mathbb{1} - P)$ then $D(\rho, \psi) < \infty$.*

Sketch of the proof: Recall that given a Hilbert space $\mathcal{H}$ the Hilbert-Schmidt space can be formed by the square root of density operators on $\mathcal{H}$. In the following the subscript HS denotes that the object in question is on a Hilbert-Schmidt space, so for A^2, B^2 density matrices on $\mathcal{H}$ satisfying $\mathrm{tr}[A^*A] < \infty$ and $\mathrm{tr}[B^*B] < \infty$, then by definition $A, B \in \mathcal{H}_{HS}$ and the inner product is denoted by $\langle A, B\rangle_{HS} := \mathrm{tr}[A^*B]$ and the norm is $||A||_{HS}^2 = \langle A, A\rangle_{HS}$. The Hilbert-Schmidt inner product satisfies Cauchy-Schwarz inequality, that is $\langle A, B\rangle_{HS} \le ||A||_{HS}||B||_{HS}$, and the norm satisfies triangle inequality $||A + B||_{HS} \le ||A||_{HS} + ||B||_{HS}$. Therefore we have, since ψ is hermitian and therefore $\psi^{1/2}$ is hermitian and since $1 = \mathrm{tr}[\psi] = ||\psi^{1/2}||$, that

$$\begin{aligned} D(\rho, \psi) &= \mathrm{tr}[\psi(\log \psi - \log \rho)] = \mathrm{tr}[\psi^{1/2}(\log \psi - \log \rho)\psi^{1/2}] = \langle \psi^{1/2}, (\log \psi - \log \rho)\psi^{1/2}\rangle_{HS} \\ &\le ||\psi^{1/2}||_{HS}^2 ||(\log \psi - \log \rho)\psi^{1/2}||_{HS}^2 = ||(\log \psi - \log \rho)\psi^{1/2}||_{HS}^2 \\ &\le \Big(||\log \psi \cdot \psi^{1/2}||_{HS} + ||\log \rho \cdot \psi^{1/2}||_{HS}\Big)^2 \end{aligned} \tag{1}$$

When the matrices are not invertible the matrix logarithms are potentially badly behaved because $\lim_{x\to 0^+} \log(x) \to -\infty$. Let us diagonalise ψ and ρ individually and and denote their eigenvalues by $\{\kappa_i^\psi\}$ and $\{\kappa_i^\rho\}$ respectively. By replacing all the zero eigenvalues with ε, where

$$0 < \varepsilon < \min\{1, \min\{\kappa_i^\psi \mid \kappa_i^\psi = 0\}, \min\{\kappa_i^\rho \mid \kappa_i^\rho = 0\}\},$$

and dividing the resulting matrices with $1 + \dim(\ker(\psi)) \cdot \varepsilon$ and $1 + \dim(\ker(\rho)) \cdot \varepsilon$ respectively (that is dividing with 1+ ε times the number of eigenvalues that are zero so that the resulting matrices still have trace 1) we obtain invertible matrices ψ_ε and ρ_ε satisfying $\lim_{\varepsilon\to 0^+} \psi_\varepsilon = \psi$ and $\lim_{\varepsilon\to 0^+} \rho_\varepsilon = \rho$. To be more explicit, let the the dimension of the Hilbert space ψ and ρ act on be equal to $N \in \mathbb{N}$, let U_ψ and U_ρ be the matrices to change to diagonal basis so

$$\psi = U_\psi \begin{pmatrix} \kappa_1^\psi & 0 & \cdots & 0 \\ 0 & \kappa_2^\psi & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \kappa_N^\psi \end{pmatrix} U_\psi^{-1}$$

$$\to \psi_\varepsilon = \frac{1}{1 + \dim(\ker(\psi))\varepsilon} U_\psi \begin{pmatrix} \max(\kappa_1^\psi, \varepsilon) & 0 & \cdots & 0 \\ 0 & \max(\kappa_2^\psi, \varepsilon) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \max(\kappa_N^\psi, \varepsilon) \end{pmatrix} U_\psi^{-1},$$

and similarly for ρ_ε. Note that the basis change matrices are not changed. Now ψ_ε and ρ_ε are invertible by construction, so the logarithms are non-singular. If we can establish that $\lim_{\varepsilon\to 0^+} D(\rho_\varepsilon, \psi_\varepsilon) < \infty$ then we would get that $D(\rho, \psi)$ is finite (the distance function D is continuous, being a continuous function of continuous functions).
Equation (1) takes the form

$$D(\rho, \psi) = \lim_{\varepsilon\to 0^+} D(\rho_\varepsilon, \psi_\varepsilon) \le \lim_{\varepsilon\to 0^+} \Big(\mathrm{tr}[(\log \psi_\varepsilon)^2 \psi_\varepsilon] + \mathrm{tr}[(\log \rho_\varepsilon)^2 \psi_\varepsilon]\Big)^2 \tag{2}$$

The first term on the right hand side of equation (2) is finite in the limit since $\lim_{\varepsilon\to 0^+} \varepsilon(\log \varepsilon)^2 = 0$. For the second term we will use the following property that holds for any $0 < \lambda < 1$:

$$\log \lambda = \lim_{t\to 0^+} t^{-1}(\lambda^t - 1). \tag{3}$$

By going to a basis for a given ε where $\log \rho_\varepsilon$ is diagonal we thus see we can write

$$\log \rho_\varepsilon = \lim_{t\to 0^+} t^{-1}\big(\rho_\varepsilon^t - \mathbb{1}\big). \tag{4}$$

In this basis, and hence in any basis (since trace is basis independent), the second term on the right hand side of equation (2) can be rewritten using (3) as

$$\mathrm{tr}\Big[\psi_\varepsilon\big(\lim_{t\to 0^+} t^{-1}(\rho_\varepsilon^t - \mathbb{1})\big)^2\Big] = \lim_{t\to 0^+} t^{-2}\mathrm{tr}\Big[\psi_\varepsilon\big(\rho_\varepsilon^t - \mathbb{1}\big)^2\Big] = \lim_{t\to 0^+} t^{-2}\mathrm{tr}\Big[\psi_\varepsilon\big(\rho_\varepsilon^{2t} - 2\rho_\varepsilon^t + \mathbb{1}\big)\Big] \tag{5}$$

It is fine to take out the limit since we are dealing with a continuous function of t. Now let us write $\psi_\epsilon = \begin{pmatrix} \psi_{\varepsilon,11} & \psi_{\varepsilon,12} \\ \psi_{\varepsilon,21} & \psi_{\varepsilon,22} \end{pmatrix}$ so that $\rho_\varepsilon = \begin{pmatrix} \rho_{\varepsilon,11} & 0 \\ 0 & \rho_{\varepsilon,22} \end{pmatrix}$. The matrix ρ_ε is block diagonal since

$$\rho = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix} = \begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix} \mathrm{diag}\{\kappa_i^\rho\} \begin{pmatrix} U_{11}^{-1} & 0 \\ 0 & U_{22}^{-1} \end{pmatrix}$$
$$\to \begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix} \mathrm{diag}\{\max\{\varepsilon, \kappa_i^\rho\}\} \begin{pmatrix} U_{11}^{-1} & 0 \\ 0 & U_{22}^{-1} \end{pmatrix}$$

