Postquantum Brègman relative entropies and nonlinear resource theories

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Abstract

We introduce the family of postquantum Brègman relative entropies, based on nonlinear embeddings into reflexive Banach spaces (with examples given by reflexive noncommutative Orlicz spaces over semi-finite W\textsuperscript{*}-algebras, nonassociative L\textsubscript{p} spaces over semi-finite JBW-algebras, and noncommutative L\textsubscript{p} spaces over arbitrary W\textsuperscript{*}-algebras). This allows us to define a class of geometric categories for nonlinear postquantum inference theory (providing an extension of Chencov’s approach to foundations of statistical inference), with constrained maximisations of Brègman relative entropies as morphisms and nonlinear images of closed convex sets as objects. Further generalisation to a framework for nonlinear convex operational theories is developed using a larger class of morphisms, determined by Brègman nonexpansive operations (which provide a well-behaved family of Mielnik’s nonlinear transmitters). As an application, we derive a range of nonlinear postquantum resource theories determined in terms of this class of operations.

1 Introduction

In this paper (which provides a further technical development of the ideas and results in [158, 159]) we discuss information geometric structures on two levels: general, with an information model \(\mathcal{M}\) understood as a set (or an object in a category [62, 199]) and an information distance \(D\) understood as a nonsymmetric function (or a functor [26, 113]) on it, and particular, with information models defined as arbitrary dimensional subsets of positive generating cones of base norm spaces (with a special interest in positive parts of preduals of W\textsuperscript{*}-algebras and JBW-algebras). We consider these geometries as a quite generic setting to develop an approach to foundations of postquantum information processing theory, understood as a theory of an intersubjective inductive inference. Due to consideration of analytic and geometric aspects of information geometry on the equal footing, as two constitutive components for a category-theoretic framework, the approach underlying this text can be considered as a

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nonlinear follow-up to the approach of Chencov [62, 63, 64, 65, 196, 197], based on replacing markovian morphisms by brègmannian projections (as the first step) and, more generally, by Brègman nonexpansive operations (as the second step).

In order to elaborate this shift in terms of more familiar concepts, we will first focus on the commutative and quantum cases. After introducing some terminology and notation, we will define two different perspectives on (statistical and quantum) information geometry, associated with two different classes of morphisms between information models (resp., coarse grainings and $D$-projections) and two different classes of distances that are well-behaved with respect to these information processing tasks (resp., $f$-distances and Brègman distances).

For a given $W^*$-algebra¹ $\mathcal{N}$, we define a quantum information model as a subset $\mathcal{M}(\mathcal{N}) \subseteq L_1(\mathcal{N})^+ \cong N_1^+$. Its elements would be called (quantum information) states. For a commutative $W^*$-algebra $\mathcal{N}$ the quantum information models $\mathcal{M}(\mathcal{N})$ turn into statistical models $\mathcal{M}(\mathcal{A}) \subseteq L_1(\mathcal{A})^+$, where $\mathcal{N} \cong L_\infty(\mathcal{A})$. Restriction to normalised states in this case gives $N_{11}^+ \cong L_1(\mathcal{A})^+_1$, and $\mathcal{A}$ is an mcb-algebra of projections in $\mathcal{N}$.

Given any set $X$, a distance is defined as a map $D : X \times X \to [0, \infty]$ such that $D(x, y) = 0 \iff x = y$. A distance is called: bounded iff $\text{ran}(D) = \mathbb{R}^+$; symmetric iff $D(x, y) = D(y, x)$; metrical iff it is bounded, symmetric and satisfies triangle inequality

$$D(x, z) \leq D(x, y) + D(y, z) \quad \forall x, y, z \in X.$$  

We will use the symbol $d$ instead of $D$ to denote metrical distances. We will use the notion information distance to refer to a distance on any set $\mathcal{M}$ that is considered as an information model (e.g., a subset of a positive generating cone of a base norm space). A quantum distance is defined as a distance on a quantum model $\mathcal{M}(\mathcal{N}) \subseteq N_1^+$, and it becomes a statistical distance if $\mathcal{N}$ is commutative.² Following Wiener’s idea that the «amount of information is the negative of the quantity defined as entropy» [259], a relative entropy is defined as a map $S : X \times X \to [-\infty, 0]$ such that $-S$ is an information distance.

The standard point of departure of commutative (statistical) and noncommutative (quantum) information geometry, as introduced and developed by Chencov [62, 64, 195, 196], is Wald’s [255, 257] unification of the approaches of Fisher [106, 107] and Neyman–Pearson [207, 208, 206]. According to it, the (conceptual and mathematical) foundation of statistical inference is decision making: given some evidential data, two information models $\mathcal{M}_1$ and $\mathcal{M}_2$, a parametrisation $\theta : \Theta \to \mathcal{M}_1$, and a prior measure $P : \mathcal{M}_1 \to [0, 1]$, one is to choose the specific morphism $\mathcal{M}_1 \to \mathcal{M}_2$ from the allowed class of morphisms, accordingly to some criteria that defines the ‘optimality’ of such decision. The allowed range of morphisms is set by default as coarse grainings: given any $W^*$-algebras $\mathcal{N}_1$ and $\mathcal{N}_2$, a coarse graining $T_* : N_{2+}^+ \to N_{1+}^+$ is defined as a Banach predual of the of the normal unital completely positive linear function $T : \mathcal{N}_1 \to \mathcal{N}_2$ (such function is called a Markov map). From unitality of $T$ it follows that $T_*$ is norm preserving. The class of Markov maps includes all conditional expectations and $\ast$-homomorphisms as special cases. The category QMod of quantum information models and their coarse grainings is a subcategory of the category QMod of quantum models and positive linear functions. When restricted to positive measures on mcb-algebras, the corresponding statistical subcategories will be denoted PMod and PMod⁺.

¹See Appendix I for some notation, notions, facts, and further references regarding the theory of operator algebras.
²The functions that we call ‘distances’ are often called ‘(information) divergences’. However, this causes very unfortunate collision of terms with well established notion of divergence used in differential calculus and differential geometry. The term ‘divergence’ was introduced and used by Kullback and Leibler [170] in the context of relative entropy, but in order to refer to an example of what we call a symmetric distance. Rényi [229] proposed to use the term ‘information gain’. Chencov [64] proposed to use the term ‘deviation’. Eguchi [100] (following Pfanzagl [222]) used the term ‘contrast functional’. We think that it is more reasonable to extend the range of the meaning of term ‘distance’, which is also in agreement with some of the prominent works in the field of information theory, e.g. [59, 79, 213].
A function $f : \mathbb{R}^+ \to \mathbb{R}$ is called **operator convex** [167] iff

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad \forall x, y \in \mathcal{B}(\mathcal{H})^+ \quad \forall \lambda \in [0, 1].$$

(2)

If $f : \mathbb{R}^+ \to \mathbb{R}$ is operator convex (hence, continuous on $]0, \infty[\)$ with $f(0) \leq 0$ and $f(1) = 0$, and if $(\mathcal{H}, \pi, J, \mathcal{N}^\pi)$ is standard representation of a $W^\pi$-algebra $\mathcal{N}^\pi$, then the $f$-**distance** [156, 218] is defined as a function $D_f : \mathcal{N}^\pi_+ \times \mathcal{N}^\pi_+ \to [0, \infty]$ such that

$$D_f(\omega, \phi) := \left\{ \begin{array}{ll}
(\zeta_\pi(\phi), f(\Delta_\omega, \phi)\zeta_\pi(\phi))_\mathcal{H} & : \omega \ll \phi \\
+\infty & : \text{otherwise},
\end{array} \right.$$

(3)

where $\zeta_\pi(\phi)$ is standard vector representative of $\phi$ in $\mathcal{H}^\pi$. In the commutative case, the analogous distance was introduced earlier in [74, 194, 10]$.^3$ By Petz’s theorem [218], if $f$ is bounded from above (hence, operator monotone decreasing), then $D_f$ given by (3) satisfies

$$D_f(\omega, \phi) \geq D_f(T_\star(\omega), T_\star(\phi)) \quad \forall \omega, \phi \in \mathcal{N}^\pi_+$$

(4)

for any unital 2-positive function $T$ such that $\text{dom}(T_\star) = \mathcal{N}^\pi_+$. This is an idempotent operation on an arbitrary information model. In the commutative case, the ‘data processing inequality’ (4) was established in [74, 75, 201, 176, 177]. In [76, 77] Csiszár provided a characterisation of the $f$-distances on finite dimensional statistical models by means of (4). The property (4) can be understood as a requirement of compatibility of the quantum distance on a quantum model with the structure of the category $Q\text{Mod}^\text{M}$, expressing the requirement that “the coarse graining of information models should always be indicated by nonincreasing of the quantification of relative information content of information states”.

On the other hand, starting from the works of Brègman [50], Chencov [63], and Hobson [130], there has emerged an alternative approach to statistical inference. Its main idea is to consider the minimisation of information distances $D$ as a process of inductive inference [260, 240], with the unique minimiser (whenever it exists) considered as a nonlinear projection onto a codomain model. In the commutative setting of measure theoretic integration, the Bayes–Laplace and Jeffreys’ updating rules [260, 58], as well as conditional expectations [27] were characterised by as such minimisers. In the noncommutative setting, both Lüders’ and quantum Jeffreys’ [128], as well as partial trace [200], were shown to be special cases of entropic projections (see [163] for details). Hence, the elementary prescriptions of statistical conditionalisation can be generalised either in the direction of positive linearity or nonlinear projections. This provides a departure point for a search of a well-defined class of nonlinear morphisms of information models that is different from coarse grainings, but also admits a legitimate information theoretic semantics.

Let $D$ be an information distance on an information model $\mathcal{M}$. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be submodels of $\mathcal{M}$. We define a $D$-**projection** as a map

$$\mathcal{P}^D_{\mathcal{Q}_2|\mathcal{Q}_1} : \mathcal{Q}_1 \ni \psi \mapsto \arg\inf_{\phi \in \mathcal{Q}_2} \{D(\phi, \psi)\} \subset \varphi(\mathcal{Q}_2),$$

(5)

where $\varphi(\mathcal{Q}_2)$ denotes the set of all subsets of $\mathcal{Q}_2$. We will use the notation $\mathcal{P}^D_{\mathcal{Q}_2|\mathcal{Q}_1}$ instead of $\mathcal{P}^D_{\mathcal{Q}_2|\mathcal{Q}_1}$ whenever the right hand side is a singleton set. From definition of $D$ it follows that $\mathcal{P}^D_{\mathcal{Q}_2|\mathcal{Q}_1}(\psi) = \psi \forall \psi \in \mathcal{Q}_1$, hence $\mathcal{P}^D_{\mathcal{Q}_2|\mathcal{Q}_1}$ is an idempotent operation on an arbitrary information model.

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$^3$Somewhat similar functionals were considered earlier in [198] under the name “generalised Hellinger integrals”, and with different assumptions on $f$ (it was considered to be a Young function).

$^4$By $\text{Mark}_+(\mathcal{N}^\pi_+)$ we will denote the space of all coarse grainings with a domain $\mathcal{N}^\pi_+$. 

3
In principle, the generalised Weierstraß–Fréchet theorem implies that the existence and uniqueness of the solution of constrained minimisation of a function $f$ on a subset $C$ of a topological space can be achieved by assuming $f$ to be convex and lower semi-continuous and assuming $C$ to be convex and compact. However, with some additional properties imposed on $f$, some of the properties of $C$ can be relaxed (and vice versa). This inseparability of choice of objects and morphisms shows the relevance of category-theoretic perspective for the description of the maximum-relative-entropic approach to information geometry. The problem is then to find a well-behaved and well-motivated class of distances $D$ and corresponding class of constraint sets $\mathcal{Q}$.

The key feature of the approach discussed in this paper is an observation that a class of distances, called Brègman distances, and denoted here as $D_\Psi$, admits a generalisation of pythagorean theorem beyond the linear framework of euclidean and Hilbert spaces, providing the additive decomposition of the nonlinear, yet ‘orthogonal’, projection onto a suitably affine class of subsets $K$ of $\mathcal{M}$.\(^5\)

\[ D_\Psi(\omega, \phi) = D_\Psi(\omega, \Psi_K^{D_\Psi}(\phi)) + D_\Psi(\Psi_K^{D_\Psi}(\phi), \phi) \quad \forall (\omega, \phi) \in K \times \mathcal{M}. \]  

This in turn allows to use geometry for the purpose of a nonlinear nonparametric “signal = data + noise” inference.

In Section 2 we consider a class $\tilde{D}_\Psi$ of two-point nonlinear functionals on vector spaces, known as Brègman functionals [50, 59, 54, 33]. Of a special interest are Brègman functionals over reflexive Banach spaces $X$ with $\Psi : X \rightarrow ]-\infty, \infty[$ being a Legendre function (see Section 2.1) [33]. This class has very good behaviour with respect to (5) and (6) with $\mathcal{M}$ substituted by $X$, and allows also to drop the norm boundedness (hence, weak compactness) requirement on $Q$, so that (5) is well defined for any convex closed subset $Q$ of $X$ (see Section 2.2). While some of the Brègman functionals are also distances, which allows to consider them as information distances in the case of finite dimensional $L_1(\mathcal{A})$ or $L_1(\mathcal{N})$ vector spaces, this framework is of limited applicability, especially when infinite dimensional (nonparametric) postquantum models are considered. More generally, there is an important gap in the theory of Brègman distances: while very nice results on existence and uniqueness of projections, generalised pythagorean theorem, as well as composability, exist for Brègman functionals on reflexive Banach spaces, the nonreflexivity of Orlicz spaces allowing for an adequate treatment of $D_1$-projections goes hand in hand with the fact that $D_1$ distances (110) are constructed from the most general definition of Brègman distance, based on the right Gâteaux derivative.