Equation (5) then reads

$$\mathrm{tr}[(\log\rho_\varepsilon)^2\psi_\varepsilon] = \lim_{t\to 0^+} t^{-2}\mathrm{tr}\Bigg[\begin{pmatrix} \psi_{\varepsilon,11} & \psi_{\varepsilon,12} \\ \psi_{\varepsilon,21} & \psi_{\varepsilon,22} \end{pmatrix}\left(\begin{pmatrix} \rho_{\varepsilon,11}^{2t} & 0 \\ 0 & \rho_{\varepsilon,22}^{2t} \end{pmatrix} - \begin{pmatrix} 2\rho_{\varepsilon,11}^{t} & 0 \\ 0 & 2\rho_{\varepsilon,22}^{t} \end{pmatrix} + \mathbb{1}\right)\Bigg]$$
$$= \lim_{t\to 0^+} t^{-2}\mathrm{tr}\Bigg[\begin{pmatrix} \psi_{\varepsilon,11} & \psi_{\varepsilon,12} \\ \psi_{\varepsilon,21} & \psi_{\varepsilon,22} \end{pmatrix}\begin{pmatrix} \rho_{\varepsilon,11}^{-1} & 0 \\ 0 & \rho_{\varepsilon,22}^{-1} \end{pmatrix}\begin{pmatrix} \rho_{\varepsilon,11}(\rho_{\varepsilon,11}^{2t} - 2\rho_{\varepsilon,11}^{t} + 1) & 0 \\ 0 & \rho_{\varepsilon,22}(\rho_{\varepsilon,22}^{2t} - 2\rho_{\varepsilon,22}^{t} + 1) \end{pmatrix}\Bigg]$$
$$\leq \lim_{t\to 0^+} t^{-2}\Bigg(\mathrm{tr}\Bigg[\begin{pmatrix} \psi_{\varepsilon,11}\rho_{\varepsilon,11}^{-1} & \psi_{\varepsilon,12}\rho_{\varepsilon,22}^{-1} \\ \psi_{\varepsilon,21}\rho_{\varepsilon,11}^{-1} & \psi_{\varepsilon,22}\rho_{\varepsilon,22}^{-1} \end{pmatrix}^*\begin{pmatrix} \psi_{\varepsilon,11}\rho_{\varepsilon,11}^{-1} & \psi_{\varepsilon,12}\rho_{\varepsilon,22}^{-1} \\ \psi_{\varepsilon,21}\rho_{\varepsilon,11}^{-1} & \psi_{\varepsilon,22}\rho_{\varepsilon,22}^{-1} \end{pmatrix}\Bigg]\times$$
$$\times\,\mathrm{tr}\Bigg[\begin{pmatrix} \rho_{\varepsilon,11}^{2t+1} - 2\rho_{\varepsilon,11}^{t+1} + \rho_{\varepsilon,11} & 0 \\ 0 & \rho_{\varepsilon,22}^{2t+1} - 2\rho_{\varepsilon,22}^{t+1} + \rho_{\varepsilon,22} \end{pmatrix}^*\begin{pmatrix} \rho_{\varepsilon,11}^{2t+1} - 2\rho_{\varepsilon,11}^{t+1} + \rho_{\varepsilon,11} & 0 \\ 0 & \rho_{\varepsilon,22}^{2t+1} - 2\rho_{\varepsilon,22}^{t+1} + \rho_{\varepsilon,22} \end{pmatrix}\Bigg]\Bigg)^{1/2}.$$

Finiteness follows if it can be shown that limit $\varepsilon \to 0^+$ and $t \to 0^+$ may be interchanged and that the subsequent limits are all finite. This may not be the case but I am not working with the optimal bound and so if this specific bound should turn out not to be finite it would not be problematic. In fact a finite bound must exist because a different proof of the minimisasation in finite dimensions has been done by Hellmann et al [12]. Furthermore Kostecki [10] has proved the result in higher generality using a method very akin to what I am using here.

Now let us establish the triangle equality in the case of a single projector.

Theorem 2.7. *Let $\mathcal{Q} = \{\omega \in \mathcal{D} | [P, \omega] = 0\}$ and let $\rho = P\psi P + (\mathbb{1} - P)\psi(\mathbb{1} - P)$, where P is a projector on $\mathcal{D}$. Then $\mathcal{Q}$ satisfies triangle equality for $\psi \in \mathcal{D}$ at ρ, that is*

$$\forall \phi \in \mathcal{Q} : D(\phi, \psi) = D(\phi, \rho) + D(\rho, \psi).$$

Proof. Since ρ and ϕ are block-diagonal so is the matrix formed by taking the log of them . Hence they commute with P and $\mathbb{1} - P$. Therefore

$$\log\rho - \log\phi = (\log\rho - \log\phi)(P + (\mathbb{1} - P)) = (\log\rho - \log\phi)(P^2 + (\mathbb{1} - P)^2)$$
$$= (P + (\mathbb{1} - P))(\log\rho - \log\phi)(P + (\mathbb{1} - P))$$

Since $P = P(P\psi P + (\mathbb{1} - P)\psi(\mathbb{1} - P))P = P\psi P + 0$, we see by the cyclicity of the trace that

$\mathrm{tr}(\psi P(\log\rho - \log\phi)P) = \mathrm{tr}(\rho P(\log\rho - \log\phi)P)$ and by the same arguments $\mathrm{tr}(\psi(\mathbb{1} - P)(\log\rho - \log\phi)(\mathbb{1} - P)) = \mathrm{tr}(\rho(\mathbb{1} - P)(\log\rho - \log\phi)(\mathbb{1} - P))$. Thereby we obtain

$$\mathrm{tr}(\psi(\log\rho - \log\phi)) = \mathrm{tr}(\rho(\log\rho - \log\phi)). \tag{6}$$

But this is equivalent with

$$\mathrm{tr}(\psi(\log\psi - \log\phi)) = \mathrm{tr}(\rho(\log\rho - \log\phi)) + \mathrm{tr}(\psi(\log\psi - \log\rho)).$$

Written in terms of D, this is

$$D(\phi, \psi) = D(\phi, \rho) + D(\rho, \psi),$$

which is the triangle equality. $\square$

With the above results we will now attempt to tackle the situation of multiple projectors all commuting with one another.

Lemma 2.8. *Let $\{P_i\}_{i\in I}$ be a set of projectors on $\mathcal{D}$ for some finite index set I satisfying $[P_i, P_j] = 0\,\forall i, j \in I$ and define for each $k \in I$ a set $\mathcal{Q}_k := \{\omega \in \mathcal{D} | [P_k, \omega] = 0\}$. If $\psi \in \mathcal{Q}_j$ then $\underset{\phi\in Q_i}{\mathrm{arginf}}\{D(\phi, \psi)\} \in Q_j$.*

Proof. From Theorem 2.7 and Lemma 2.8 we have

$$\underset{\phi\in\mathcal{Q}_i}{\mathrm{arginf}}\{D(\phi, \psi)\} = P_i\psi P_i + (\mathbb{1} - P_i)\psi(\mathbb{1} - P_i).$$

Since $\psi \in \mathcal{Q}_j$ we have $[\psi, P_j] = 0$. This means, since $[P_i, P_j] = 0$, that

$$[P_i\psi P_i + (\mathbb{1} - P_i)\psi(\mathbb{1} - P_i), P_j] = P_i[\psi, P_j]P_i + (\mathbb{1} - P_i)[\psi, P_j](\mathbb{1} - P_j) = 0.$$

Hence by definition of $\mathcal{Q}_j$, $\underset{\phi\in Q_i}{\mathrm{arginf}}\{D(\phi, \psi)\} \in Q_j$. $\square$

With the next Lemma we will find an expression for the minimiser of $D(\phi, \psi)$ for ϕ commuting with P_i for all i. Once this is established we are ready to prove that this minimiser is given by weak collapse rule.

Lemma 2.9. *Let $\{P_i\}_{i\in\{1,\ldots,n\}}$ be projectors satisfying $[P_i, P_j] = 0\,\forall i, j \in \{1, \ldots, n\}$, define for each $i \in [1, \ldots, n]$ sets $\mathcal{Q}_i = \{\omega \in \mathcal{D}\,|\,[P_i, \omega] = 0\}$ and define the set $\mathcal{Q} = \{\omega \in \mathcal{D}|[P_i, \omega] = 0\,\forall i \in \{1, \ldots, n\}\}$. Then we have*

$$\underset{\phi\in\mathcal{Q}}{\mathrm{arginf}}\{D(\phi, \psi)\} = \rho_n,$$

where $\rho_k = \underset{\phi\in\mathcal{Q}_k}{\mathrm{arginf}}\{D(\phi, \rho_{k-1})\}$, $\rho_0 = \psi$, $\mathcal{Q} = \cap_{i=1}^n \mathcal{Q}_i$ satisfies triangle equality at ρ_n, and $D(\rho_n, \psi) < \infty$.

Proof. This will be proved by induction. Assume $\rho_k = \underset{\phi \in \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_k}{\operatorname{arginf}} \{D(\phi, \psi)\}$ and that $\cap_{i=1}^k \mathcal{Q}_i$ satisfies triangle equality at ρ_k and $D(\phi, \rho_k) < \infty$. Then we want to show that $\rho_{k+1} = \underset{\phi \in \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_{k+1}}{\operatorname{arginf}} \{D(\phi, \psi)\}$, that $\cap_{i=1}^{k+1} \mathcal{Q}_i$ satisfies triangle equality at ρ_{k+1}, and that $D(\phi, \rho_{k+1}) < \infty$.
Note that for any $\sigma \in \cap_{i=1}^k \mathcal{Q}_i$ we have $\underset{\phi \in \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_{k+1}}{\operatorname{arginf}} \{D(\phi, \sigma)\} \in \cap_{i=1}^k \mathcal{Q}_i$ by lemma 2.8. Therefore, by definition,

$$\rho_{k+1} = \underset{\phi \in \mathcal{Q}_{k+1}}{\operatorname{arginf}} \{D(\phi, \rho_k)\} = \underset{\phi \in \mathcal{Q}_{k+1}}{\operatorname{arginf}} \left\{ D\Big(\phi, \underset{\phi' \in \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_k}{\operatorname{arginf}} \{D(\phi', \psi)\}\Big) \right\} = \underset{\phi \in \mathcal{Q}_1 \cap \ldots \cap \mathcal{Q}_{k+1}}{\operatorname{arginf}} \{D(\phi, \psi)\}.$$

Then, by Theorem 2.7, we have that $\mathcal{Q}_{k+1}$ satisfies triangle equality for ρ_k at ρ_{k+1}. Now choose a $\phi \in \cap_{i=1}^{k+1} \mathcal{Q}_i$. Then the assumed triangle equality for $\mathcal{Q}_k$ for ψ at ρ_k gives

$$\begin{aligned} D(\phi, \psi) &= D(\phi, \rho_k) + D(\rho_k, \psi), \\ D(\phi, \rho_k) &= D(\phi, \rho_{k+1}) + D(\rho_{k+1}, \rho_k). \end{aligned}$$

Adding these two gives

$$\begin{aligned} D(\phi, \psi) + D(\phi, \rho_k) &= D(\phi, \rho_k) + D(\rho_k, \psi) + D(\phi, \rho_{k+1}) + D(\rho_{k+1}, \rho_k) \\ \Leftrightarrow \quad D(\phi, \psi) &= D(\phi, \rho_{k+1}) + D(\rho_k, \psi) + D(\rho_{k+1}, \rho_k) \\ \Leftrightarrow \quad D(\phi, \psi) &= D(\phi, \rho_{k+1}) + D(\rho_{k+1}, \psi). \end{aligned}$$

The last line follows from the triangle equality of $\cap_{i=1}^k \mathcal{Q}_i$ for ψ at ρ_k, and since $\rho_{k+1} \in \cap_{i=1}^{k+1} \mathcal{Q}_i$, so in particular is in $\cap_{i=1}^k \mathcal{Q}_i$. This is what we wanted, and Theorem 2.7 ensures that $\mathcal{Q}_1$ satisfies triangle equality for ψ at ρ_k. If Conjecture 2.6 holds then we have finiteness since $D(\rho_{k+1}, \psi) = D(\rho_{k+1}, \rho_k) + D(\rho_k, \psi)$. The second term is finite by assumption and the first is finite by Conjecture 2.6 since
$\rho_{k+1} = \underset{\phi \in \mathcal{Q}_{k+1}}{\operatorname{arginf}} \{D(\phi, \rho_k)\} = P_{k+1} \rho_k P_{k+1} + (\mathbb{1} - P_{k+1}) \rho_k (\mathbb{1} - P_{k+1})$. □

We are now ready to achieve the weak collapse rule. We further assume that $P_i P_j = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$.

Theorem 2.10. Weak collapse rule: *Let* $\{P_i\}_{i \in \{1,\ldots,n\}}$ *be projectors on* $\mathcal{D}$ *and let* $P_i P_j = \delta_{ij} \forall i, j$ *and let* $\sum_{i=1}^n P_i = \mathbb{1}$ *and* $\mathcal{Q}_W = \{\omega \in \mathcal{D} | [P_i, \omega] = 0 \forall i\}$. *Then* $\underset{\phi \in \mathcal{Q}}{\operatorname{arginf}} \{D(\phi, \psi)\} = \sum_{i=1}^n P_i \psi P_i$.

Proof. We are going to apply Lemma 2.9. Take $\rho_0 = \psi$ and $\rho_1 = \underset{\phi \in \mathcal{Q}_1}{\operatorname{arginf}} \{D(\phi, \psi)\}$. From Theorem 2.7 we know that $\rho_1 = P_1 \psi P_1 - (\mathbb{1} - P_1) \psi (\mathbb{1} - P_1)$. We are now going to continue by using Lemma 2.9, Theorem 2.7 and induction. So assume that
$\rho_k = \sum_{i=1}^k P_i \psi P_i + (\mathbb{1} - \sum_{i=1}^k P_i) \psi (\mathbb{1} - \sum_{i=1}^k P_i)$. Because of the condition $P_i P_j = \delta_{ij}$ we have

that $(\mathbb{1}-\sum_{i=1}^{k}P_i)$ is a projector, so we can use Theorem 2.7 to achieve that

$$\begin{aligned}\rho_{k+1} &= P_{k+1}\rho_k P_{k+1} + (\mathbb{1}-P_{k+1})\rho_k(\mathbb{1}-P_{k+1})\\ &= \Big(P_{k+1}+(\mathbb{1}-P_{k+1})\Big)\Big(\sum_{i=1}^{k}P_i\psi P_i + (\mathbb{1}-\sum_{i=1}^{k}P_i)\psi(\mathbb{1}-\sum_{i=1}^{k}P_i)\Big)\Big(P_{k+1}+(\mathbb{1}-P_{k+1})\Big)\\ &= \sum_{i=1}^{k+1}P_i\psi\sum_{i=1}^{k+1}P_i + (\mathbb{1}-\sum_{i=1}^{k+1}P_i)\psi(\mathbb{1}-\sum_{i=1}^{k+1}P_i)\end{aligned}$$

For this n'th term, since $\sum_{i=1}^{n}P_i=\mathbb{1}$, we have $\rho_n=\sum_{i=1}^{n}P_i\psi P_i$. By Lemma 2.9

$$\rho_n = \underset{\phi\in\mathcal{Q}_W}{\operatorname{arginf}}\{D(\phi,\psi)\}$$

□

We now continue to strong collapse. The strategy will be first to prove the so-called quantum Jeffrey's rule, the following theorem. Strong collapse then follows in a limit. The following proofs follow closely that of Kostecki [10].

Theorem 2.11. Quantum Jeffrey's rule: *Let $\{P_i\}_{i=1}^{n}$ be projectors on $\mathcal{D}$ and let $P_iP_j=\delta_{ij}\forall i,j$ and let $\sum_{i=1}^{n}P_i=\mathbb{1}$ and also assume that $tr(\psi P_i)\neq 0\,\forall i$ and let $\mathcal{Q}_{QJ}=\{\omega\in\mathcal{D}|[\omega,P_i]=0, tr(\omega P_i)=\lambda_i\,\forall i\}$, where $\lambda_i>0\,\forall i$ and $\sum_i\lambda_i=1$. Then*

$$\underset{\phi\in\mathcal{Q}_{QJ}}{\operatorname{arginf}}\{D(\phi,\psi)\} = \sum_{i=1}^{n}\lambda_i\frac{P_i\psi P_i}{tr(\psi P_i)}.$$

Proof. Let $\rho,\phi\in\mathcal{Q}_{QJ}$ and let $\rho_i=\lambda_i\frac{P_i\psi P_i}{\mathrm{tr}(P_i\psi P_i)}\,\forall i$ and $\rho:=\bigoplus_i\rho_i$ (that is a block diagonal matrix with ρ_i in the i-th block and zero elsewhere) and let $\phi:=\bigoplus_i\phi_i$ with $\mathrm{tr}(\phi_i)=\lambda_i$. We will show that ρ uniquely minimises D with respect to any ϕ of the above sort.
We have that

$$\log\rho-\log\phi=\bigoplus_{i=1}^{n}(\log\rho_i-\log\phi_i).$$

This means that

$$\mathrm{tr}(\psi(\log\rho-\log\phi)) = \sum_{i=1}^{n}\mathrm{tr}(P_i\psi P_i(\log\rho_i-\log\phi_i)) = \sum_{i=1}^{n}\mathrm{tr}\Big(\frac{\lambda_i}{\lambda_i}\frac{\mathrm{tr}(P_i\psi P_i)}{\mathrm{tr}(P_i\psi P_i)}P_i\psi P_i(\log\rho_i-\log\phi_i)\Big).$$

Write $\log\rho_i=\log\lambda_i\frac{\rho_i}{\lambda_i}=\log\lambda_i\mathbb{1}\frac{\rho_i}{\lambda_i}$. Now since ρ_i is positive and $[\mathbb{1},\rho_i]=0$, we have $\lambda_i\mathbb{1}\frac{\rho_i}{\lambda_i}=\exp(\log(\lambda_i\mathbb{1})+\log(\rho_i\lambda_i))$. Similarly with $\log\phi_i$, so

$$\log\rho_i-\log\phi_i=\log\frac{\rho_i}{\lambda_i}-\log\frac{\phi_i}{\lambda_i}$$

With this and with $\tilde{\rho}_i:=\frac{\rho_i}{\lambda_i}$ and $\tilde{\phi}_i:=\frac{\phi_i}{\lambda_i}$, we have

$$\mathrm{tr}(\psi(\log\rho-\log\phi))=\sum_{i=1}^{n}\lambda_i\mathrm{tr}(P_i\psi P_i)\mathrm{tr}(\tilde{\rho}_i(\log\tilde{\rho}_i-\log\tilde{\phi}_i))=\sum_{i=1}^{n}\lambda_i\mathrm{tr}(P_i\psi P_i)D(\tilde{\phi}_i,\tilde{\rho}_i)\geq 0$$

$$\Leftrightarrow\quad \mathrm{tr}(\psi(\log\psi-\log\rho))\leq\mathrm{tr}(\psi(\log\psi-\log\phi)$$

or

$$D(\rho,\psi)\leq D(\phi,\psi)$$

Equality only holds for $\rho=\phi$. This means that ρ uniquely minimises D. □

Remark 2.12. ***Strong collapse:*** Strong collapse follows from quantum Jeffreys in the limit $\lambda_i\to 0$ for all $i\neq k$ and $\lambda_k\to 1$.