In order to investigate the possibilities of bridging this gap, in Sections 3.1 and 3.2 we develop a theory of general abstract Brègman distances, without embedding them into topological, bornological, or differential framework. The key elements of this construction are the Young–Fenchel inequality, dual pairs of coordinate systems\(^6\) and a suitable generalisation of the bijective Legendre transform to the infinite dimensional case. This approach includes the large part of theory of Brègman (and Alber) functionals as a special case. We think that this study can serve as a good point of departure for a future research on the “optimal” definition of Brègman distance that would unify the reflexive and nonreflexive approaches by balancing better the convex and topological structure\(^7\).

Taking some lessons from this general investigation, in Section 3.3 we return back to the particular, introducing a Brègman–Legendre distance, which is an abstract Brègman distance

\(^5\)These sets are required to be affine, in a specific sense: the affinity condition is applied to the image of their nonlinear embedding into a suitably chosen linear space that is used to define $D_\Psi$ and also determines what ‘orthogonality’ means. See below and Section 3.

\(^6\)A research on the role of coordinate embeddings (translating between a distance on nonlinear model and a functional on a linear space) for establishing the existence and uniqueness of projections has been initiated by Nagaoka and Amari [202, 15], and our work can be understood as an investigation of the nonsmooth functional analytic foundations for this approach.

\(^7\)Possibly by inducing the latter from the bornology determined by bounded level sets, as in [175].
determined by composition of the concrete Brègman distance (with Legendre $\Psi$ over reflexive $X$) with the bijective nonlinear embeddings $\tilde{\ell}$ of an ambient space into the subset of $X$. This allows for construction of a category $\text{Cvx}(\tilde{\ell}, \Psi)$ of $D_{\Psi}$-projections onto $\tilde{\ell}$-closed $\tilde{\ell}$-convex models, and its full subcategory $\text{Aff}(\tilde{\ell}, \Psi)$ of $D_{\Psi}$-projections onto (closed) $\tilde{\ell}$-affine models (for which the generalised pythagorean theorem holds globally). The $D_{\Psi}$-projections inside each homset of $\text{Aff}(\tilde{\ell}, \Psi)$ form a poset which has a structure of an ordered commutative monoid, so it is a resource theory in the sense of [111] (see [162] for more discussion). We specify further the domain of $\tilde{\ell}$ to spaces of postquantum and quantum information states, and give an example provided by the noncommutative Orlicz spaces $L_\gamma(N)$ over semi-finite $W^*$-algebras $\mathcal{N}$, which allow us to define a class $D_{\gamma\mathcal{N}}$ of quantum Brègman–Orlicz distances. The resulting categories $\text{Cvx}(\tilde{\ell}_\gamma, t\mathcal{T})$ can be seen as an elementary brègmannian alternative to markovian categories $\text{QMod}_\mathcal{M}$.

The families of $f$-distances and Brègman distances are widely regarded as two most important classes of information distances (cf. e.g. [77, 80, 81]). This leads to ask about the class of quantum information distances that belong to both families. Amari showed [13] that for the finite dimensional statistical models $L_1(\mathcal{X}, \mathcal{O}(\mathcal{X}), \mu)^+$ this intersection is characterised, up to a multiplicative constant, by the Liese–Vajda family [176] of $\gamma$-distances. In Section 4 we use the Falcone–Takesaki theory [103] of noncommutative integration and $L_p(\mathcal{N})$ spaces over arbitrary $W^*$-algebras (without a restriction to semi-finite $\mathcal{N}$) to construct the canonical family $D_{\gamma}$ of quantum $\gamma$-distances, which provides a common generalisation of the Jenčová–Ojima family [210, 139] and the Liese–Vajda family of $\gamma$-distances. We prove that the family of $D_{\gamma}$ belongs to an intersection of quantum $f$-distances $D_f$ and quantum Brègman distances $D_{\Psi}$. Following Amari’s result, we conjecture that this property characterises $\gamma$-distances on $L_1(\mathcal{N})^+ \cong \mathcal{N}_1^+$. Similarly to characterisation of quantum $f$-distances by the monotonicity under coarse grainings, and characterisation of Brègman distances by the generalised pythagorean equation, the proof of this conjecture remains an open problem. We discuss also the conditions of existence, uniqueness, and stability of the solutions to the corresponding constrained distance minimisation problems.

We end Section 4 with construction of a nonassociative analogue of $D_\gamma$ family, defined over preduals of semi-finite JBW-algebras $A$. The Banach space properties of nonassociative $L_{1/\gamma}$ spaces imply the corresponding properties of nonassociative $D_\gamma$ distances. This provides an example of a family of nonquantum postquantum Brègman–Legendre distances.

In quantum resource theories (see e.g. [69] for a detailed review, c.f. also definitions and references in Section 5), given any $W^*$-algebra $\mathcal{N}$, one considers free operations as given by some submonoid $\mathcal{T}$ of the monoid $\text{Mark}_\gamma(\mathcal{N}_1^+)$ of coarse grainings. It is often the case that the resource monotones (with respect to $\mathcal{T}$) are selected among $f$-distances $D_f$ as they are known to satisfy (4). However, in principle, by restricting to the subclass $\mathcal{T}$, the larger classes of distances $D$ (beyond $D_f$) can be admitted in the role of resource monotones. With $T \in \mathcal{T}$, the alternative nonexpansivity conditions

(i) $D(T(x), T(y)) \leq D(x, y) \quad \forall x, y \in \mathcal{N}_1^+$,

(ii) $D(x, T(y)) \leq D(x, y) \quad \forall (x, y) \in \text{Fix}(T) \times \mathcal{N}_1^+,$

where $\text{Fix}(T) := \{ x \in \mathcal{N}_1^+ \mid T(x) = x \} \neq \emptyset$,

can be seen either (a) as conditions on $D$, given a class $\mathcal{T}$, or (b) as conditions on $T$, given a choice of a specific family of resource monotones. If (a) is chosen, then the family of distances resulting from (i) deserves to be denoted $D_\mathcal{T}$. Because the class $D_\mathcal{T}$ can be larger than $D_f$, its intersection with Brègman distances $D_{\Psi}$ may admit more members than just the scalar multiples of $D_f$. It is an interesting open problem to characterise this intersection for some specific quantum resource theories, such as quantum entanglement [133]. The resulting information geometries (also in smooth case, with dually flat hessian geometries arising from the Taylor expansion of information distance) can be seen as a direct ‘minimal’ extension of
Chencov’s programme to quantum (and postquantum) resource theories, with Markov monotonicity relaxed to $\mathcal{T}$-nonexpansivity. Alternatively, if (b) is chosen, then one is immediately led to an observation that the conditions (i) and (ii) are equally well applicable to any monoids of operations on $\mathcal{N}_+^*$, not necessarily linear or completely positive. It turns out that when $D$ is taken to be the quantum Brègman–Legendre distance $D_\Psi$, there is a large theory of the so-called Brègman nonexpansive operations, that becomes available at our disposition. This forms a departure for development of a nonlinear postquantum resource theory, introduced and discussed in Section 5. This framework is presented briefly, less on its own right, and more in order to show that the brègmannian paradigm provides a full-fledged alternative to markovian one also at the level of applications to sub- and sup- quantum theories, far beyond the special case of morphisms given by $D_\Psi$-projections. In brègmannian resource theory, $D_\Psi$ is no longer seen primarily as a generator of $D_\Psi$-projections, but rather as a tool for controlling nonexpansivity of the essentially wider class of nonlinear maps (with $D_\Psi$-projections being just their special case). This way Chencov’s idea of information geometric categories obtains a further, ‘large’, extension, with $D_\Psi$ used for geometric control of the range of available nonlinear nonexpansive morphisms. This is somewhat characteristically dual to markovian case, where: (1) the task of information geometry was seen as to restrict the class of geometries by means of their nonexpansivity under all markovian morphisms; (2) a direction of development of a resource theory is to restrict the class of operations, which leads to a proliferation of resource monotones.

2 Brègman functionals

At least five different inequivalent general notions of a Brègman functional are present in the literature (we review them below, to a reasonable extent determined by our later applications). The substantial part of the theory of Brègman functionals is developed for the reflexive Banach spaces. However, this excludes the discussion of the most interesting case of $L_1$ spaces as well as nonreflexive Orlicz spaces, which are naturally related with $D_1$ distances. For that case, there are at least three approaches possible: the general approach based on one-sided Gâteaux derivatives on arbitrary Banach spaces, the measure theoretic approach based on integrals over premeasurable spaces and pointwise composition of gradients over $\mathbb{R}^n$ with $\mathbb{R}^n$-valued measure functions, and the intermediate approach, which can be applied to arbitrary Banach space, but requires Fréchet differentiability.

2.1 From Fenchel duality to Legendre functions

A dual pair is defined [92, 93, 181] as a triple $(X, X^d, [\cdot, \cdot]_{X \times X^d})$, where $X$ and $X^d$ are vector spaces over $K \in \{\mathbb{R}, \mathbb{C}\}$, equipped with a bilinear duality pairing $[\cdot, \cdot]_{X \times X^d} : X \times X^d \to K$ satisfying

$$[x, y]_{X \times X^d} = 0 \forall x \in X \Rightarrow y = 0, \quad [x, y]_{X \times X^d} = 0 \forall y \in X^d \Rightarrow x = 0. \quad (7)$$

An example of a dual pair is given by a Banach space $X$, $X^d = X^*$, and the dual pairing given by the Banach space duality. A function $f : X \to [-\infty, \infty]$ on a set $X$ is called proper iff it never takes the value $-\infty$ and $\text{efd}(f) := \{x \in X \mid f(x) \neq \infty\} \neq \emptyset$. The Fenchel subdifferential [104, 193, 52] of a proper $\Psi : X \to [-\infty, +\infty]$ at $x \in \text{efd}(\Psi)$ is a set

$$\partial \Psi(x) := \{\hat{y} \in X^d \mid \Psi(x) - \Psi(z) \geq \text{re} \{z - x, \hat{y}\}_{X \times X^d} \forall z \in X\}. \quad (8)$$

*We use here the general setting of dual vector spaces, and do not restrict our considerations to locally convex topological vector spaces, because we have in mind the possible future use of convenient vector spaces [112, 168] and stereotype spaces [4].

6
For $x \in X \setminus \text{efd}(\Psi)$ one defines $\partial \Psi(x) := \emptyset$. The elements of $\partial \Psi(x)$ are called \textbf{Fenchel subgradients} at $x$. The \textbf{Fenchel dual} of $\Psi$ is defined as $\Psi^F : X^d \to [-\infty, +\infty]$ such that \cite{44, 182, 104}

$$\Psi^F(y) := \sup \{ \text{re} \left[ x, y \right]_{X \times X^d} - \Psi(x) \} \quad \forall y \in X^d. \quad (9)$$

Given $X^dd$ such that $(X^d, X^dd, [\cdot, \cdot]_{X^d \times X^d})$ is a dual pair and $X \subseteq X^dd$, one defines $\Psi^{FF} : X \to [-\infty, +\infty]$ by $\Psi^{FF} := (\Psi^F)^F$. The functions $\Psi^F$ and $\Psi^{FF}$ are convex for any $\Psi$, and $\Psi^{FF}|_X \leq \Psi$. If $\text{efd}(\Psi) \neq \emptyset$, then $\Psi^F(x) > -\infty \forall x \in X^d$. If $(X, X^t)$ is a dual pair of locally convex topological vector spaces, equipped with weak-* and weak topologies, respectively, and $\Psi$ is proper, then $\Psi^F$ is weakly-* lower semi-continuous, $\Psi^{FF}$ is weakly lower semi-continuous, and $(\Psi^{FF}|_X = \Psi$ holds iff $\Psi$ is weakly lower semi-continuous and convex) \cite{131, 51}. A lower semi-continuous convex $\Psi$ on $X$ is proper iff $\Psi^F$ on $X^t$ is proper. If $X$ is a Banach space and $\Psi : X \to [-\infty, +\infty]$ is proper, convex, then it is lower semi-continuous in norm topology of $X$ iff it is lower semi-continuous in weak topology on $X$. In what follows, we will always assume $\text{efd}(\Psi) \neq \emptyset$. If $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ is convex and $\hat{y} \in X^d$, then the \textbf{Young–Fenchel inequality} \cite{262, 104}

$$\Psi(x) + \Psi^F(\hat{y}) - \text{re} \left[ x, \hat{y} \right]_{X \times X^d} \geq 0 \quad (10)$$

holds, with equality iff $\hat{y} \in \partial \Psi(x)$. If $(X, X^t)$ is a dual pair of locally convex topological vector spaces, and $\Psi$ is proper, convex, and lower semi-continuous, then equality in (10) holds iff $x \in \partial \Psi^F(\hat{y})$. There exist various criteria for nonemptiness of Fenchel subdifferential.

The key role of Fenchel subdifferential $\partial \Psi(x)$ is to characterise minimisers of $\Psi$ at $x$. In particular, if $X$ is a Banach space, $x \in X$, and $\Psi : X \to [-\infty, +\infty]$ is proper and convex, then

$$x_0 \in \text{arg inf}_{x \in X} \{ \Psi(x) \} \iff 0 \in \partial \Psi(x_0). \quad (11)$$

If $\Psi$ is also lower semi-continuous with respect to norm topology on $X$, then the conditions in (11) are equivalent to $\partial \Psi^F(0) \cap X^* \neq \emptyset$, where $\Psi^F$ is a Fenchel dual with respect to the Banach duality of $X$ and $X^*$.