2.3 Partial trace from entropic projection

Now we turn to the original contributions of this essay. In this Section we will examine whether partial trace is also a special case of minimisation of distance. The proof is complete in the case of invertible matrices. However in the case of general matrices there is still work left in figuring out if the distance $D(\rho,\psi)$ is always finite.
We are interested in maps $\mathcal{D}_{\mathcal{H}_A\otimes\mathcal{H}_B}\to\mathcal{D}_{\mathcal{H}_A\otimes\mathcal{H}_B}$ which erase information about Hilbert space $\mathcal{H}_B$, so the constraint set we choose is $\mathcal{Q}=\{\omega\in\mathcal{D}_{\mathcal{H}_A\otimes\mathcal{H}_B}\,|\,\omega=\omega_A\otimes\mathbb{1}_B,\,\omega_A\in\mathcal{D}_{\mathcal{H}_A}\}$ where $\mathbb{1}_B$ is the identity on $\mathcal{H}_B$.

Let us begin by looking at the issue of finite distance between an arbitrary state and the proposed minimiser $\rho=\mathrm{tr}_B\psi\otimes\mathbb{1}_B$.

Conjecture 2.13. *Let $\mathcal{D}=\mathcal{D}_{\mathcal{H}_A\otimes\mathcal{H}_B}$ be the set of density matrices for the bipartite system given by the finite dimentional Hilbert spaces $\mathcal{H}_A\otimes\mathcal{H}_B$ and let $\psi\in\mathcal{D}_{\mathcal{H}_A\otimes\mathcal{H}_B}$ be arbitrary and $\rho=tr_B\psi\otimes\mathbb{1}_B$. Then $D(\rho,\psi)<\infty$.*

Sketch of the proof: Let us use the same estimates as in the sketch of the proof of Conjecture 2.6 and the same trick of taking $\psi,\rho\to\psi_\varepsilon,\rho_\varepsilon$ with $0<\varepsilon<\min\{1,\min\{\kappa_i^\psi\},\min\{\kappa_i^\rho\}\}$, where $\{\kappa_i^\psi\}$ are the eigenvales of ψ and $\{\kappa_i^\rho\}$ are the eigenvalues of ρ. Let us replace eigenvalues that are equal to 0 with ε such that ψ_ε and ρ_ε are invertible, and $\lim_{\varepsilon\to0^+}\psi_\varepsilon=\psi$ and $\lim_{\varepsilon\to0^+}\rho_\varepsilon=\rho$. In such case we have

$$D(\rho,\psi)\leq\lim_{\varepsilon\to0^+}\Big(\mathrm{tr}[(\log\psi_\varepsilon)^2\psi_\varepsilon]+\mathrm{tr}[(\log\rho_\varepsilon)^2\psi_\varepsilon]\Big)^2 \tag{7}$$

Again the first term is finite in the limit, and using the same logarithm identity as before we need to check that $\lim_{\varepsilon\to0^+}\lim_{t\to0^+}t^{-2}\mathrm{tr}\Big[\psi_\varepsilon\big(\rho_\varepsilon^t-\mathbb{1}\big)^2\Big]$ is finite. Note that the function under the limit operation is a continuous function for all $t,\varepsilon>0$.

The sufficient assumption that will give us what we want, is that we may interchange the limits $\varepsilon \to 0^+$ and $t \to 0^+$.
One way this could be made true is to prove that the convergence in the limit $t \to 0^+$ is uniform. Using this assumption the second term in equation (7) reads

$$\lim_{t\to 0^+}\lim_{\varepsilon\to 0^+} t^{-2}\mathrm{tr}\Big[\psi_\varepsilon\big(\rho_\varepsilon^t-\mathbb{1}\big)^2\Big] = \lim_{t\to 0^+} t^{-2}\mathrm{tr}\Big[\psi\big(\rho^t-\mathbb{1}\big)^2\Big] = \lim_{t\to 0^+} t^{-2}\mathrm{tr}\Big[\psi\big(\rho^{2t}-2\rho^t+\mathbb{1}\big)\Big]$$

$$= \lim_{t\to 0^+} t^{-2}\mathrm{tr}\Big[\psi\Big(\big((\mathrm{tr}_B\psi)^{2t}-2(\mathrm{tr}_B\psi)^t+\mathbb{1}_A\big)\otimes\mathbb{1}_B\Big)\Big] = \lim_{t\to 0^+} t^{-2}\mathrm{tr}_A\Big[\mathrm{tr}_B\psi\big((\mathrm{tr}_B\psi)^{2t}-2(\mathrm{tr}_B\psi)^t+\mathbb{1}_A\big)\Big]$$

$$= \lim_{t\to 0^+} t^{-2}\mathrm{tr}_A\Big[(\mathrm{tr}_B\psi)^{2t+1}-2(\mathrm{tr}_B\psi)^{t+1}+\mathrm{tr}_B\psi\Big].$$

Here we used the fact from Remark 2.15 to take the trace over system A only. Let us now denote the index set of non-zero eigenvalues of $\mathrm{tr}_B\psi$ by I and denote the non-zero eigenvalues $\{\lambda_i\}_{i\in I}$. We can then calculate the limit above explicitly by using L'Hôpital's rule twice:

$$\lim_{t\to 0^+}\frac{\sum_{i\in I}\Big(\lambda_i^{2t+1}-2\lambda_i^{t+1}+\lambda_i\Big)}{t^2} = \lim_{t\to 0^+}\frac{\sum_{i\in I}\Big(2\lambda_i^{2t+1}-2\lambda_i^{t+1}\Big)\log\lambda_i}{2t}$$

$$= \lim_{t\to 0^+}\frac{\sum_{i\in I}\Big(4\lambda_i^{2t+1}-2\lambda_i^{t+1}\Big)(\log\lambda_i)^2}{2} = \sum_{i\in I}\lambda_i(\log\lambda_i)^2<\infty.$$

Note that in the case that both ψ and ρ are non-singular we have $\rho_\varepsilon=\rho$ and $\psi_\varepsilon=\psi$ so the limit exchange is trivially true. Thus the finiteness is proven to hold in this case. It is left for future work to figure out if $\lim_{\varepsilon\to 0^+}(\rho_\varepsilon,\psi_\varepsilon)<\infty$ in the general case.

In order to show that the proposed minimiser satisifies triangle equality we will utilize a decomposition into tensor factors of matrices with trace 1 as explained in the following Lemma. I will take $\mathrm{Lin}(\mathcal{H}_A,\mathcal{H}_B)$ to denote the space of linear maps from the Hilbert space $\mathcal{H}_A$ to the Hilbert space $\mathcal{H}_B$.

Lemma 2.14. *Let $\psi\in\mathcal{D}=\mathcal{D}_{\mathcal{H}_A\otimes\mathcal{H}_B}$. Then there exist a countable index set I_{AB} and sets $\{T_i^A\}_{i\in I_{AB}}$ with $T_i^A\in Lin(\mathcal{H}_A,\mathcal{H}_A)$ and $tr[T_i^A]=1$ $\forall i$ and $\{T_i^B\}_{i\in I_{AB}}$ with $T_i^B\in Lin(\mathcal{H}_B,\mathcal{H}_B)$ and $tr[\psi_i^B]=1$ $\forall i$ and a set $\{\alpha_i\}_{i\in I_{AB}}$ with $\alpha_i\in\mathbb{C}$ $\forall i$ such that*

$$\psi=\sum_{i=1}^n\alpha_i T_i^A\otimes T_i^B$$

Proof. We have $\psi\in\mathrm{Lin}(\mathcal{H}_A\otimes\mathcal{H}_B,\mathcal{H}_A\otimes\mathcal{H}_B)$. For a vector space V let $V^\star$ denote the dual space of V. With this,

$$\mathrm{Lin}(\mathcal{H}_A\otimes\mathcal{H}_B,\mathcal{H}_A\otimes\mathcal{H}_B)=\mathcal{H}_A^\star\otimes\mathcal{H}_B^\star\otimes\mathcal{H}_A\otimes\mathcal{H}_B=\mathrm{Lin}(\mathcal{H}_A,\mathcal{H}_A)\otimes\mathrm{Lin}(\mathcal{H}_B,\mathcal{H}_B)$$

However we can find an index set I_A and I_B and matrices $\{T_i^A\}_{i\in I_A}$ and $\{T_j^B\}_{j\in I_B}$ with unit trace such that $\{T_i^A\}_{i\in I_A}$ spans $\mathrm{Lin}(\mathcal{H}_A,\mathcal{H}_A)$ and $\{T_j^B\}_{j\in I_B}$ spans $\mathrm{Lin}(\mathcal{H}_B,\mathcal{H}_B)$.

To be more concrete, if $\dim(\mathcal{H}_A) = n$ and $\dim(\mathcal{H}_B) = m$ choose $I_A = \{1, \dots, n^2\}$ and $I_B = \{1, \dots, m^2\}$ and pick the matrices

$$\{T_i^A\}_{i \in I_A} = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \dots, \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \dots \right\}$$

These are matrices that have either a 1 in the $(1,1)$-entry or a 1 in the $(1,1)$-entry as well as a 1 in one different entry while retaining a unit trace. Pick similar matrices for $\{T_i^B\}_{i \in I_B}$. Now a basis for $\mathcal{D}$ is simply all tensor products of T_i^A and T_j^B, so that there exists a $m \times n$-matrix α with complex entries such that

$$\psi = \sum_{i \in I_A, j \in I_B} \alpha_{i,j} T_i^A \otimes T_j^B$$

Now simply define a new index set I_{AB} that is isomorphic to $I_A \times I_B$ and we have the desired

$$\psi = \sum_{i \in I_{AB}} \alpha_i T_i^A \otimes T_i^B$$

$\square$

Remark 2.15. If ψ is an arbitrary density matrix then $\mathrm{tr}_B \psi \geq 0$. This follows from the fact that for any hermitian matrix $M \in \mathrm{Lin}(\mathcal{H}_A, \mathcal{H}_A)$ on subsystem A then $\mathrm{tr}[(M \otimes \mathbb{1}_B)\psi] = \mathrm{tr}_A[M \mathrm{tr}_B \psi]$, see for instance Nielsen and Chuang [1]
This same fact, along with linearity of partial trace, also ensures that in the decomposition of Lemma 2.14 we have

$$0 \leq \mathrm{tr}_B \psi = \mathrm{tr}_B \Big[\sum_{i \in I_{AB}} \alpha_i T_i^A \otimes T_i^B \Big] = \sum_{i \in I_{AB}} \alpha_i T_i^A \cdot \mathrm{tr}_B [T_i^B] = \sum_{i \in I_{AB}} \alpha_i T_i^A$$

Proposition 2.16. *Let $\mathcal{D} = \mathcal{D}_{\mathcal{H}_A \otimes \mathcal{H}_B}$ be as in Lemma 2.13, $\psi \in \mathcal{D}$ be arbitrary and let $\mathcal{Q} = \{\omega \in \mathcal{D} \mid \omega = \omega_A \otimes \mathbb{1}_B,\ \omega_A \in \mathcal{D}_{\mathcal{H}_A}\}$ where $\mathbb{1}_B$ is the identity on $\mathcal{H}_B$. Then $\mathcal{Q}$ satisfies triangle equality for $\psi \in \mathcal{D}_{A \otimes B}$ at $\rho := \frac{1}{\dim B} tr_B \psi \otimes \mathbb{1}_B \in \mathcal{Q}$ for D.*

Proof. Notice first that $\rho \in \mathcal{Q}$ since $\mathrm{tr}[\rho] = \mathrm{tr}[\frac{1}{\dim B} \mathrm{tr}_B[\psi] \otimes \mathbb{1}_B] = \frac{1}{\dim B} \mathrm{tr}_A[\mathrm{tr}_B \psi] \mathrm{tr}_B[\mathbb{1}_B] = \mathrm{tr}[\psi] \frac{\dim B}{\dim B} = 1$ and since $\mathrm{tr}_B \psi \geq 0$ as explaned in Remark 2.15.
We use Lemma 2.14 to write $\psi = \sum_{i \in_{AB}} \alpha_i T_i^A \otimes T_i^B$. Observe that by definition of partial trace (for cleanliness I drop the specification that $i \in I_{AB}$ in the sums),

$$\mathrm{tr}_B[\psi] = \mathrm{tr}_B \Big[\sum_i \alpha_i T_i^A \otimes T_i^B \Big] = \sum_i \alpha_i T_i^A \cdot \mathrm{tr}_B [T_i^B] = \sum_i \alpha_i T_i^A$$

Using that $\log(A \otimes \mathbb{1}) = \log(A) \otimes \mathbb{1}$ for any Hermitian matrix A, and the fact that $\mathrm{tr}[M] = \mathrm{tr}_A[\mathrm{tr}_B[M]]$ for any matrix
$M \in \mathrm{Lin}(\mathcal{H}_A \otimes \mathcal{H}_B, \mathcal{H}_A \otimes \mathcal{H}_B)$, we can write

$$\begin{aligned}
D(\rho,\psi) &= \mathrm{tr}[\psi\log\psi] - \mathrm{tr}[\psi\log\rho] = \mathrm{tr}[\psi\log\psi] - \mathrm{tr}[\psi(\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)\otimes \mathbb{1}_B)]\\
&= \mathrm{tr}[\psi\log\psi] - \mathrm{tr}\Big[\sum_i \alpha_i T_i^A\otimes T_i^B(\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)\otimes \mathbb{1}_B)\Big]\\
&= \mathrm{tr}[\psi\log\psi] - \sum_i \mathrm{tr}_A\Big[\alpha_i T_i^A\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)\Big]\cdot \mathrm{tr}_B[T_i^B]\\
&= \mathrm{tr}[\psi\log\psi] - \mathrm{tr}_A\Big[\sum_i \alpha_i T_i^A\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)\Big]\cdot \mathrm{tr}_B[T_i^B]\\
&= \mathrm{tr}[\psi\log\psi] - \mathrm{tr}_A\Big[\mathrm{tr}_B[\psi]\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)\Big].
\end{aligned}$$

Let $\phi = \phi_A\otimes \mathbb{1}_B\in\mathcal{Q}$ be arbitrary. Then

$$\begin{aligned}
D(\phi,\rho) &= \mathrm{tr}[\rho\log\rho] - \mathrm{tr}[\rho\log\phi]\\
&= \mathrm{tr}_A\Big[\frac{1}{\dim B}\mathrm{tr}_B[\psi]\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)\Big]\mathrm{tr}_B[\mathbb{1}_B] - \mathrm{tr}_A\Big[\frac{1}{\dim B}\mathrm{tr}_B[\psi]\log(\phi_A)\Big]\mathrm{tr}_B[\mathbb{1}_B]\\
&= \mathrm{tr}_A[\mathrm{tr}_B[\psi]\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)] - \mathrm{tr}_A[\mathrm{tr}_B[\psi]\log(\phi_A)].
\end{aligned}$$

We also have

$$\begin{aligned}
D(\phi,\psi) &= \mathrm{tr}[\psi\log\psi] - \mathrm{tr}[\psi\log\phi] = \mathrm{tr}[\psi\log\psi] - \mathrm{tr}\Big[\sum_i \alpha_i T_i^A\otimes T_i^B\log(\phi_A)\otimes\mathbb{1}_B\Big]\\
&= \mathrm{tr}[\psi\log\psi] - \mathrm{tr}_A[\mathrm{tr}_B[\psi]\log(\phi_A)].
\end{aligned}$$

With this we have

$$\begin{aligned}
D(\rho,\psi)+D(\phi,\rho) &= \mathrm{tr}[\psi\log\psi] - \mathrm{tr}_A\Big[\mathrm{tr}_B[\psi]\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)\Big]\\
&\quad + \mathrm{tr}_A[\mathrm{tr}_B[\psi]\log\Big(\frac{1}{\dim B}\mathrm{tr}_B[\psi]\Big)] - \mathrm{tr}_A[\mathrm{tr}_B[\psi]\log(\phi_A)]\\
&= \mathrm{tr}[\psi\log\psi] - \mathrm{tr}_A[\mathrm{tr}_B[\psi]\log(\phi_A)]\\
&= D(\phi,\psi). \qquad (8)
\end{aligned}$$

□

Remark 2.17. I have used explicitly the fact that we are considering finite dimensional Hilbert spaces. For instance the difinition of ρ in Proposition 2.16 is ill defined in infinite dimesions.

Theorem 2.18. *Let $\mathcal{D}_{\mathcal{H}_A\otimes\mathcal{H}_B}$ denote the set of density matrices on a joint Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$ and take $\psi\in\mathcal{D}_{\mathcal{H}_A\otimes\mathcal{H}_B}$. Then $D(\rho,\psi) = tr[\psi\log\psi - \psi\log\rho]$ for $\rho\in\mathcal{Q}$ is minimised uniquely by $\rho = tr_B[\psi]\otimes\mathbb{1}_B$.*

Proof. Follows immediately from Lemma 2.2 and Proposition 2.16. □

3 The strategy for a potential proof in infinite dimensions

Now we turn our attention to a much more complicated problem: Proving the same as above for infinite dimensional systems. A general proof in case of the collapse rules have been provided by Kostecki [10]. This section will discuss a different strategy that should work both for the collapse rules and for partial trace. This strategy combines a proposal by Carlos Guedos [22] with the use of the formula 3.1 (proposed by an anonymous referee of [10]). The actual proofs are not finished however and remain as future work. The strategy is quite general and the result would fit well into a generalisation of quantum mechanics without Hilbert spaces.
The strategy relies on the mathematical concept of W^*-algebra[1] which generalises the notion of algebras of bounded operators on a Hilbert space. In this section I will begin with a review of what W^*-algebras are, what some of their properties are and some relevant constructions needed to employ them. This will include a brief introduction to relative modular operators which will be used in a generalised definition of a distance that reduces to the one used in the previous section in the case of finite dimensional systems.
My aim is to give a general understanding of what the constructions are and what some of their key properties are. Much of the material can be found in any introductory text on C^*-algebra, such as in Davidson [3].