If $(X, X^d, [\cdot, \cdot]_{X \times X^d})$ is a dual pair and $\Psi : X \to ]-\infty, +\infty]$ is proper, then:

$$\text{efd}(\partial \Psi) := \{ x \in \text{efd}(\Psi) \mid \partial \Psi(x) \neq \emptyset \}, \quad \text{(12)}$$

$$\text{efe}(\partial \Psi) := \{ \hat{y} \in X^d \mid \hat{y} \in \partial \Psi(x), x \in \text{efd}(\partial \Psi) \}, \quad \text{(13)}$$

$$\partial \Psi^{-1} : X^d \ni \hat{y} \mapsto (\partial \Psi)^{-1}(\hat{y}) := \{ x \in X \mid \hat{y} \in \partial \Psi(x) \} \ni \varphi(X), \quad \text{(14)}$$

where $\varphi(X)$ denotes a power set of $X$. If $(X^d, X^dd, [\cdot, \cdot]_{X^d \times X^d})$ is a dual pair and $X \subseteq X^dd$, then $\Psi$ is called \textbf{adequate} \cite{253} iff $\text{efd}((\partial \Psi)^{-1}) = \text{efd}(\Psi^F) \neq \emptyset$ and $(\partial \Psi)^{-1}(\hat{y}) = \{ \hat{y} \}$ $\forall \hat{y} \in \text{efd}((\partial \Psi)^{-1})$. If $X$ is a Banach space, $X^d = X^*$, and $\Psi$ is proper, convex, and lower semi-continuous in norm topology on $X$, then $\text{int}(\text{efd}(\Psi)) \subseteq \text{efd}(\partial \Psi)$, and $\text{efd}(\partial \Psi)$ is dense in $\text{efd}(\Psi)$, with int denoting a topological interior of a set.

If $X$ is a vector space over $\mathbb{K}$, $t \in \mathbb{R}$, and $\Psi : X \to ]-\infty, +\infty]$ is proper then the \textbf{right Gâteaux derivative} of $\Psi$ at $x \in X$ in the direction $h \in X$ reads

$$X \times X \ni (x, h) \mapsto D^+_G \Psi(x; h) := \lim_{t \to +0} (\Psi(x + th) - \Psi(x))/t \in [0, +\infty]. \quad (15)$$

If $x$ is fixed and (15) exists for all $h \in X$, then $\Psi$ is called \textbf{Gâteaux differentiable at} $x$. If $\Psi : X \to ]-\infty, +\infty]$ is convex and Gâteaux differentiable at $x$, then $D^+_G \Psi(x; \cdot) \in \partial \Psi(x)$. If $\Psi : X \to ]-\infty, +\infty]$ is convex and continuous at $x$, then $\partial \Psi(x) = \{ \ast \}$ if $\Psi$ is Gâteaux differentiable at $x$. If $\Psi : X \to ]-\infty, +\infty]$ is convex, lower semi-continuous, and Gâteaux differentiable at $x$, then it is continuous at $x$. If $X$ is a Banach space and $\Psi$ is convex and lower semi-continuous, then $D^+_G \Psi(x; \cdot)$ is convex on $X$, and continuous on $\text{int}(\text{efd}(\Psi))$, while $D^+_G \Psi(\cdot, \cdot)$ is finite and upper semi-continuous on $\text{int}(\text{efd}(\Psi)) \times X$. If $x \in \text{efd}(\Psi)$ and $D^+_G \Psi(x; \cdot)$
is continuous at some \( h \in X \), then \( \partial \Psi(x) \neq \emptyset \). If \( X \) is a Banach space and \( \Psi \) is Gâteaux differentiable at \( x \in X \), then \( D^G_x \Psi(x; y) = [y, D^G_x \Psi]_{X \times X^*} \). For \( y \in X \) defines the \textbf{Gâteaux derivative} \([114, 115, 116] \), \( D^G_x \Psi(x) = D^G_x \Psi \in X^* \) of \( \Psi \) at \( x \). A function \( \Psi \) is called \textbf{Gâteaux differentiable} iff \( \text{int(efd}(\Psi)) \neq \emptyset \) and \( \Psi \) is Gâteaux differentiable for all \( x \in \text{int}(\text{efd}(\Psi)) \).

If \( X \) is a Banach space, \( \Psi : X \to [-\infty, +\infty] \) is proper, convex, and lower semi-continuous in norm topology, then: \( i \) if \( \Psi^F \) (with respect to Banach space duality) is strictly convex at all elements of \( \text{efd}(\Psi^F) \), then \( \Psi \) is Gâteaux differentiable; \( ii \) if \( \Psi^F \) is Gâteaux differentiable at all \( x \in X^* \), then \( \Psi \) is strictly convex at all elements of \( \text{int}(\text{efd}(\Psi)) \).

Given a normed space \( X \), a Fréchet derivative of \( \Psi : X \to [-\infty, +\infty] \) at \( x \in X \) will be denoted as \( D^F_x \Psi \). If \( \Psi \) is Fréchet differentiable at all \( x \in \text{int}(\text{efd}(\Psi)) \), then it is also norm continuous and Gâteaux differentiable. For \( \dim X < \infty \) these two notions of derivative coincide.

A Banach space \( X \) is called: \textbf{strictly convex} \([108, 70] \) iff

\[
\forall x, y \in X \quad \|x + y\| = \|x\| + \|y\|, \quad x \neq 0 \neq y \Rightarrow \exists \lambda > 0 \ y = \lambda x; \quad (16)
\]

\textbf{Gâteaux differentiable} \([21, 187] \) iff \( \|\cdot\| \) is Gâteaux differentiable at every \( x \in X \setminus \{0\} \); \textbf{uniformly convex} \([70] \) iff

\[
\forall \epsilon_1 > 0 \quad \exists \epsilon_2 > 0 \quad \forall x, y \in X \quad \|x\| = 1, \quad \|x - y\| \geq \epsilon_1 \Rightarrow \|x + y\| \leq 2 - \epsilon_2; \quad (17)
\]

\textbf{uniformly Fréchet differentiable} \([239] \) iff

\[
\forall \epsilon_1 > 0 \quad \exists \epsilon_2 > 0 \quad \forall x, y \in X \quad \|x\| = 1, \quad \|y\| \leq \epsilon_2 \Rightarrow \|x + y\| + \|x - y\| \leq 2 + \epsilon_1 \|y\|; \quad (18)
\]

\textbf{reflexive} \([122] \) iff the map \( j : X \to X^{**} \), defined by \( j(x)(y) := y(x) \forall x \in X \forall y \in X^* \) is an isometric isomorphism. If \( X \) (resp. \( X^* \)) is Gâteaux differentiable, then \( X^* \) (resp. \( X \)) is strictly convex \([238, 155] \). A Banach space \( X \) is uniformly convex (resp. uniformly Fréchet differentiable) iff \( X^* \) is uniformly Fréchet differentiable (resp. uniformly convex) \([85] \). If \( X \) is uniformly convex (resp. uniformly Fréchet differentiable), then it is also strictly convex (resp. Gâteaux differentiable). If \( X \) is uniformly convex or uniformly Fréchet differentiable, then it is reflexive \([192, 148, 216, 239] \).

If \( X \) is Gâteaux differentiable, then there exists a norm-to-weak-* continuous map \( X \to \text{efd}(\Psi) \), \( \{x \in X \mid \|x\| = 1\} \to \{x \in X^* \mid \|x\| = 1\} \) that is uniquely determined by a condition \( [x, y] : X \times X^* \to \mathbb{R} \).

Let \( X \) be a Banach space with a norm \( \|\cdot\| \). In what follows, we will refer to Banach spaces assuming implicitly that they are over \( \mathbb{R} \). For Banach spaces over \( \mathbb{C} \) all definitions and results require to replace \( \|\cdot\|_X \times X^* \) by \( \|\cdot\|_{X \times X^*} \). A function \( T : X \to \varphi(X^*) \) is called \textbf{locally bounded} at \( x \in X \) iff \([242] \)

\[
\exists \epsilon > 0 \sup \{\|T(x + \epsilon y)\| \mid y \in X, \|y\| \leq 1\} < +\infty. \quad (17)
\]

If \( \Psi : X \to [-\infty, +\infty] \) is proper, then

\[
(\partial \Psi)^{-1}(\hat{y}) = \arg \min_{x \in X} \{\Psi(x) - [x, \hat{y}]_{X \times X^*}\}. \quad (18)
\]

A function \( \Psi : X \to [-\infty, +\infty] \) is called \textbf{coercive} iff \( \lim_{\|x\| \to +\infty} \Psi(x) = +\infty \). A Banach space \( X \) is reflexive iff every proper, convex, coercive function that is lower semi-continuous in norm topology attains its minimum on \( X \). If \( \Psi : X \to [-\infty, +\infty] \) is proper, convex, lower semi-continuous and \( \Psi^F \) denotes its Fenchel dual with respect to the Banach space duality of \( X \) and \( X^* \), then \( \Psi \) is called \([231, 33, 47, 49] \):

\begin{itemize}
  \item \textbf{essentially Gâteaux differentiable} iff \( \partial \Psi \) is locally bounded on \( \text{efd}(\partial \Psi) \) or \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \) and \( \partial \Psi(x) = \{*\} \forall x \in \text{efd}(\partial \Psi) \);
  \item \textbf{essentially strictly convex} iff \( (\partial \Psi)^{-1} \) is locally bounded on \( \text{efd}(\partial \Psi)^{-1} \) and \( \Psi \) is strictly convex on every convex subset of \( \text{efd}(\partial \Psi) \);
\end{itemize}
• **Legendre** iff $\Psi$ is essentially Gâteaux differentiable and essentially strictly convex;

• **essentially Fréchet differentiable** iff it is essentially Gâteaux differentiable and Fréchet differentiable for all $x \in \text{int}(\text{efd}(\Psi))$;

• **Fréchet–Legendre** iff $\Psi$ and $\Psi^F$ are essentially Fréchet differentiable.

If $\Psi$ is continuous and is Gâteaux differentiable at all $x \in X$ then it is essentially Gâteaux differentiable. If $\Psi$ is essentially Gâteaux differentiable then $\text{int}(\text{efd}(\Psi)) \neq \emptyset$ and $\Psi$ is Gâteaux differentiable on $\text{int}(\text{efd}(\Psi))$ [33]. If $X$ is reflexive, then $\Psi$ is essentially Gâteaux differentiable (resp. Legendre, Fréchet–Legendre) iff $\Psi^F$ is essentially strictly convex (resp. Legendre, Fréchet–Legendre). If $X$ is reflexive and $\Psi$ is Legendre, then

$$D^G\Psi : \text{int}(\text{efd}(\Psi)) \rightarrow \text{int}(\text{efd}(\Psi^F))$$

is bijective, $(D^G\Psi)^{-1} = D^G(\Psi^F)$, and both $D^G\Psi$ and $D^G(\Psi^F)$ are norm-to-weak continuous and locally bounded on their respective domains [33]. If $X$ is an arbitrary Banach space, $\Psi : X \rightarrow ]-\infty, +\infty]$ is proper and weakly lower semi-continuous, and $\text{efd}((\partial \Psi)^{-1})$ is open, then [253]

1) if $\Psi$ is essentially Gâteaux differentiable, then $\Psi$ is adequate,

2) if $X$ is reflexive, then $\Psi^F$ is essentially Gâteaux differentiable iff $\Psi$ is adequate.

### 2.2 Concrete Brègman functionals

Let $X$ be a Banach space, and let $\Psi : X \rightarrow ]-\infty, +\infty]$ be proper. Then the **Brègman functional** $D_\Psi : X \times X \rightarrow [0, +\infty]$ can be defined in any of the following *inequivalent* ways (see also [53]):

(B1) for $\Psi$ convex, with $\text{efd}(\Psi) \neq \emptyset$ [152, 153, 154, 54, 56]:

$$D_\Psi : X \times X \ni (x, y) \mapsto \begin{cases} \Psi(x) - \Psi(y) - \mathcal{D}_y^G\Psi(y; x - y) : y \in \text{efd}(\Psi) \\ +\infty : \text{otherwise}; \end{cases}$$

(B2) for $\Psi$ convex and lower semi-continuous, with $\text{int}(\text{efd}(\Psi)) \neq \emptyset$ [33]:

$$D_\Psi : X \times X \ni (x, y) \mapsto \begin{cases} \Psi(x) - \Psi(y) - \mathcal{D}_y^G\Psi(y; x - y) : y \in \text{int}(\text{efd}(\Psi)) \\ +\infty : \text{otherwise}; \end{cases}$$

(B3) for $\Psi$ convex, lower semi-continuous, and Gâteaux differentiable on $\text{int}(\text{efd}(\Psi)) \neq \emptyset$ [8]:

$$D_\Psi : X \times X \ni (x, y) \mapsto \begin{cases} \Psi(x) - \Psi(y) - \left[ [x - y, \mathcal{D}_y^G\Psi] \right]_{X \times X^*} : y \in \text{int}(\text{efd}(\Psi)) \\ +\infty : \text{otherwise}; \end{cases}$$

(B4) for $\Psi$ convex, lower semi-continuous, and Fréchet differentiable on $\text{int}(\text{efd}(\Psi)) \neq \emptyset$ [110, 109] (here we generalise the definition given in these papers):

$$D_\Psi : X \times X \ni (x, y) \mapsto \begin{cases} \Psi(x) - \Psi(y) - \left[ [x - y, \mathcal{D}_y^G\Psi] \right]_{X \times X^*} : y \in \text{int}(\text{efd}(\Psi)) \\ +\infty : \text{otherwise}; \end{cases}$$

(B5) for $\text{MeFun}(\mathcal{X}, \mathcal{U}(\mathcal{X}); \mathbb{R}^+)$ denoting the space of $\mathcal{U}(\mathcal{X})$-measurable functions $h : \mathcal{X} \rightarrow \mathbb{R}^+$, $\tilde{\mu}$ denoting a countably additive finite measure on $\mathcal{U}(\mathcal{X})$, $\Psi : \mathbb{R} \rightarrow ]-\infty, +\infty]$ proper, strictly convex, and differentiable on $]0, +\infty[$ with $\Psi(0) = \lim_{t \rightarrow 0^+} \Psi(t)$ and $t < 0 \Rightarrow \Psi(t) = +\infty$, $X$ given by a suitable Banach space of some elements of
The definitions (B₁) are replaced by

\[\forall x, y \in X \quad (\Psi(x) - \Psi(y)) + \sup_{\dot{\psi} \in \partial \Psi(y)} \{ [y - x, \dot{\psi}]_{X \times X^*} \},\]

for all \(x \in X\) such that

\[\|y - x, \dot{\psi}\|_{X \times X^*} = \sup_{\dot{\psi} \in \partial \Psi(y)} \{ [y - x, \dot{\psi}]_{X \times X^*} \};\] (28)

4) if \(\Psi\) is Gâteaux differentiable at \(y\), then

\[\tilde{D}_\Psi(x, y) = \Psi(x) - \Psi(y) - [x - y, \nabla \Psi]_{X \times X^*} = \Psi(x) + \Psi^F(\nabla \Psi) - [x, \nabla \Psi]_{X \times X^*};\] (29)

5) if \(\Psi\) is essentially strictly convex, then

\[\tilde{D}_\Psi(x, y) = 0 \iff x = y;\] (30)

6) if \(\Psi\) is Gâteaux differentiable at \(x\) and essentially strictly convex, then

\[\tilde{D}_\Psi(x, y) = \tilde{D}_{\Psi^F}(\nabla \Psi, \nabla \Psi) \quad \forall x \in \text{int}(\text{efd}(\Psi)).\] (31)

We can conclude that the Brègman functional can be considered a distance if \((\Psi\) is strictly convex, one of the conditions (25)-(26) holds, and (B₁) is used) or \((\Psi\) is essentially strictly convex, \(X\) is reflexive, and (B₂) is used).

If \(X\) is a Banach space and \(\Psi : X \to ] - \infty, +\infty]\) is proper, then an *Alber functional* on \(X\) is defined as [5, 6, 7]

\[W_\Psi : X \times X^* \ni (x, \dot{y}) \mapsto \Psi(x) + \Psi^F(\dot{y}) - \|x, \dot{y}\|_{X \times X^*} \in [0, +\infty].\] (32)
The condition \((\Psi\text{ is Gâteaux differentiable at } x \text{ and } \tilde{y} = \mathcal{D}_x^G \Psi)\) is equivalent to \(W_\Psi(x, \tilde{y}) = 0\). If \(\Psi\) is also convex, lower semi-continuous, and Gâteaux differentiable on \(\text{int}(\text{efd}(\Psi)) \neq \emptyset\), and \(X\) is reflexive, then the Young–Fenchel inequility (10) gives

\[
\Psi(x) + \Psi^F(\mathcal{D}_x^G \Psi) - \left[ [x, \mathcal{D}_x^G \Psi] \right]_{X \times X^*} = 0 \quad \forall x \in \text{int}(\text{efd}(\Psi))
\]  

(33)

and

\[
W_\Psi(x, \mathcal{D}_y^G \Psi) = \Psi(x) + \Psi^F(\mathcal{D}_y^G \Psi) - \left[ [x, \mathcal{D}_y^G \Psi] \right]_{X \times X^*} = \Psi(x) - \Psi(y) - \left[ x - y, \mathcal{D}_y^G \Psi \right]_{X \times X^*} = \tilde{D}_\Psi(x, y)
\]  

(34)

for all \(x, y \in \text{int}(\text{efd}(\Psi))\), with \(\tilde{D}_\Psi\) given by (B3). These equations are special cases of (29).

If (B3) is used, then for every \(x, y \in X\) and \(z, w \in \text{int}(\text{efd}(\Psi))\) [61, 36, 34]

\[
\tilde{D}_\Psi(z, w) + \tilde{D}_\Psi(w, z) = \left[ [z - w, \mathcal{D}_z^G \Psi - \mathcal{D}_w^G \Psi] \right]_{X \times X^*},
\]

(35)

\[
\tilde{D}_\Psi(x, w) + \tilde{D}_\Psi(w, z) = \tilde{D}_\Psi(x, z) + \left[ [x - w, \mathcal{D}_x^G \Psi - \mathcal{D}_w^G \Psi] \right]_{X \times X^*},
\]

(36)

\[
\tilde{D}_\Psi(x, w) + \tilde{D}_\Psi(y, z) - \tilde{D}_\Psi(x, z) - \tilde{D}_\Psi(y, w) = \left[ [x - y, \mathcal{D}_x^G \Psi - \mathcal{D}_y^G \Psi] \right]_{X \times X^*}.
\]

(37)

The equation (36) is an instance of a \textit{generalised cosine equation}, while the equation (37) is an instance of a \textit{quadrilateral equation}.

A \textit{Brègman functional projection} [59, 31, 33, 34] from a set \(C_1 \subseteq X\) onto a set \(C_2 \subseteq X\) is the function \(\tilde{\mathcal{P}}_{C_2|C_1}^\Psi\) defined by

\[
C_1 \ni y \mapsto \left\{ x \in C_2 \cap \text{efd}(\Psi) \mid \tilde{D}_\Psi(x, y) = \inf_{z \in C_2} \left\{ \tilde{D}_\Psi(z, y) \right\} < +\infty \right\} \in \varphi(C_2).
\]

(38)

For \(C_1 = X\) we denote \(\tilde{\mathcal{P}}_{C_2|X}^\Psi := \tilde{\mathcal{P}}_{C_2|X}^\Psi\). If \(\tilde{\mathcal{P}}_{C_2|X}^\Psi(y) = \{x\}\), then we will use the notation \(\tilde{\mathcal{P}}_{C_2|X}^\Psi(y) = x\).

The main problems considered in the context of Brègman functional projections are their existence, uniqueness, characterisation, and stability (which means the behaviour of sequences converging to the unique solution of the minimisation problem). Various results, depending on different sets of assumptions, are present in the literature. Here we will present the main existence, uniqueness and characterisation results obtained for the Banach space setting and the measure theoretic setting (which generalise earlier results of [59, 86, 60, 244, 97, 31], obtained for \(\mathbb{R}^n\)).

(P1) [8, 7]. If (B3) is used, \(\Psi\) is strictly convex on \(\text{efd}(\Psi)\), \(C \subseteq X\) is convex, and \(C \cap \text{efd}(\Psi) \neq \emptyset\), then \(\tilde{\mathcal{P}}_{C|X}^\Psi(y)\) contains at most one element. If, in addition, \(X\) is reflexive and \(C\) is nonempty and weakly closed\(^9\), then \(\tilde{\mathcal{P}}_{C|X}^\Psi(y) = \{\ast\} \forall y \in \text{int}(\text{efd}(\Psi))\) whenever \((C \cap \text{efd}(\Psi))\) is norm bounded or \(\lim_{\|x\| \to +\infty} \frac{\Psi(x)}{\|x\|} \to +\infty \forall x \in (C \cap \text{efd}(\Psi))\). Moreover, if \(X\) is an arbitrary Banach space, (B3) is used, \(\Psi\) is strictly convex, \(C \subseteq X \) is nonempty and convex, \(y \in X, x \in C\), then equivalent are:

\[
\tilde{D}_\Psi(z, x) + \tilde{D}_\Psi(x, y) \leq \tilde{D}_\Psi(z, y) \quad \forall z \in C,
\]

(39)

\[
\left[ [z - x, \mathcal{D}^G_x \Psi - \mathcal{D}^G_y \Psi] \right]_{X \times X^*} \leq 0 \quad \forall z \in C,
\]

(40)

\[
x = \tilde{\mathcal{P}}_{C|X}^\Psi(y).
\]

(41)

If \(X\) and \(\Psi\) are as above, \(K\) is a vector subspace of \(X\), then

\[
\tilde{D}_\Psi(x, y) = \tilde{D}_\Psi(x, \tilde{\mathcal{P}}_{K|X}^\Psi(y)) + \tilde{D}_\Psi(\tilde{\mathcal{P}}_{K|X}^\Psi(y), y) \quad \forall (x, y) \in K \times X.
\]

(42)

\(^9\text{Note that, by Mazur’s theorem, each convex set in a Banach space is norm closed iff it is weakly closed.}\)
(P2) [33, 34, 35]. If (B2) is used, \( \Psi \) is Legendre (or if \( \Psi \) is strictly convex, essentially strictly convex, and Gâteaux differentiable at \( y \)), \( y \in \text{int}(\text{efd}(\Psi)) \), \( X \) is reflexive, and \( C \subseteq X \) is nonempty convex closed, \( C \cap \text{int}(\text{efd}(\Psi)) \neq \emptyset \), then \( \hat{P}_C^\Psi(y) = \{ * \} \) and
\[
x = \hat{P}_C^\Psi(y) \iff \left( x \in C \text{ and } C \subseteq \left\{ z \in X : \|[z - x, \mathcal{D}_x^G \Psi - \mathcal{D}_y^G \Psi]_{X \times X^*} \| \leq 0 \right\} \right)
\]
\( \forall x, y \in \text{int}(\text{efd}(\Psi)) \), which is equivalent to characterisation of \( \hat{P}_C^\Psi(y) \) as a unique \( x \in C \cap \text{int}(\text{efd}(\Psi)) \) that satisfies
\[
\hat{D}_\Psi(z, x) + \hat{D}_\Psi(x, y) \leq \hat{D}_\Psi(z, y) \quad \forall z \in C.
\] (43)

(P3) [55, 230, 57, 7]. A function \( \Psi : X \to ]-\infty, +\infty] \) is called \textbf{totally convex} [54, 55, 57] at \( y \in \text{efd}(\Psi) \) iff
\[
\inf\{ \hat{D}_\Psi(x, y) : x \in \text{efd}(\Psi), \|x - y\| = t \} > 0 \quad \forall t > 0,
\] (44)
where (B1) is used. For \( C \subseteq X \) and \( \hat{y} \in X^* \) its \textbf{Alber projection} reads
\[
\hat{P}_C^\Psi(\hat{y}) := \arg\inf_{x \in C} \{ W_\Psi(x, \hat{y}) \}.
\] (45)
If \( X \) is reflexive, \( \Psi \) is strictly convex and lower semi-continuous on \( \text{efd}(\Psi) \), \( \hat{y} \in \text{efd}(\Psi^F) \), \( C \subseteq \text{efd}(\Psi) \) is nonempty, convex, and closed, and the set \( \{ z \in \text{efd}(\Psi) : W_\Psi(z, \hat{y}) \leq \lambda \} \) is bounded for any \( \lambda \in \mathbb{R}^+ \) (which holds if \( \lim_{\|x\| \to +\infty} \frac{\Psi(x)}{\|x\|} = +\infty \), or if \( \hat{y} \in \text{efc}(\partial \Psi) \) and \( \Psi \) is totally convex at each \( x \in \text{efd}(\Psi) \)), then \( \hat{P}_C^\Psi(\hat{y}) = \{ * \} \). If, in addition, \( \Psi \) is Gâteaux differentiable on \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \), then \( \hat{P}_C^\Psi(\hat{y}) \) is characterised as such \( x \in C \) that is a unique solution in \( C \) of
\[
\|[x - z, \hat{y} - \mathcal{D}_x^G \Psi]_{X \times X^*} \| \leq 0 \quad \forall z \in C,
\] (46)
or, equivalently, of
\[
W_\Psi(x, \hat{y}) + W_\Psi(z, \mathcal{D}_x^G \Psi) \leq W_\Psi(z, \hat{y}) \quad \forall z \in C.
\] (47)
From \( \mathcal{D}_x^G \Psi \in \text{int}(\text{efd}(\Psi^F)) \) \( \forall x \in \text{int}(\text{efd}(\Psi)) \) it follows that
\[
\hat{P}_C^\Psi(x) = \hat{P}_C^\Psi(\mathcal{D}_x^G \Psi) = \left( \hat{P}_C^\Psi \circ \mathcal{D}^G \Psi \right)(x).
\] (48)
This implies existence and uniqueness of the Brègman functional projection for (B3) under the above conditions, with \( x = \hat{P}_C^\Psi(y) \) for \( y \in \text{int}(\text{efd}(\Psi)) \) characterised as a unique solution in \( C \) of (39) or, equivalently, of (40).

(P4) [253]. If (B3) is used, \( C \subseteq X \), \( C \cap \text{efd}(\Psi) \neq \emptyset \), \( \mathcal{D}^G \Psi(\text{int}(\text{efd}(\Psi))) = X^* \), and \( y \in \text{int}(\text{efd}(\Psi)) \), then
\[
(\hat{P}_C^\Psi(y) = \{ * \}) \iff (\Psi + 1_C \text{ is adequate})
\] (49)
\[
\iff (\Psi + 1_C \text{ is essentially strictly convex})
\] (50)
\[
(\Psi + 1_C \text{ is adequate}) \iff (\Psi + 1_C \text{ is essentially strictly convex})
\] (51)
where
\[
1_C : X \ni x \mapsto \begin{cases} \ 0 : & x \in C \\ +\infty : & x \not\in C \end{cases}
\] (52)
is called an \textbf{indicator function}, (P1\textsubscript{4}) denotes an additional assumption that \( X \) is reflexive, while (P2\textsubscript{4}) denotes an additional assumption that \( X \) reflexive, \( \Psi \) is Legendre and cofinite, and \( C \) is weakly closed set with \( C \subseteq \text{int}(\text{efd}(\Psi)) \).
Equation (42) is an instance of a **generalised pythagorean equation**, discovered independently by Chencov [63, 64] in the case of $D_1$ distance and by Brègman in the case of his original distance (88). In [89, 30] an instance of (42) has been established for $\tilde{\Psi}^\Psi_C(y)$ with $\tilde{D}_\Psi$ defined by (B2), $X = \mathbb{R}^n$, $\Psi$ Legendre, and (P2) with $C = K + x_0$, where $K \subseteq X$ is a closed vector subspace and $x_0 \in \text{int}(\text{efd}(\Psi))$. Another instance of a generalised pythagorean equation, independent of (42), was established in a measure theoretic setting of (B5) in [82].