3.1 Review of W^*-algebra, representations and relative modular theory

3.1.1 C^*-algebras, states and W^*-algebras

I'll begin by reminding that a vector space A over a field $\mathbb{K}$ which is equipped with a distributive binary operation $A \times A \to A$ is called an algebra. The binary operation has to satisfy that $(ax) \cdot (by) = ab(x \cdot y)$ for $a, b \in \mathbb{K}$ and $x, y \in A$. A Banach algebra $\mathfrak{A}$ is an algebra over $\mathbb{R}$ or $\mathbb{C}$ that is also a Banach space whose norm satisfies $||x \cdot y|| \leq ||x|| \cdot ||y||$. If a Banach algebra $\mathfrak{A}$ is an algebra over $\mathbb{C}$ and is also equipped with a mapping $^* : \mathfrak{A} \to \mathfrak{A}$ satisfying the following properties

$$
\begin{aligned}
(x+y)^* &= x^* + y^* \\
(\lambda x)^* &= \bar{\lambda} x^* \\
x^{**} &= x \\
xy &= y^* x^*, \\
||x^* x|| &= ||x||^2,
\end{aligned}
$$

[1] The word von Neumann algebra may be more familiar; a von Neumann algebra is a specific representation of a W^* algebra which every W^*-algebra admits.

where $x, y \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, with $\bar{\lambda}$ denoting complex conjugate of λ and dropping $\cdot$ in products of algebra elements when there is no risk of confusion, then $\mathfrak{A}$ is called a C^*-algebra. I will take $\mathfrak{A}$ to always refer to a C^*-algebra in this section.
The textbook example of a C^* algebra is bounded operators $\mathcal{B}(\mathcal{H})$ on a complex Hilbert space $\mathcal{H}$.

We will restrict our attention to unital C^* algebras, that is C^* algebras with a unit element $\mathbb{1}$ satisfying for any $x \in \mathfrak{A}$ that $\mathbb{1}x = x\mathbb{1} = x$.
With the spectrum $\sigma(x)$ of an element $x \in \mathfrak{A}$ is meant[2]

$$\sigma(x) := \{\lambda \in \mathbb{C} | x - \lambda\mathbb{1} \text{ is not invertible}\}$$

An element $x \in \mathfrak{A}$ is called self-adjoint if $x = x^*$. The self-adjoint elements of the C^*-algebra will play the roles of observables for our quantum theory. An element $x \in \mathfrak{A}$ is called positive if it is self-adjoint and $\sigma(x) \subset [0, \infty)$, and the set of positive elements of $\mathfrak{A}$ is denoted $\mathfrak{A}^+$. Positive elements satisfy a few nice properties the first on being that every $x \in \mathfrak{A}^+$ has a unique positive square root $x^{1/2} \in \mathfrak{A}^+$. Furthermore it is always possible to find a $y \in \mathfrak{A}$ for any positive $x \in \mathfrak{A}^+$ so that $x = y^*y$.
One of the suprising and really neat fact is that for any C^* algebra $\mathfrak{A}$ the norm on $\mathfrak{A}$ is given entirely from the algebraic properties, namely

$$\forall x \in \mathfrak{A}: \ ||x|| = \sqrt{\sup\{|\lambda| \ \Big| \ \lambda \in \sigma(x)\}} \tag{9}$$

Given two different C^*-algebras a question one might have is whether they have the same algebraic structure. They do in the case that there exists a so-called $*$-isomorphism between them. A $*$-isomorphism between two C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is a mapping $f : \mathfrak{A}_1 \to \mathfrak{A}_2$ such that for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $x_1, x_2 \in \mathfrak{A}_1$

$$\begin{aligned} f(\lambda_1 x_1 + \lambda_2 x_2) &= \lambda_1 f(x_1) + \lambda_2 f(x_2) \\ f(x_1 x_2) &= f(x_1) f(x_2) \\ f(x_1^*) &= f(x_1)^* \end{aligned}$$

In light of (9) it is clear that $*$-isomorphisms preserve norm-topological properties since the norm is given just by the algebraic sgtructure.
We now have algebraic objects that will play the role of bounded operators on Hilbert spaces. We would like to also have something analogous to a state that can provide us with expectation values of these objects. They can be thought of as dual to the algebra elements.
The Banach dual $\mathfrak{A}^\star$ of $\mathfrak{A}$ is defined as the set of all $\mathbb{C}$-linear functionals $\mathfrak{A} \to \mathbb{C}$ that are continuous in the norm topology of $\mathfrak{A}$. An element $\omega \in \mathfrak{A}^\star$ said to be positive iff $x \in \mathfrak{A}^+$ implies $\omega(x) \geq 0$; tracial iff $\omega(xy) = \omega(yx)$; self-adjoint iff $\omega^*(x) := (\omega(x^*))^* = \omega(x)$; faithful iff $\omega(x) = 0 \Rightarrow x = 0$ for any $x \geq 0$ and normalized iff $\omega(\mathbb{1}) = 1$. The set of positive,

[2]The definition works for any unital Banach algebra of $\mathbb{K}$ with $\lambda \in \mathbb{K}$ instead.

linear, continuous functionals on $\mathfrak{A}$ is denoted $\mathfrak{A}^{\star +}$ and the set of positive, normalized, linear, continuous functionals on $\mathfrak{A}$ is denoted $\mathfrak{A}^{\star}{}_1^+$.
Suppose for a Banach algebra $\mathcal{B}$ that there exists a Banach space $\mathcal{B}_\star$ such that $(\mathcal{B}_\star)^\star = \mathcal{B}$. In that case $\mathcal{B}_\star$ is called a predual of $\mathcal{B}$. A remarkable fact is that fora C^*-algebra $\mathfrak{A}$ the predual $\mathfrak{A}_\star$ is unique. The elements of $\mathfrak{A}_\star^+ := \mathfrak{A}^{\star +} \cap \mathfrak{A}_\star$ are called **states** and elements of $\mathfrak{A}_\star^+{}_1 := \mathfrak{A}^{\star}{}_1^+ \cap \mathfrak{A}_\star$ are called **normalized states**. A C^*-algebra $\mathfrak{A}$ for which a predual $\mathfrak{A}_\star$ exists is called a W^* alegbra. In this text the symbol $\mathcal{N}$ will be used to describe W^* algebras.
For a W^*-algebra $\mathcal{N}$, set of projectors on $\mathcal{N}$ is denoted $\text{Proj}(\mathcal{N}) = \{x \in \mathcal{N} \,|\, x^2 = x\}$. From the definition of positivity we obtain a partial order on $\mathcal{N}$ defined such that $x \leq y$ implies $y - x \geq 0$. Using these definitions we can define the support-projection of a normalized state $\omega \in \mathcal{N}_\star{}_1^+$ as

$$\text{supp}(\omega) = \mathbb{1} - \sup\{P \in \text{Proj}(\mathcal{N}) \,|\, \omega(P) = 0\},$$

where the supremum is taken with respect to the partial order.

The definition extents to all semi-finite, normal weights on W^*-algebras. A weight is functional $\omega : \mathcal{N}^+ \to [0, \infty]$ such that $\omega(0) = 0$, $\omega(x+y) = \omega(x)+\omega(y)$, and $\lambda \geq 0 \Rightarrow \omega(\lambda x) = \lambda\omega(x)$. A weight ω is furthermore defined to be normal if it satisfies that $\omega(\sup\{x_i\}) = \sup\{\omega(x_i)\}$ for any increasing net $\{x_i\} \subseteq \mathfrak{A}^+$ and a semi-finite if it satisfies $\forall x \in \mathfrak{A}^+ \exists y \in \mathfrak{A}^+ : \omega(x) = \infty \Rightarrow (x \geq y \text{ and } 0 < \omega(y) < \infty)$.