The composability of Brègman projection holds when they are **zone consistent** [59, 60, 31]: that is, when the projection onto convex set (whenever it exists and is unique) is within a domain of applicability of this projection onto any other convex set (again, with existence and uniqueness). According to [33], if the conditions (P2) for $\tilde{\Psi}^\Psi_C(y) = \{+\}$ are used with $\Psi$ Legendre, then $\tilde{\Psi}^\Psi_C(y)$ is zone consistent (meaning: $\tilde{\Psi}^\Psi_C(y) \in (\text{efd}(\Psi))$ and $\tilde{\Psi}^\Psi_C(y) = \tilde{\Psi}^\Psi_C(y)$. According to [7], if the conditions (P1) are used, then $\tilde{\Psi}^\Psi_C(x) = \tilde{\Psi}^\Psi_C(\tilde{\Psi}^\Psi_K(x)) = \tilde{\Psi}^\Psi_K(\tilde{\Psi}^\Psi_K(x))$ for nonempty convex closed $C$, a vector subspace $K$ of $X$ with $C \subseteq K$, and $x \in \text{int}(\text{efd}(\Psi))$.

The Brègman functional (B5) has been characterised in [141] by means of a generalised pythagorean equation. The Brègman functional (B3) has been characterised in finite dimensional case of $X = \mathbb{R}^n$ (for which it coincides with (B4)) in [77] by a set of conditions which have geometric character, and in [27] by the condition that

$$\arg\inf_{y \in X} \left\{ \int_X \tilde{\mu}(x) \tilde{D}_\Psi(x(x), y) \right\} = \int_X \tilde{\mu}(x)x(x)$$

for some measure space $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \tilde{\mu})$ and $\tilde{\mu}$-integrable function $x : \mathcal{X} \to X$. Generalisation of equation (53) (but not of the associated characterisation) to (B4) in arbitrary dimension, under some additional conditions, was provided in [109, 105]. The equality (53) was proved for the family of Liese–Vajda $\gamma$-distances (93) in [268].

### 3 Abstract Brègman distances

Our main objects of interest are not Brègman **functionals**, but Brègman **distances**, considered over information models. Most of research deals with Brègman functionals on vector spaces, as presented in the previous section. Here we follow the idea considered to a various extent in [15, 263, 139, 14, 13, 159], according to which Brègman distances shall be defined in terms of Brègman functionals on vector spaces composed with (nonlinear) embeddings of statistical or quantum models. Apart from requirement $D_\Psi(\psi, \phi) = 0 \iff \phi = \psi$ imposed from scratch, this approach stresses that a Brègman distance is an information distance defined by means of some choice of representation of an information model in a linear space, which forms a domain for corresponding Brègman functional. This exposes the dualistic properties of Brègman distance that are responsible for generalised cosine and pythagorean theorems.\(^\text{10}\) The novel aspect of our work is a systematic treatment of an extension of this approach to infinite dimensional case. The main idea is to introduce abstract generalisations of Legendre transformation and Brègman **functional** (Section 3.1), using the Young–Fenchel inequality (10), and to subsequently define a Brègman distance over an arbitrary set $Z$, using this functional together with a pair of (not necessarily linear) embeddings $(\ell, \ell^0) : Z \times Z \to X \times X^d$ into a dual pair of vector spaces (Section 3.2). In Section 3.3 we specify the domain $Z$ to be given by a generating positive cone of a base norm space or a positive part of a predual of $W^*$-algebra, introducing this way the postquantum and quantum Brègman–Legendre distances, respectively. As a concrete new example, we construct the special case of the latter class,\(^\text{10}\)Also, on more general level, this implements Lawvere’s quality–quantity distinction [172, 173], which corresponds, e.g., to use of these nonlinear embeddings to introduce a local smooth geometric structure on state spaces, c.f. [139] and references therein.
the Brègman–Orlicz distances, based on the embeddings into noncommutative Orlicz spaces over semi-finite W*-algebras. These examples provide a suitable compromise between two approaches: they are abstract Brègman distances which are determined by (B2) with reflexive X and Legendre Ψ on the level of representation. This way the abstract approach is used to handle the duality with respect to nonlinear embeddings, while the concrete approach is used to handle the existence and uniqueness of the projections.

3.1 Abstract Legendre transformation

Given a dual pair \((X, X^d, \langle \cdot, \cdot \rangle_{X \times X^d})\) over \(K \in \{\mathbb{R}, \mathbb{C}\}\) and a convex proper \(\Psi : X \to \mathbb{R} \cup \{+\infty\}\), let us define a \textit{generalised Alber functional} as a map

\[
X \times X^d \ni (x, \hat{y}) \mapsto W_\Psi(x, \hat{y}) := \Psi(x) + \Psi^F(\hat{y}) - \text{re} \ [x, \hat{y}]_{X \times X^d} \in [0, \infty].
\]  

(54)

By definition and (10),

i) \(W_\Psi(x, \hat{y})\) is convex in each variable separately,

ii) \(W_\Psi(x, \hat{y}) \geq 0 \ \forall (x, \hat{y}) \in X \times X^d\),

iii) \(W_\Psi(x, \hat{y}) = 0 \iff (\hat{y} \in \partial \Psi(x) \text{ and } x \in \text{efd}(\partial \Psi))\).

If \(X\) is a Banach space and \(X^d = X^*\) with duality given by Banach space duality, then a generalised Alber functional (54) coincides with an Alber functional (32).\footnote{We have proposed the definition (54) in [159], while being unaware of Alber’s work (which is summarised in Section 2).}

For a given dual pair \((X, X^d, \langle \cdot, \cdot \rangle_{X \times X^d})\) a \textit{dual coordinate system} on a set \(Z\) is defined as a map

\[
(\ell, \ell^0) : Z \times Z \ni (\omega, \phi) \mapsto (\ell(\omega), \ell_0^0(\phi)) \in X \times X^d.
\]  

(55)

If \(W_\Psi : X \times X^d \to [0, \infty] \) is a generalised Alber functional and \((\ell_\Psi, \ell_\Psi^0) : Z \times Z \to X \times X^d\) is a dual coordinate system such that

\[
\begin{align*}
\partial \Psi(x) \neq \emptyset & \quad \forall x \in \text{efd}(\partial \Psi) \cap \text{cod}(\ell_\Psi) \\
\ell_\Psi^0(\omega) & \in \partial \Psi(\ell_\Psi(\omega)) \quad \forall \omega \in Z,
\end{align*}
\]  

(56)

then a \textit{Brègman pre-distance} is defined as a function

\[
D_\Psi : Z \times Z \ni (\omega, \phi) \mapsto D_\Psi(\omega, \phi) := W_\Psi(\ell_\Psi(\omega), \ell_\Psi^0(\phi)) \in [0, \infty].
\]  

(57)

The conditions (56) can be understood either as constraints on allowed dual coordinate systems if \(\Psi\) is given, or as constraints on \(\Psi\) if \((\ell_\Psi, \ell_\Psi^0)\) is given. By definition, \(D_\Psi(\omega, \phi)\) is convex in each variable separately, \(D_\Psi(\omega, \phi) \geq 0 \ \forall \omega, \phi \in Z\), and \(\omega = \phi \Rightarrow D_\Psi(\omega, \phi) = 0 \ \forall \omega \in Z\). This weakening of the usual property of distance \((\omega = \phi \iff D(\omega, \phi) = 0)\) is caused by restriction of domain of \(W_\Psi\) to \(\text{cod}(\ell_\Psi) \times \text{cod}(\ell_\Psi^0)\). In order to impose an implication in the opposite direction, one would have to impose additional conditions that are not natural at this level of generality (they will be discussed below).

Definition (57) exposes the dualistic and variational structures underlying Brègman distances. However, the standard definition of Brègman distance uses only a single coordinate system instead of a dual pair, exposing geometric properties of Brègman distance and imposing \(D_\Psi(x, y) = 0 \iff x = y\) at the price of nontrivial restrictions on the domain of duality and convexity. Usually these restrictions are introduced in order to adapt to presupposed topological and differential framework (e.g. of a reflexive Banach space), which imposes some specific restrictions on Brègman distance (as exemplified by various definitions of Brègman functional in previous section), and requires one to prove that such Brègman distance encodes
the Legendre case of the Fenchel duality with the dual variable \( y \in X^d \) given by some suitably defined notion of derivative (e.g. Fréchet, Gâteaux, right Gâteaux), see e.g. [59, 31, 57, 48] for standard examples in commutative case, [221] for an example in the finite dimensional noncommutative case, and [139] for an example in the infinite dimensional noncommutative case. We do not assume any fixed framework for continuity or smoothness, so we can consider general properties of the relationship between explicitly abstract Brégman distance and its standard (hence, restricted) version, which has both arguments represented on the same space. The transition between these two formulations in the real finite dimensional case is provided by means of bijective Legendre transformation \( L_\Psi : \emptyset \rightarrow \Xi \), which acts between suitable open subsets \( \emptyset \subset \mathbb{R}^n \) and \( \Xi \subset \mathbb{R}^n \), and is given by the gradient,

\[
L_\Psi : \emptyset \ni \theta \mapsto \eta := \text{grad} \Psi (\theta) \in \Xi.
\]  

(58)

In the coordinate-dependent form this reads

\[
\eta_i = (L_\Psi(\theta))_i := \frac{\partial \Psi(\theta)}{\partial \theta^i}, \quad \theta^i = (L_\Psi^{-1}(\eta))^i := \frac{\partial \Psi^F(\eta)}{\partial \eta_i}.
\]  

(59)

whenever the duality pairing is given by

\[
[\cdot, \cdot] : \mathbb{R}^n \times \mathbb{R}^n \ni (\theta, \eta) \mapsto \theta \cdot \eta^\top := \sum_{i=1}^n \theta^i \eta_i \in \mathbb{R}.
\]  

(60)

We will now construct a general framework for conversion between these two forms of the Brégman distance, which is independent of any particular assumptions about continuity or differentiability. The key element in this setting is the (generally nonlinear) dualiser function. It will provide also an infinite dimensional generalisation of the bijective transformation between the dual coordinate systems that strengthens (56).

The relationship between dual coordinate systems is in the infinite dimensional case is more complicated than just replacing gradient by the Gâteaux derivative. It involves characterisation in terms of subdifferential, and depends on the function \( \Psi \) and on the specific structure of the dual pair \((X, X^d)\), \([\cdot, \cdot]_{X \times X^d}\) of vector spaces. In [159] we have proposed the following generalisation of the Legendre transformation to the case of arbitrary dual pair of vector spaces of arbitrary dimension, which preserves its bijective character without any fixed choice of topological background. The generalisation of (58) is provided by the dualiser, defined as a map \( L_\Psi : X \rightarrow X^d \) associated with a convex proper function \( \Psi : X \rightarrow \mathbb{R} \cup \{+\infty\} \) such that there exists a nonempty set \( \Theta_\Psi \subseteq \text{efd}(\Psi) \) satisfying:

(i) \( L_\Psi \) is a bijection on \( \Theta_\Psi \),

(ii) \( \Psi^F(L_\Psi(y)) - \Psi(y) = \text{re} \left[ y, L_\Psi(y) \right]_{X \times X^d} \forall y \in \Theta_\Psi \),

(iii) \( L_\Psi(y) \in \partial \Psi(x) \iff x = y \forall x, y \in \text{efd}(\partial \Psi) \).

If such \( L_\Psi \) exists, then \( \Theta_\Psi \) will be called an admissible domain of \( L_\Psi \) and denoted \( \text{add}(L_\Psi) \), while \( \text{adc}(L_\Psi) \equiv \Xi_\Psi := L_\Psi(\Theta_\Psi) \) will be called its admissible codomain. The function \( \Psi \) will be called dualisable with respect to \((X, X^d)\) iff there exists at least one dualiser \( L_\Psi \). Each triple \( (\Theta_\Psi, \Xi_\Psi, L_\Psi) \) will be called an abstract Legendre transformation. A bijection

\[
L_\Psi : X \ni \emptyset \ni \Xi_\Psi \subseteq X^d,
\]  

(61)

is a generalisation of (58), and allows to determine an abstract Legendre dual of \( \Psi \) as \( \Psi^L := \Psi^F \circ L_\Psi : \emptyset \rightarrow ]-\infty, +\infty[ \). A change of domain \( X \) or a change of duality structure \([\cdot, \cdot]_{X \times X^d} \) on \( X \) changes the available dualisers. Also, there might be several different dualisers
for a given quadruple \(((X, X^d, [\cdot, \cdot]_{X \times X^d}), \Psi)\). The existence of different dualisers is equivalent to \(\partial \Psi\) being a nonsingleton, nonempty, set-valued function.

Given an abstract Legendre transformation \((\Theta_\Psi, \Xi_\Psi, L_\Psi)\), we can define the Brégman functional \(D_\Psi : X \times X \to [0, +\infty]\) associated to a generalised Alber functional \(W_\Psi\) [159],

\[
D_\Psi(x, y) := \begin{cases} 
W_\Psi(x, L_\Psi(y)) = \Psi(x) - \Psi(y) - \Re \langle x - y, L_\Psi(x) \rangle_{X \times X^d} : y \in \Theta_\Psi \\
+\infty : \text{otherwise.}
\end{cases}
\]

(62)

The equality above follows from the property (ii) of \(L_\Psi\). The bounded version of this functional is given by restriction of the domain of (62) to \(\tilde{D}_\Psi : \text{efd}(\Psi) \times \Theta_\Psi \to [0, \infty[\) from the property (iii) of \(L_\Psi\) it follows that \(\tilde{D}_\Psi\) satisfies

\[
\tilde{D}_\Psi(x, y) = 0 \iff x = y \quad \forall (x, y) \in X \times X,
\]

(63)
or for all \((x, y) \in \text{efd}(\Psi) \times \Theta_\Psi\) whenever \(\tilde{D}_\Psi\) is bounded. The equivalence appears here at the price of loss of convexity of \(\tilde{D}_\Psi\) in the second variable (it is a common feature in standard treatments of Brégman functionals, see e.g. [32]). This is because using the inverse of a dualiser \(L_\Psi\) may not preserve the convexity properties. From \(\Theta_\Psi \subseteq \text{efd}(\Psi)\) it follows that the definition (62) is a generalisation of (B3). We will call this definition (B\(D_\)) and consider it as an alternative to (B2), aimed at preservation of convex and dualistic properties without reducing them to the setting of topological differentiability. From the results discussed in the previous section it follows that (B2) with reflexive \(X\) and Legendre \(\Psi\) is a special case of (B\(D_\)). More precisely, if \(X\) is a reflexive Banach space, \(X^d = X^*\), \(\Psi\) is convex, proper, lower semi-continuous, and Legendre, then \((\Theta_\Psi, \Xi_\Psi, L_\Psi)\) is given by \((\text{int}(\text{efd}(\Psi)), \text{int}(\text{efd}(\Psi^F)), \Xi^G)\) due to (19), and in such case (62) reduces to (22). Properties (62) and (63) follow then from (34), and property 5) in Section 2, respectively. A restriction of (B\(D_\)) onto \(\Theta_\Psi \times \Theta_\Psi\) will be called a strong Brégman functional. If

\[
\Psi^F|_{\Theta_\Psi} = \Psi \quad \text{and} \quad L_\Psi \circ L_\Psi^{-1} = L_\Psi.
\]

(64)

then the strong Brégman distance satisfies

\[
\tilde{D}_\Psi(x, y) = \tilde{D}_\Psi(x, y).
\]

(65)

While every strong Brégman functional \(\tilde{D} : \Theta \times \Theta \to \mathbb{R}\) is determined by a triple \((\Psi, \Theta, L_\Psi)\), there may be different choices of \((\Psi, L_\Psi)\) implementing its the representation of \(\tilde{D}\). If the conditions (64) are satisfied, we will call the corresponding pair \((\Psi, L_\Psi)\) a well-adapted Brégman representation.

### 3.2 Abstract Brégman distances

Let \((X, X^d, [\cdot, \cdot]_{X \times X^d})\) be a dual pair, let \(\Psi : X \to \mathbb{R} \cup \{+\infty\}\) be a convex proper function, let \((\Theta_\Psi, \Xi_\Psi, L_\Psi)\) be an abstract Legendre transformation, let \(Z\) be a set, and let \((\ell_\Psi, \ell_\Psi^0) : Z \times Z \to X \times X^d\) be a dual coordinate system such that \(\text{cod}(\ell_\Psi) \subseteq \Xi_\Psi\). Then we define the abstract Brégman distance on \(Z\) as a function \(D_\Psi : Z \times Z \to [0, \infty]\) such that

\[
D_\Psi(\omega, \phi) := W_\Psi(\ell_\Psi(\omega), \ell_\Psi^0(\phi)) = D_\Psi(\ell_\Psi(\omega), L_\Psi^{-1} \circ \ell_\Psi^0(\phi)) = \Psi(\ell_\Psi(\omega)) - \Psi(L_\Psi^{-1} \circ \ell_\Psi^0(\phi)) - \Re \langle \ell_\Psi(\omega) - L_\Psi^{-1} \circ \ell_\Psi^0(\phi), \ell_\Psi^0(\phi) \rangle_{X \times X^d}.
\]

(66)

It is possible to weaken the above definition by weakening the condition (iii) of definition of \(L_\Psi\) by replacing \(\text{efd}(\partial \Psi)\) and \(L_\Psi(y)\) by \(\text{efd}(\partial \Psi) \cap \text{cod}(\ell_\Psi)\) and \(L_\Psi(y) \cap \text{cod}(\ell_\Psi^0)\) respectively. In such case we will speak of weak Brégman distance. Both definitions imply

\[
D_\Psi(\omega, \phi) = 0 \iff \omega = \phi \quad \forall \omega, \phi \in Z.
\]

(67)
It follows that a single Brègman pre-distance (57) may have several different representations in terms of abstract (or weak) Brègman distances, depending on the choice of the dualiser \(L_\Psi\) (66), corresponding to the choice of the abstract Legendre transformation \((\Theta_\Psi, \Xi_\Psi, L_\Psi)\). If \(D_{\Psi, L_1}\) and \(D_{\Psi, L_2}\) are two Brègman functionals defined from a single generalised Alber functional \(W_\Psi\) by two dualisers \(L_1\) and \(L_2\) of \(\Psi\), then they are equal to each other on \(V \subseteq \text{add}(L_1) \cap \text{add}(L_2)\) iff there exists a dualiser \(L_3\) of \(\Psi\) such that \(\text{add}(L_3) = V\). Every choice of a triple \((\Theta_\Psi, \Xi_\Psi, L_\Psi)\) that turns Brègman pre-distance to an abstract Brègman distance can be considered as a \textit{localisation} of the former.

Especially interesting case of an abstract Brègman distance (66) is when the equality

\[
\ell_\Psi^0 = L_\Psi \circ \ell_\Psi
\]  

holds for all elements of \(Z\). Equation (68) is a special case of the conditions (56) and allows to rewrite (66) as

\[
D_{\Psi}(\omega, \phi) = D_{\Psi}(\ell_\Psi(\omega), \ell_\Psi(\phi)) = \Psi(\ell_\Psi(\omega)) - \Psi(\ell_\Psi(\phi)) - \Re \left[ \ell_\Psi(\omega) - \ell_\Psi(\phi), L_\Psi \circ \ell_\Psi(\phi) \right]_{X \times X^a},
\]

which does not depend on \(\ell_\Psi^0\). Functional of the form (69) will be called a \textit{standard Brègman distance}. If \(\ell_\Psi\) and \(\ell_\Psi^0\) are bijections on \(\Theta_\Psi\) and \(\Xi_\Psi\), respectively, so the diagram

\[
\begin{array}{ccc}
\mathbb{R} \cup \{+\infty\} & \xrightarrow{\Psi} & \Xi_\Psi \\
\downarrow & \searrow \swarrow & \downarrow \\
\Theta_\Psi & \downarrow \ell_\Psi^{-1} & L_\Psi \\
\downarrow & \searrow \swarrow & \downarrow \\
Z & \ell_\Psi^0 & \Xi_\Psi \\
\end{array}
\]

commutes, then we will call an associated distance (69) a \textit{strong Brègman distance}. Each strong Brègman distance is uniquely determined by a strong Brègman functional and the bijective map \(\ell_\Psi : Z \to \Theta_\Psi\), so it is determined by a triple \((\Psi, L_\Psi, \ell_\Psi)\).

From the definitions (57) and (54) it follows that every abstract Brègman distance \(D_{\Psi}\) with its corresponding dual coordinate system \((\ell_\Psi, \ell_\Psi^0)\) satisfies the \textit{quadrilateral equation}

\[
D_{\Psi}(z_1, z_2) + D_{\Psi}(z_4, z_3) - D_{\Psi}(z_1, z_3) - D_{\Psi}(z_4, z_2) = \\
\Re \left[ \left[ \ell_\Psi(z_1) - \ell_\Psi(z_4), \ell_\Psi^0(z_3) - \ell_\Psi^0(z_2) \right] \right]_{X \times X^a},
\]

and the \textit{generalised cosine equation}

\[
D_{\Psi}(z_1, z_2) + D_{\Psi}(z_2, z_3) - D_{\Psi}(z_1, z_3) = \\
\Re \left[ \left[ \ell_\Psi(z_1) - \ell_\Psi(z_2), \ell_\Psi^0(z_3) - \ell_\Psi^0(z_2) \right] \right]_{X \times X^a},
\]

for all \(z_1, z_2, z_3, z_4 \in Z\) (cf. [264]). From the definition (62) of bounded Brègman functional \(\bar{D}_{\Psi}\) it follows that \(\bar{D}_{\Psi}\) satisfies the generalised cosine equation that generalises (36).

\[
\bar{D}_{\Psi}(x_1, x_2) + \bar{D}_{\Psi}(x_2, x_3) - \bar{D}_{\Psi}(x_1, x_3) = \\
\Re \left[ x_1 - x_2, L_\Psi(x_3) - L_\Psi(x_2) \right]_{X \times X^a}
\]  

\forall x_1, x_2, x_3 \in \text{add}(L_\Psi), and it also satisfies the corresponding generalisation of the quadrilateral relation (37). From (73) it follows that for any given \(x, y, \bar{y} \in \text{add}(L_\Psi)\), the \textit{generalised orthogonal decomposition}

\[
\bar{D}_{\Psi}(x, \bar{y}) + \bar{D}_{\Psi}(\bar{y}, y) = \bar{D}_{\Psi}(x, y) \quad \forall x \in \text{add}(L_\Psi)
\]

is equivalent with the \textit{orthogonality condition},

\[
\Re \left[ x - \bar{y}, L_\Psi(y) - L_\Psi(\bar{y}) \right]_{X \times X^a} = 0.
\]
Moreover, the equivalence holds also if \( = \) is replaced by \( \geq \) in (74) and \( = \) is replaced by \( \leq \) in (75). As we will see below, under suitable assumptions that guarantee the existence and uniqueness of solution of the corresponding variational problem, the generalised orthogonal decomposition can be turned into a theorem stating the existence and uniqueness of generalised additive decomposition of information distance under projection onto subspace (submodel), known as the generalised Pythagorean theorem (or equation).

Let \( y \in \text{add}(L_\Psi) \), let \( C \subseteq \text{add}(L_\Psi) \) be nonempty, convex, and containing at least one element \( z \) such that \( \bar{D}_\Psi(z, y) < \infty \), let \( x \in C \). In such case an abstract \( \bar{D} \)-projection of \( y \), providing an abstract analogue of (38), reads

\[
y \in \bar{\Psi}_C^\Psi(y) = \arg \inf_{x \in C} \{ \bar{D}_\Psi(x, y) \}.
\]  

(76)

In general, \( \bar{\Psi}_C^\Psi(y) \) might not exist or might be nonunique. If there exists a unique \( y = \bar{\Psi}_C^\Psi(y) \) for \( y \in \text{add}(L_\Psi) \), such that \( (y, \bar{y}) \) satisfies the orthogonality condition (75), then \( y = \bar{\Psi}_C^\Psi(y) \) is called orthogonal. Property (74) generalises in such case the additive decompositions of norm under linear projections on closed convex subsets in the Hilbert space to the class of nonlinear projections \( \bar{\Psi}_C^\Psi \) onto closed convex subsets \( C \) in the linear space \( X \). Note that the ‘orthogonality’ of projection is understood in the sense of the bilinear duality pairing \( \langle \cdot, \cdot \rangle_{X \times X^d} \), while the nonlinearity of projection \( \bar{\Psi}_C^\Psi \) corresponds to the nonlinear dualiser \( L_\Psi \). In particular, if \( D_\Psi \) is given by (B3), then condition (75) turns to equality in (40), so the orthogonality condition (75) satisfied by \( \bar{y} = \bar{\Psi}_C^\Psi(y) \) turns to generalised Pythagorean equation (42).

Given an abstract Brègman distance \( D_\Psi \) on \( Z \) and \( K_1, K_2 \subseteq Z \), we define an abstract \( D_\Psi \)-projection as a map

\[
\Psi_{K_2|K_1}^{D_\Psi} : K_1 \ni \phi \mapsto \arg \inf_{\omega \in K_2} \{ D_\Psi(\omega, \phi) \} \subseteq \psi(K_2).
\]  

(77)

If \( \ell_\Psi \times \ell_\Psi^\circ \) is bijective on \( K_2 \times K_1 \), then the existence (resp., uniqueness) of \( \Psi_{K_2|K_1}^{D_\Psi}(\phi) \) follows from the existence (resp., uniqueness) of \( \Phi_{\ell_\Psi(K_2)|L_\Psi^{-1}(\ell_\Psi^\circ(K_1))}^{D_\Psi}(\ell_\Psi(\phi)) \). If \( \Psi_{K_2|K_1}^{D_\Psi}(\phi) = \{ * \} \), then we will denote it by \( \Phi_{K_2}^{D_\Psi} \). The generalised cosine equation (73) and the above discussion leads us to call an abstract Brègman projection \( \Phi_{K}^{D_\Psi}(\psi) \) orthogonal iff it is a singleton and satisfies

\[
\text{re} \left[ \langle \ell_\Psi(\phi) - \ell_\Psi(\Phi_{K}^{D_\Psi}(\psi)), \ell_\Psi(\psi) - \ell_\Psi(\Phi_{K}^{D_\Psi}(\psi)) \rangle \right]_{X \times X^d} = 0 \ \forall \phi \in K,
\]  

(78)

which is equivalent the generalised Pythagorean equation

\[
D_\Psi(\phi, \Phi_{K}^{D_\Psi}(\psi)) + D_\Psi(\Phi_{K}^{D_\Psi}(\psi), \psi) = D_\Psi(\phi, \psi) \ \forall \phi \in K.
\]  

(79)

The characterisation of orthogonal \( \Phi_{K}^{D_\Psi} \) for a given \( D_\Psi \) and \( K \) requires to introduce additional structure on the space \( X \) that would allow for identification of singletons in (77), and depends on implementation. If \( D_\Psi \) is a standard Brègman distance, then, given the topological and convexity conditions guaranteeing the existence and uniqueness of \( \Phi_{K}^{D_\Psi} \), the corresponding conditions on \( \Phi_{K}^{D_\Psi} \) can be deduced via the relative topology induced on \( Z \) by \( \ell_\Psi \), assuming the suitable convexity (and, eventually, compactness) of \( \ell_\Psi(K) \). From the study of special cases (discussed in Section 2.2), it also follows that the necessary condition for (79) to hold is the affinity of \( \ell_\Psi(K) \).