3.1.2 The GNS construction, relative modular operators and quantum relative entropy

It would be nice to see that all these abstract objects allow to reconstruct familiar quantum mechanical structure. This can be done through the GNS construction.
A general representation π of a C^*-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ is defined as a $*$-homomorphism of $\mathfrak{A}$ onto $\mathcal{B}(\mathcal{H})$. A vector $\xi \in \mathcal{H}$ is said to be cyclic for a $\pi(\mathfrak{A})$ iff $\pi(\mathfrak{A})\xi := \cup_{x\in\pi(\mathfrak{A})}\{x\xi\}$ is norm dense in $\mathcal{B}(\mathcal{H})$. Given a C^*-algebra $\mathfrak{A}$ and $\omega \in \mathfrak{A}^{\star +}$ the GNS construction is a representation π_ω for a state ω onto a Hilbert space $\mathcal{H}_\omega$ along with a cyclic vector Ω_ω such that $||\Omega\omega||_{\mathcal{H}} = ||\omega||$ such that for $\forall x, y \in \mathfrak{A}$

$$\omega(x) = \langle \Omega_\omega, \pi_\omega(x)\Omega_\omega \rangle$$
$$\omega(y^* x) = \langle \pi_\omega(y)\Omega_\omega, \pi_\omega(x)\Omega_\omega \rangle$$

A triplet $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ is called a GNS representation for $\mathfrak{A}$. If ω is faithful, then Ω_ω is also seperating for $\mathcal{H}_\omega$, meaning that $\forall x \in \mathfrak{A} \pi_\omega(x)\Omega_\omega = 0 \Rightarrow x = 0$. Given a weight ϕ on a W^*-algebra $\mathcal{N}$ we can define a left ideal $\mathcal{I}_\phi$ called the Gelfand ideal as

$$\mathcal{I}_\phi = \{x \in \mathcal{N} \,|\, \phi(x^* x) = 0\}$$

From this ideal we can define equivalence classes of elements of $\mathcal{N}$ by mapping them to the quotient space $\mathcal{N}/\mathcal{I}_\phi$ as

$$\mathcal{N} \ni x \mapsto \{x + n \,|\, n \in \mathcal{I}_\phi\} := [x]_\phi \in \mathcal{N}/\mathcal{I}_\phi$$

Consider the scalar form $\langle \cdot, \cdot \rangle_\omega$ given by $\langle x, y \rangle_\omega = \omega x^* y \forall x, y \in \mathfrak{A}$. The Hilbert space $\mathcal{H}_\omega$ is then constructed as the completion of $\mathfrak{A}/\mathcal{I}_\omega$ in the topology generated by $\langle \cdot, \cdot \rangle_\omega$ equipped with the inner product $\langle \cdot, \cdot \rangle_\omega$. The representation π_ω is defined so $\pi_\omega(x) : [y]_\omega \mapsto [xy]_\omega$ and the cyclic vector Ω_ω is defined as $\Omega_\omega := [\mathbb{1}]_\omega$.

Now I will introduce the topic of relative modular operators. They are used for achieving Arakis definition of quantum relative entropy that handles arbitrary W^*-algebras and reduces to the Umegaki entropic distance we have used in the previous sections when considering bounded operators on finite dimensional Hilbert spaces. The topic is quite involved and I will not go into full details.

The relative modular operators arise as polar decompositions of operators that relate abstract algebra elements x to their starred elements x^*. That they are polar decompositions ensures their non-negativity.

A polar decomposition is intuitively like rewriting a complex number $z = re^{i\theta}$ with $\theta \in \mathbb{R}$ and $r \in [0, \infty)$. It works for any densely defined, closed, linear operator, that is for operators whose domain is a dense subset of $\mathcal{H}$ and for which the graph $\cup_{\xi \in \mathrm{dom}(x)}(\xi, x\xi) \subseteq \mathcal{H} \oplus \mathcal{H}$ is closed. An operator x for which the closure of said graph is a closed subset of $\mathcal{H} \oplus \mathcal{H}$ is called closable and the operator defined by the closure is called closed and is denoted $\overline{x}$. The polar decomposition of a densely defined closed operator x is $x = u|x|$ where u is a partial isometry on $\mathcal{B}(\mathcal{H})$, meaning that u^*u is a projection (if x is invertible then u is unitary).

Suppose we have two faithful states ρ and ψ. We can then define the following mapping $R_{\rho,\psi} : \mathcal{N}/\mathcal{I}_\psi \mapsto \mathcal{N}/\mathcal{I}_\rho$ by

$$R_{\rho,\psi} : \mathcal{H}_\psi \ni \pi_\psi(x)\Omega_\psi \mapsto \pi_\rho(x^*)\Omega_\rho \in \mathcal{H}_\phi,$$

where $(\mathcal{H}_\psi, \pi_\psi, \Omega_\psi)$ and $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ are GNS constructions of ψ and ρ respectively. Since Ω_ψ is cyclic $R_{\rho,\psi}$ is densely defined and it can be shown to be closable so we can take the unique polar decomposition

$$\overline{R}_{\rho,\psi} = J_{\rho,\psi}\Delta_{\rho,\psi}^{1/2}$$

The operator $\Delta_{\rho,\psi} = R^*_{\rho,\psi}\overline{R}_{\rho,\psi}$ is called the relative modular operator and the operator $J_{\rho,\psi}$ is called the relative modular conjugation and can be shown to be a conjugation. This means that $J_{\rho,\psi}$ is antilinear ($J_{\rho,\psi}(ax + by) = a^* J_{\rho,\psi}(x) + b^* J_{\rho,\psi}(y)$ for $x, y \in \mathrm{dom}(J_{\rho,\psi})$ and $a, b \in \mathbb{C}$), isometric ($J^*J = \mathbb{1}$) and involutive ($J^2 = \mathbb{1}$). The operator $\Delta_{\rho,\psi}$ is going to be used for our generalised definition of quantum relative entropy

One thing that would be really nice is if we could have just a single Hilbert space that all the states can be embedded in. This problem is solved by the so-called standard representation of the W^*-algebra in question [17]. Such a representation is given by a quadruple $(\mathcal{H}, \pi, J, \mathcal{H}^\natural)$, where $\mathcal{H}$ is a Hilbert space, $\pi : \mathcal{N} \to \mathcal{B}(\mathcal{H})$ is s $*$-homomorphism, J is a conjugation on $\mathcal{H}$, and

$\mathcal{H}^\natural$ is a self-polar cone of $\mathcal{H}$ (meaning that $\forall \xi \in \mathcal{H}^\natural \forall \lambda \geq 0 \lambda\xi \in \mathcal{H}^\natural$ and $\mathcal{H}^\natural = \{\zeta \in \mathcal{H} \,|\, \langle \zeta, \xi \rangle_{\mathcal{H}} \geq 0 \forall \xi \in \mathcal{H}^\natural\}$), such that

1. $J\pi(\mathcal{N})J = \pi(\mathcal{N})^\bullet := \{x \in \pi(\mathcal{N}) \,|\, xy = yx \; \forall y \in \pi(\mathcal{N})\}$
2. $\xi \in \mathcal{H}^\natural \Rightarrow J\xi = \xi$
3. $\pi(x)J\pi(x)J\mathcal{H}^\natural \subseteq \mathcal{H}$
4. $\pi(x) \in \pi(\mathcal{N}) \cap \pi(\mathcal{N})^\bullet \Rightarrow J\pi(x)J = \pi(x)^*$

A standard representation, if it exists, satisfies a few very nice properties [17]:

1. $\forall \phi \in \mathcal{N}_\star^+ \exists! \zeta_\pi(\phi) \in \mathcal{H}^\natural : \forall x \in \mathcal{N} \phi(x) = \langle \zeta_\pi(\phi), \pi(x)\zeta_\pi(\phi)\rangle_{\mathcal{H}}$
2. $\mathcal{H}^\natural$ is closed and convex and $\xi \in \mathcal{H}^\natural \Rightarrow J\xi \in \mathcal{H}^\natural$.
3. If $\phi \in \mathcal{N}_\star^+$ is a faithful state then $\zeta_\pi(\phi)$ is cyclic and seperating for $\pi(\mathcal{N})$.
4. For any W^*-algebra a standard representation unique up to unitary equivalence

So, given a standard representation, all states on the W^*-algebra can be mapped to the elements of the cone of a single Hilbert space. Notice that all the previous constructions in principle are completely independent of Hilbert space strucure. To see how they relate to conventional quantum mechanical structures of finite dimensional systems let $\mathcal{N} = \mathcal{B}(\mathcal{K})$ for som Hilbert space $\mathcal{K}$. A standard representation of $\mathcal{N}$ can then be constructed [11] by looking at the Hilbert-Schmidt space $\mathcal{H}_{HS}$ as discussed in section 2 where $\zeta_\pi(\rho) = \rho^{1/2}$ and equipping the space with the conjugation operation J given by $J\xi = \xi^*$ for $\xi \in \mathcal{H}_{HS}$, the left multiplication mapping $\pi(x) = \mathscr{L}(x) : \mathcal{H}_{HS} \ni \xi \mapsto x\xi \in \mathcal{H}_{HS}$ and taking the cone as all the positive elements $\mathcal{H}^\natural = \mathcal{H}_{HS}^+$. From this it is clear if $\forall \xi \in \mathcal{H}^\natural$ then $J\xi = \xi$.
Consider the mapping $R_{\rho,\psi}$ for faithful states ρ, ψ defined by $R_{\rho,\psi}[x]_\psi = [x^*]_\rho$ in this representation. This can be shown to be closable and its closure admits a polar decomposition into a relative modular conjugation operator $J_{\rho,\psi}$ and a relative modular operator $\Delta_{\rho,\psi}$ in the way described above since it is densely defined. It can be shown that one can have $J_{\rho,\psi} = J$ as defined above for alle faithful states ρ, ψ, and $\Delta_{\rho,\psi} = \mathscr{L}(\zeta_\pi(\rho)) \circ \mathscr{R}(\zeta_\pi(\psi^{-1}))$ where $\mathscr{R}$ is right multiplication.