The existence and uniqueness can follow from various assumptions, which generally involve topological and convexity conditions imposed on spaces \( X \), constraints \( C \), and functions \( \Psi \). In particular, if \( X \) is a locally convex space, \( C \) is weakly compact, and \( \bar{D}_\Psi \) is weakly lower semi-continuous, then the existence follows from Bauer’s theorem [29]. On the other hand, if \( X \) is a reflexive Banach space, \( C \) is closed, \( \bar{D}_\Psi \) is lower semi-continuous, strictly convex, and
Gâteaux differentiable at \( y \), with \( \text{int}(\text{efd}(\bar{D}_\Psi)) \neq \emptyset \), \( C \cap \text{efd}(\bar{D}_\Psi) \neq \emptyset \) and \( y \in \text{int}(\text{efd}(\bar{D}_\Psi)) \), then \( \bar{D}_\Psi(y) \) is at most a singleton [48]. The conjunction of these two conditions is sufficient to guarantee the existence and uniqueness of \( \bar{D}_\Psi(y) \).

### 3.3 Brègman–Legendre distances

From the former Sections, it follows that there are few different candidates for the general notion of a Brègman distance on a general Banach space:

1. **(BD\(_1\))** the Brègman functional \( \bar{D}_\Psi \) defined by (B\(_1\)) under additional assumptions that \( \Psi \) is strictly convex on \( \text{efd}(\Psi) \) and that one of the equations (25)-(26) holds;
2. **(BD\(_2\))** the Brègman functional \( \bar{D}_\Psi \) defined by (B\(_2\)) for reflexive \( X \) and \( \Psi \) essentially strictly convex on \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \);
3. **(BD\(_3\))** the Brègman functional \( D_\Psi \) defined by (B\(_D\)), i.e. (62), with duality given by Banach space duality;
4. **(BD\(_4\))** the abstract Brègman distance (66), which is defined as a composition of (B\(_D\)) with a dual coordinate system, shifting domain to the space \( Z \), which in turn can be an arbitrary subset of a Banach space;
5. **(BD\(_5\))** defined as (BD\(_2\)), but with an additional assumption of essential Gâteaux differentiability on \( \text{int}(\text{efd}(\Psi)) \). This is a special case of both (BD\(_1\)) and (BD\(_3\)). The essential Gâteaux differentiability implies Gâteaux differentiability on \( \text{int}(\text{efd}(\Psi)) \neq \emptyset \), so this definition coincides with (B\(_3\)) on \( \text{int}(\text{efd}(\Psi)) \).

In principle, there are four main properties that one would expect from a general notion of the Brègman distance:

1. **(E\(_1\))** it should be a distance;
2. **(E\(_2\))** it should possess well defined existence and uniqueness properties for the Brègman projections onto a well defined class of subsets;
3. **(E\(_3\))** it should allow for generalised cosine and pythagorean theorems;
4. **(E\(_4\))** it should allow for composable projections.

All above candidates satisfy (E\(_1\)). The conditions (E\(_2\)) and (E\(_4\)) can be guaranteed at the level of (BD\(_5\)). The condition (E\(_3\)) requires either to strengthen (B\(_2\)) in (BD\(_4\)) with an additional assumption of strict convexity of \( \Psi \), in order to use (P\(_1\)), or to use (BD\(_3\)) with an additional orthogonality condition (78). However, the condition (78) is abstract and requires some additional conditions for it to hold. On the other hand, using (BD\(_5\)) as a Brègman distance restricts the underlying Banach space to be reflexive. Our solution is to plug (BD\(_5\)) into (BD\(_4\)) as a concrete model of (B\(_D\)). This way all expectations (E\(_1\))-E\(_4\) are met, while the underlying domain of a distance may be a subset of a nonreflexive Banach space.

**Definition 3.1.** Let \( U \) be a set, \( X \) be a reflexive Banach space, \( Z \) be a subset of \( X \), \( \Psi : X \to ]-\infty, +\infty[ \) be Legendre, \( \bar{\ell} : U \to Z \) be a bijection, \( \bar{\ell}(U) \subseteq \text{int}(\text{efd}(\Psi)) \), and let \( \bar{D}_\Psi \) be a Brègman functional defined by (21), i.e. (B\(_2\)). Then the Brègman–Legendre distance on \( U \) reads

\[
D_\Psi(\phi, \psi) := \bar{D}_\Psi(\bar{\ell}(\phi), \bar{\ell}(\psi)) \quad \forall \phi, \psi \in U.
\]

(80)

**Proposition 3.2.** \( D_\Psi \) given by (80) is a standard Brègman distance in the sense of (69) with \( L_\Psi : \Theta_\Psi \to \Xi_\Psi \) given by (19). Furthermore, if \( C \subseteq V \subseteq N^*_\psi \) is nonempty and closed in the topology induced by \( \bar{\ell} \) from the weak topology of \( X \), \( \bar{\ell}(C) \subseteq \text{int}(\text{efd}(\Psi)) \subseteq X \) is convex, and \( \psi \in V \), then:
Proof. Follows directly from (P$_2$).

\begin{proof}
\end{proof}

Remark 3.3. Note that in the above proposition $C$ does not have to be convex (resp.: closed, affine) in $\mathcal{N}_i$; this what matters is only whether it becomes convex (resp.: closed, affine) under the coordinate system $\tilde{\ell}$. We will use the terminology $\tilde{\ell}$-convex (resp.: $\tilde{\ell}$-closed, $\tilde{\ell}$-affine) to refer to this property. If $\tilde{\ell}(C)$ is convex, weakly closed, and norm bounded in $X$, then it is weakly compact, and in such case $C$ will be called $\tilde{\ell}$-compact (by definition, it is also $\tilde{\ell}$-closed).

Remark 3.4. Due to zone consistency and idempotency of $\Psi^{D\phi}$, the collection of $\tilde{\ell}$-closed $\tilde{\ell}$-convex subsets of $U$, equipped with $D\phi$-projections between them, forms a reasonable territory for introducing categorical constructions. However, given arbitrary $\tilde{\ell}$-closed $\tilde{\ell}$-convex subsets $C_0, C_1, K_1, C_2, K_2$ of $U$, satisfying $K_1 \subseteq C_1$ and $K_2 \subset C_2$, there is no guarantee that there always exists an $\tilde{\ell}$-closed $\tilde{\ell}$-convex $K_3 \subseteq C_2$ such that $\Psi^{D\phi}_{K_3} \circ \Psi^{D\phi}_{K_1}(C_0) = \Psi^{D\phi}_{K_2}(C_0)$, if the composition of two $D\phi$-projections is understood as an application of the second projection on the set-theoretic codomain of the former one. However, there is a way to circumvent this obstacle, by defining the composition $\Psi^{D\phi}_{K_3} \circ \Psi^{D\phi}_{K_1}$ to be given by $\Psi^{D\phi}_{K_2}$ whenever $K_1 \cap K_2 \neq \emptyset$. In such case, $K_1 \cap K_2$ is $\tilde{\ell}$-closed $\tilde{\ell}$-convex, so the composite arrow is uniquely defined, and its associativity (as well as an additional feature: $\Psi^{D\phi}_{K_3} \circ \Psi^{D\phi}_{K_1} = \Psi^{D\phi}_{K_2}$) follows from the associativity of intersection of closed convex sets in $X$. The quantitative evaluation of such arrow can be performed by means of an algorithm given in [35] (valid for any countable family $\{K_i\}_{i \in I}$ and any $\Psi$ that is totally convex on bounded sets, cf. [59, 55, 57]).

This composition rule allows us to define the category $\mathcal{Cvx}(\tilde{\ell}, \Psi)$ of $D\phi$-projections onto $\tilde{\ell}$-closed $\tilde{\ell}$-convex subsets (thus, hom$_{\mathcal{Cvx}(\tilde{\ell}, \Psi)}(\cdot, C)$ consists of $\Psi^{D\phi}_{K}$ with $K$ varying over all $\tilde{\ell}$-closed $\tilde{\ell}$-convex subsets of $\tilde{\ell}$-closed $\tilde{\ell}$-convex set $C$), as well as its subcategory $\mathcal{Aff}(\tilde{\ell}, \Psi)$ of $D\phi$-projections onto $\tilde{\ell}$-closed $\tilde{\ell}$-affine sets. Under restriction of the composition to only such cases when $K_2 \subseteq K_1$, the (infinitarily cyclic) algorithmic aspect of the evaluation can be dropped out, and thus it is convenient to define also the corresponding categories with the restricted composition rule, denoted $\mathcal{Cvx}^C(\tilde{\ell}, \Psi)$ and $\mathcal{Aff}^C(\tilde{\ell}, \Psi)$, respectively.

**Definition 3.5.** Let $Y$ be a base norm space [99, 101] with a generating positive cone $Y^+$. Then setting $U = Y^+$ in (80) defines a postquantum Bregman–Legendre distance. If $Y = \mathcal{N}_i$ for a $W^\ast$-algebra $\mathcal{N}$, then such $D\phi$ becomes a quantum Bregman–Legendre distance on $\mathcal{N}_i$.

Let $X$ be a real locally convex topological vector space, and $X^t$ its topological dual. A function $\Upsilon : X \to [0, \infty]$ that is convex, lower semi-continuous, satisfying $\Upsilon \neq +\infty$, $\Upsilon \neq 0$, $\Upsilon(0) = 0$, $\Upsilon(\lambda x) = 0 \forall \lambda > 0 \Rightarrow x = 0$, $x \neq 0 \Rightarrow \lim_{\lambda \to +\infty} \Upsilon(\lambda x) = +\infty$, is called a **Young function** [262, 44]. A **Young–Birnbaum–Orlicz dual** of a Young function $\Upsilon$ is defined as [45, 182]

$$
\Upsilon^Y : X^t \ni y \mapsto \Upsilon^Y(y) := \sup_{x \in X} \{ \langle x, y \rangle_{X \times X^t} - \Upsilon(x) \},
$$

and it is a special case of the Fenchel dual (9). If $\{y \in X^t \mid \langle x, y \rangle = 0 \ \forall x \in \text{efd}(\Upsilon)\} = \emptyset$ then $\Upsilon^Y$ is also a Young function [140]. A Young function $\Upsilon$ is said to satisfy global $\Delta_2$ condition iff [44] $\exists \lambda > 0 \ \forall x \geq 0 \ \Upsilon(2x) \leq \lambda \Upsilon(x)$. Given arbitrary $W^\ast$-algebra $\mathcal{N}$, we define
a noncommutative Orlicz space over $N$ as $L_Y(N) := N_Y^{\text{sa}}$, using a Young function $\Upsilon : N^{\text{sa}} \to [0, \infty]$:

\[
N_Y^{\text{sa}} := \{ x \in N^{\text{sa}} \mid \lim_{\lambda \to +0} \Upsilon(\lambda x) = 0 \} = \{ x \in N^{\text{sa}} \mid \exists \lambda > 0 \; \Upsilon(\lambda x) < \infty \}, \tag{84}
\]

\[
| \cdot |_{\Upsilon} : N_Y^{\text{sa}} \ni x \mapsto \inf \{ \lambda > 0 \mid \Upsilon(\lambda^{-1} x) \leq 1 \} \in \mathbb{R}^+. \tag{85}
\]

If $\Upsilon$ satisfies a global $\triangle_2$ condition, then \((L_Y(N))^* \cong L_{\Upsilon^Y}(N)\), where $\Upsilon^Y$ is calculated with respect to the Banach duality between $N_Y^{\text{sa}}$ and $N_Y^{\text{sa}*}$. In such case $L_Y(N)$ becomes a noncommutative analogue of the Morse–Transue–Krasnosel’skii–Rutickii space \([198, 165, 166]\), with $\exists \lambda > 0$ in (84) replaced by $\forall \lambda > 0$. If, furthermore, $N$ is a semi-finite $W^*$-algebra, equipped with a faithful normal semi-finite trace $\tau$, then $L_Y(N)$ is isometrically isomorphic to the quantum Brégman–Orlicz distance is defined as

\[
D_{\Upsilon \tau} : N^+_N \times N^+_N \ni (\phi, \psi) \mapsto \tilde{D}_{\Upsilon \tau}(\tilde{\Upsilon}(\phi), \tilde{\Upsilon}(\psi)) \in [0, \infty], \tag{87}
\]

where $\tilde{D}_{\Upsilon \tau}$ is given by (B2), $\tilde{\Upsilon} : N^*_N \ni \omega \mapsto u_{\phi} \tilde{\Upsilon}^{-1}(h_{|\phi|}) \in L_{\Upsilon}(N, \tau)$ is a homeomorphism $N^*_N \to \tilde{\Upsilon}(N)$, $\phi = |\phi|(u \cdot)$ is a unique polar decomposition, and $h_{|\phi|}$ is a unique noncommutative Radon–Nikodým density of $\phi$ with respect to $\tau$ \([96, 235, 223]\).

**Definition 3.6.** Let $N$ be a semi-finite $W^*$-algebra, let $\tau$ be a faithful normal semi-finite trace on $N$, let $\Upsilon : N^{\text{sa}} \to [0, \infty]$ be a Young function satisfying a global $\triangle_2$ condition, and corresponding to and Orlicz function $\tilde{\Upsilon}$ via $\Upsilon(x) = \tau(\tilde{\Upsilon}(x)) \forall x \in N^+$. Let $t \in \mathbb{R}$, let $\Psi := t \tau \circ \tilde{\Upsilon}$ be Legendre on $X = L_{\Psi}(N, \tau)$, and assume that $\Psi$ is invertible. Then a quantum Brégman–Orlicz distance is defined as

\[
D_{t \Upsilon \tau} : N^+_N \times N^+_N \ni (\phi, \psi) \mapsto \tilde{D}_{t \Upsilon \tau}(\tilde{\Upsilon}(\phi), \tilde{\Upsilon}(\psi)) \in [0, \infty],
\]

where $\tilde{D}_{t \Upsilon \tau}$ is a generalisation of the concept of “generalised Amari embedding”, introduced in \([118]\) for measure spaces.