Now we can define the quantum relative entropy on normalized states of a W^*-algebra in a standard representation $(\mathcal{H}, \pi, J, \mathcal{H}^\natural)$ as [6][7]

$$D(\rho,\psi) = \begin{cases} -\langle \zeta_\pi(\psi), \log(\Delta_{\rho,\psi})\zeta_\pi(\psi)\rangle_{\mathcal{H}} & \mathrm{supp}(\rho) \leq \mathrm{supp}(\psi) \\ +\infty & \text{otherwise} \end{cases}. \tag{10}$$

This distance satisfies that $D(\rho,\psi) \geq 0 \forall \rho, \psi \in \mathcal{N}_\star^+$ and $D(\rho,\psi) = 0 \Leftrightarrow \rho = \psi$.
When $\mathcal{N} = \mathcal{B}(\mathcal{H})$ with the standard representation as explained above with functionals $\omega = \mathrm{tr}(\zeta_\pi(\omega)\cdot)$ one can show that $\mathrm{supp}(\rho) \leq \mathrm{supp}(\psi)$ is equivalent to $\mathrm{ran}(\zeta_\pi(\rho)) \subseteq \mathrm{ran}(\zeta_\pi(\pi))$. Note

that for the constructions above we needed the states to be faithful. From the definition of D we only need ψ to be faithful since we are restricting to using the constructions on domains. A standard way of handling the definition of $D(\rho,\psi)$ when ψ is not faithful is to restrict to a subalgebra where ψ is faithful.
In the case of finite dimensional Hilbert spaces it can be shown that (10) reduces to[18]

$$D(\rho,\psi) = \mathrm{tr}[\psi(\log\psi - \log\rho)]$$

3.1.3 Conditional expectations

Let $\mathcal{N}$ be a W^*-algebra and let $\mathcal{N}_0 \subseteq \mathcal{N}$ be a subalgebra. A conditional expectation is defined as a map $E : \mathcal{N} \to \mathcal{N}_0$ such that

1. $E(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 E(x_1) + \lambda_2 E(x_2)\ \forall x_1, x_2 \in \mathcal{N}\ \forall \lambda_1, \lambda_2 \in \mathbb{C}$
2. $E(x) = x\ \forall x \in \mathcal{N}_0$
3. $x \geq 0 \Rightarrow E(x) \geq 0$

If furthermore E satisfies

4. For any normalized and positive $x \in \mathcal{N}$ $E(x) = 0 \Rightarrow x = 0$
5. $\sup_i\{x_i\} = x \Rightarrow \sup_i\{E(x_i)\} = E(x)$ for every bounded increasing net $\{x_i\} \subseteq \mathcal{N}_1^+$.

E is said to be faithful and normal, respectively.

Umegaki and Nakamura showed [19] that weak Lüders' rule is a conditional expectation and Carlen [20] showed recently that partial trace in finite dimensions is a conditional expectation.

3.2 Triangle equality derived from conditional expectations

The following theorem by Ohya and Petz [2] supplies the relevance that conditional expectations have for our goal.

Theorem 3.1. *Let $\mathcal{N}$ be a W^*-algebra of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{N}_0 \subseteq \mathcal{N}$ and let $E : \mathcal{N} \to \mathcal{N}_0$ be a conditional expectation. Then it holds $\forall \psi, \phi \in \mathcal{M}(\mathcal{N})$ that*

$$D(\phi \circ E, \psi) = D(\phi|_{\mathcal{N}_0}, \psi|_{\mathcal{N}_0}) + D(\psi \circ E, \psi)$$

where $D(\cdot,\cdot)$ is defined in (10)*.*

Before the formula can be applied the following conjecture would need to be true.

Conjecture 3.2. *With the conditions as in Theorem 3.1*

$$D(\phi|_{\mathcal{N}_0}, \psi|_{\mathcal{N}_0}) = D(\phi \circ E, \psi \circ E)$$

Notice first that for any functional f on $\mathcal{N}$ it holds that $f|_{\mathcal{N}_0} = (f \circ E)|_{\mathcal{N}_0}$ since $E(x) = x \, \forall x \in \mathcal{N}_0$ and furthermore note that $f|_{\mathcal{N}_0}$ and $f \circ E$ have the same range. If the conjecture is proven we would immediately have triangle equality:

Theorem 3.3. *Let $\mathcal{N}$ be as above, let $E : \mathcal{N} \to \mathcal{N}_0$ be a conditional expectation, let $\mathcal{Q} = \{\omega \in \mathcal{N}_\star^+ \mid \omega \circ E = \omega\}$ and let $\psi \in \mathcal{N}_\star^+$ be arbitrary. Then there is a $\rho \in \mathcal{Q}$ so $\mathcal{Q}$ satisfies triangle equality for ψ at ρ. That is*

$$\forall \psi \in \mathcal{N}_\star^+ \exists \rho \in \mathcal{Q} : \forall \psi \in \mathcal{Q} \; D(\phi, \psi) = D(\phi, \rho) + D(\rho, \psi)$$

Proof. Let $\psi \in \mathcal{N}_\star^+$ be arbitrary. Since E is a conditional expectation by definition $E(x) = x \forall x \in \mathcal{N}_0$. Therefore for any $x \in \mathcal{N}$ we have $E \circ E(x) = E(x)$. This means that $\psi \circ E \in \mathcal{Q}$. Let $\rho = \psi \circ E$. Theorem 3.1 holds for any $\phi \in \mathcal{N}_\star^+$. In particular then

$$\forall \phi \in \mathcal{Q} : D(\phi, \psi) = D(\phi, \rho) + D(\rho, \psi)$$

$\square$

Here is the proposed strategy for a proof that partial trace maximises relative entropy also in infinite dimensions. First it would have to be established that partial trace E in this generality is also a conditional expectation. Secondly it would have to be demonstrated that $D(\psi \circ E, \psi) < \infty$. If this could be done the result would follow from Theorem 3.3.

4 Discussion

In the last two sections we have seen how quantum mechanical state update might be achieved through constrained maximisation of quantum relative entropy. In the first of these we examined finite dimensional Hilbert spaces in order to show that the familiar Lüders' collapse rules and partial trace emerge uniquely by maximisation of relative entropy. The proofs are not completed, as it still needs to be demonstrated that the distances between inital states and updated ones are finite. For the collapse rules it is known that such a finite bound must exist since Kostecki [10] and Hellmann et al [12] have demonstrated that the collapsed states minimise the entropic distance. For partial trace it is still unknown whether it is bounded in general but it was shown explicitly to be the case when the initial and updated state are both non-singular. Thus what is left to check is a measure 0 subset.

In the last Section we saw how one might approach a general proof that holds in any dimension. This relies on W^*-algebra and builds upon a result by Ohya and Petz. What is to be proven is firstly that the state updates are conditional expectations and secondly showing that the entropic distances between the original states and the updated ones are finite. Since the strategy is defined in terms of W^*-algebras, of which regular quantum mechanical structure is a special case, the success of this proof may provide a new possible framework for measurements in algebraic generalisations of quantum mechanics based on constrained maximisation of relative entropy.

Another potential use of the positive outcome of the project is the formulation of completely positive trace preserving maps (the current paradigm for the most general quantum channels) purely in terms of entropy. If unitary evolution is postulated then the projective measurements along with partial trace provide the necessary tools for expressing any completely positive trace preserving map in terms of constrained maximisation of relative entropy.
Lastly the result is of potential interest for quantum bayesianism. The framework that quantum bayesianism is attempting to construct is supposed to be devoid of objects with ontic properties. Therefore if quantum measurements may be expressed without reference to external objects then that is a step on the way towards realising this framework.

5 Conclusion

The main original result of this essay is a proof that the partial trace operation is a special case of the procedure of constrained maximisation of quantum relative entropy. The proof holds for invertible states and for finite dimensional quantum mechanics only. A method was proposed to check if it also holds in the case of non-invertible states.

Besides this the quantum mechanical collapse rules were showed to maximise quantum relative entropy. Uniqueness wasn't shown here but it has been shown for instance in full generality by Kostecki [10].

Finally we discussed a method that might be useful for proving the same results in the formulation of W^*-algebras, the success of which would give that the collapse rules and partial trace uniquely maximise the relative entropy in infinite dimensional quantum mechanics theory.

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