**Proposition 3.8.** The distance $D_{t \Upsilon \tau}$ is a quantum Brégman–Legendre distance on $N^+_N$, and it does not depend on the choice of trace $\tau$, only on the $W^*$-algebra $N$, the Young function $\Upsilon$, and the multiplicative constant $t$.

**Proof.** In \([25]\) (cf. \([68]\)) it was shown that for any two semi-finite traces $\tau_1, \tau_2 \in \mathcal{W}_0(N)$ the spaces $L_{\Upsilon}(N, \tau_1)$ and $L_{\Upsilon}(N, \tau_2)$ are isometrically isomorphic, provided that $\Upsilon$ satisfies a global $\triangle_2$ condition. (The same follows from isometric isomorphisms of both these spaces with $L_{\Upsilon}(N)$.) This is satisfied by the definition of $D_{t \Upsilon \tau}$. \(\square\)

---

12This is a natural generalisation of the construction proposed in \([140]\) to the case of arbitrary Young function. In principle, this construction dates back to \([203, 204]\), where (84) is defined for $N^{\text{sa}}$ replaced by an arbitrary real Banach space $X$, and \([198, 204, 180, 258]\) where the norm (85) was introduced. Thus, for any real Banach space, $L_{\Upsilon}(X) := X^{\text{sa}}$ is an abstract Orlicz space over $X$. For example, for any JBW-algebra $A$ (see Appendix 2) we can define this way an abstract nonassociative Orlicz space $L_{\Upsilon}(A)$.

13In both cases it is more precise to speak of analogues rather than generalisations, because the original measure theoretic Orlicz \([211]\) and MTKR spaces are abstract Orlicz spaces over $L_0(A, \mathcal{O}(A), \mu; \mathbb{R})$, which is larger than $N^{\text{sa}} = L_{\infty}(A, \mathcal{O}(A), \mu; \mathbb{R})$.

14The statement of the sufficient condition for reflexivity in Corollary 4.3 of \([232]\) is missing the requirement of the global $\triangle_2$ condition for $\Upsilon^Y$. For example, $\Upsilon(x) = (1 + |x|) \log(1 + |x|) - |x|$ satisfies global $\triangle_2$ condition, but its YBO dual, $\Upsilon^Y(x) = e^{|x|} - |x| - 1$, does not.
Corollary 3.9. Every triple \((\mathcal{N}, \Upsilon, t)\) of a semi-finite \(W^*\)-algebra \(\mathcal{N}\), Young function \(\Upsilon : \mathcal{N}^{sa} \to [0, \infty]\), and a constant \(t \in \mathbb{R}\), determines uniquely the corresponding category \(\text{Cov}(\mathcal{E}_t, t\Upsilon)\) (resp., \(\text{Aff}(\mathcal{E}_t, t\Upsilon)\)) of Brègman–Orlicz projections onto \(\mathcal{E}_t\)-closed \(\mathcal{E}_t\)-convex (resp. \(\mathcal{E}_t\)-affine) subsets of \(\mathcal{N}^+_t\).

Remark 3.10. In the finite dimensional case, there are available several simplifications of the above constructions, allowing also to deal directly with \(L_1(\mathcal{N})\) spaces, while dropping the embeddings \(\ell\).

In particular, if \(X = X^d = \mathbb{R}^n\) with duality given by \((60)\), \(\Psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) is convex and proper, \(L_\Psi\) is given by the Legendre transformation \((58)\), \(D_\Psi\) is given by a functional introduced originally by Brègman in \([50]\),

\[
D_\Psi(x, y) = \Psi(x) - \Psi(y) - [x - y, \nabla \Psi(y)]_{\mathbb{R}^n \times \mathbb{R}^n},
\]

(88)

so the abstract Legendre transformation is determined by such \((\Theta_\Psi, \Xi_\Psi)\) that cod\((\ell_\Psi) \subseteq \Xi_\Psi\), then the associated standard Brègman distance reads

\[
D_\Psi(\omega, \phi) = \Psi(\ell_\Psi(\omega)) - \Psi(\ell_\Psi(\phi)) - \sum_{j=1}^n (\ell_\Psi(\omega) - \ell_\Psi(\phi))^j (\nabla \Psi(\ell_\Psi(\phi)))_j.
\]

(90)

If \(\mathcal{A}\) is represented in terms of a measurable space \((\mathcal{X}, \mathcal{U}(\mathcal{X}), \mathcal{U}^0(\mathcal{X}))\), \(\phi_1\) and \(\phi_2\) are densities in MeFun\((\mathcal{X}, \mathcal{U}(\mathcal{X}); \mathbb{R}^n)\) with respect to a fixed measure \(\tilde{\mu}\) on \((\mathcal{X}, \mathcal{U}(\mathcal{X}))\) such that \(\mathcal{U}(\mathcal{X}) = \mathcal{U}^0(\mathcal{X})\), so that they can be identified with the elements of \(L_1(\mathcal{X}, \mathcal{U}(\mathcal{X}), \tilde{\mu}; \mathbb{R}^n)\), \(\ell_\Psi\) is taken to be identity mapping, and

\[
\Psi(\phi_i) = \int_{\mathcal{X}} \tilde{\mu}(x) \tilde{\Psi}(\phi_i(x)),
\]

(91)

then \((90)\) takes the form \((B_5)\), with domain of \(\tilde{\Psi}\) generalised from \(\mathbb{R}^+\) to \((\mathbb{R}^+)^n\). If \(\mathcal{N} = \mathcal{B}(\mathcal{H})\) with dim \(\mathcal{H} < \infty\), \(\phi_1, \phi_2 \in \mathcal{G}_1(\mathcal{H})^0\), and \(\ell_\Psi\) is taken to be identity, then a condition analogous to \((91)\) reads \([124, 91, 221]\)

\[
\Psi(\phi_i) = \text{tr}_\mathcal{H}(\tilde{\Psi}(\phi_i)),
\]

(92)

where \(\tilde{\Psi} : \mathbb{R} \to ] - \infty, +\infty[\) is proper, operator strictly convex function, differentiable on \([0, +\infty[\) with \(\tilde{\Psi}(0) = \lim_{t \to +\infty} \tilde{\Psi}(t)\) and \(t < 0 \Rightarrow \tilde{\Psi}(t) = +\infty\), and it is applied to density operator \(\phi_i\) in terms of functional calculus on its spectrum.

Note that the relations \((89), (68), \) and \((56)\) correspond to three sectors of the information geometry theory: finite dimensional, infinite dimensional with good duality properties, and generally infinite dimensional. Dually strong Brègman distances \((70)\) provide very good duality properties in any dimensions.

In what follows, we will consider a family \(D_\gamma\), which provides an example of quantum Brègman–Orlicz distances without a restriction of \(W^*\)-algebras to semi-finite case. We will also consider its generalisation to semi-finite JBW-algebras, providing a nonquantum example of a postquantum Brègman–Legendre distance.

4 Quantum and nonassociative \(\gamma\)-distances

By imposing the condition of monotonicity under coarse graining on the abstract Brègman distances (or on the corresponding standard Brègman distances), one obtains a strong restriction on the allowed forms of the ‘generating’ function \(\Psi\) and the corresponding dual coordinate
A direct calculation shows that these functions satisfy
\[ D_1(ω, φ) := \left\{ \begin{array}{ll}
\int \frac{1}{\gamma(1-\gamma)} \left( \gamma μ_ω + (1-\gamma)ν_φ - ν_φ \left( \frac{μ_ω}{ν_φ} \right)^\gamma \right) & : \gamma \in [0,1], \ μ_ω \ll ν_φ \\
\int \lim_{\gamma \to \pm \gamma} \frac{1}{\gamma(1-\gamma)} \left( \gamma μ_ω + (1-\gamma)ν_φ - ν_φ \left( \frac{μ_ω}{ν_φ} \right)^\gamma \right) & : \gamma \in (0,1], \ μ_ω \ll ν_φ
\end{array} \right. \] (93)

where the right limit, \( \gamma \to + \gamma \), is considered for \( \gamma = 0 \), while the left limit, \( \gamma \to - \gamma \), is considered for \( \gamma = 1 \). Here \( μ_ω \) and \( ν_φ \) are finite positive measures corresponding to the positive integrals \( ω \) and \( φ \). The boundary cases take the form
\[ D_1(ω, φ) = \left\{ \begin{array}{ll}
\int (μ_ω - ν_φ + ν_φ \log \frac{μ_ω}{ν_φ}) & : \mu_ω \ll ν_φ \\
\int \lim_{\gamma \to \pm \gamma} (μ_ω - ν_φ + ν_φ \log \frac{μ_ω}{ν_φ}) & : \text{otherwise}
\end{array} \right. \] (94)

and
\[ D_0(ω, φ) = \left\{ \begin{array}{ll}
\int (μ_ω - ν_φ - ν_φ \log \frac{μ_ω}{ν_φ}) & : \mu_ω \ll ν_φ \\
\int \lim_{\gamma \to \pm \gamma} (μ_ω - ν_φ - ν_φ \log \frac{μ_ω}{ν_φ}) & : \text{otherwise}
\end{array} \right. \] (95)

It follows directly that \( D_γ \) satisfies
\( i) \ ν \ll μ \ll ν \Rightarrow D_γ(μ, ν) = D_{1-γ}(μ, ν) \ \forall γ \in [0,1], \)
\( ii) \ D_γ(\lambda μ, λ ν) = λ D_γ(μ, ν) \ \forall λ \in ]0,∞[. \)

A direct calculation shows that \( D_γ \) is an \( f \)-distance with
\[ f_γ(t) = \left\{ \begin{array}{ll}
\frac{1}{1-\gamma} t - \frac{1}{\gamma(1-\gamma)} t^\gamma & : γ \in ]0,1[ \\
\log t - (t - 1) & : γ = 1 \\
-\log t + (t - 1) & : γ = 0.
\end{array} \right. \] (96)

These functions satisfy
\[ f_0(t) = \lim_{γ \to +0} f_γ(t), \] (97)
\[ f_1(t) = \lim_{γ \to -1} f_γ(t). \] (98)

Under restriction to \( L_1(A)_{++} \), \( D_1(ω, φ) \) becomes the **Wald–Good–Kullback–Leibler distance** [256, 119, 170, 169] (cf. [120, 28]).

\[ D_1|_{L_1(A)_{++}}(ω, φ) = \left\{ \begin{array}{ll}
\int μ_ω \log \frac{μ_ω}{ν_φ} & : μ_ω \ll ν_φ \\
\int \lim_{\gamma \to \pm \gamma} μ_ω \log \frac{μ_ω}{ν_φ} & : \text{otherwise}
\end{array} \right. \] (99)

All above properties hold for the domain of \( γ \) extended from \([0,1]\) to \( R \) with the conditions satisfied for \( γ \in ]0,1[ \) extending to \( γ \in R \setminus \{0,1\} \). Nevertheless, we will consider this extension separately.

The Liese–Vajda \( γ \)-distances are generalised Brègman distances for \( γ \in ]0,1[ \) (see below), while for \( γ \in ]0,1[ \) and \( \dim(L_1(A)) = n < ∞ \) they are standard Brègman distances in the sense of (B_5) and (90) with \( X = R^n \) and \( Ψ_{γ^{-1}}(x) = \sum_{i=1}^{n}(x_i \log(x_i) - x_i + 1) \).

Amari [13] has shown that the Liese–Vajda \( γ \)-distances with \( γ \in R \) can be characterised in the finite dimensional case as a unique (up to a multiplicative constant) class of standard Brègman distances that are monotone under coarse grainings.\(^{15}\)

\(^{15}\)The assumption of **decomposability** used in Amari’s proof is a discrete version of (91), so, together with (89), it amounts to a choice of a specific dual coordinate system.
has shown that under restriction to $L_1(A)_+^+$ the uniqueness result is stronger, characterising the pair $\{D_1|_{L_1(A)_+^+}, D_0|_{L_1(A)_+^+}\}$. So far no corresponding characterisation results for the noncommutative case are known.\(^\dagger\)

Consider the $\gamma$-embedding functions on $N_+^+$ valued in $L_{1/\gamma}(N)_+$ spaces:

$$\ell_\gamma : N_+^+ \ni \omega \mapsto \ell_\gamma(\omega) := \gamma^{-1} \omega^{\gamma} \in L_{1/\gamma}(N), \quad (100)$$

with $\gamma \in [0, 1]$. These functions arise as restrictions of

$$\tilde{\ell}_\gamma : N_+ \ni \omega \mapsto \tilde{\ell}_\gamma(\omega) := \gamma^{-1} u|\omega|^{\gamma} \in L_{1/\gamma}(N), \quad (101)$$

which are bijections due to uniqueness of the polar decomposition $\omega = |\omega|(\cdot u)$. In particular, $\tilde{\ell}_{1/2}$ maps bijectively $N_+$ onto Hilbert space $L_2(N)$. The special case of the function (100) was introduced by Nagaoka and Amari [202] in commutative finite dimensional setting,

$$\ell_\gamma : M(\mathcal{X}, \mathcal{U}(\mathcal{X}), \hat{\mu}) \ni p(x) \mapsto \ell_\gamma(p(x)) := \left\{ \begin{array}{ll} \frac{1}{2} p(x)^{\gamma} & : \gamma \in [0, 1] \\
\log p(x) & : \gamma = 0 \end{array} \right\} \in L_{1/\gamma}(M(\mathcal{X}, \mathcal{U}(\mathcal{X}), \hat{\mu})^+. \quad (102)$$

Since then it became a standard tool of information geometry theory. However, the Nagaoka–Amari formulation (102), as well as its noncommutative generalisations [124, 15],\(^\ddagger\)

$$\ell_\gamma : \mathcal{G}_1(\mathcal{H})^+ \ni \rho \mapsto \ell_\gamma(\rho) := \left\{ \begin{array}{ll} \frac{1}{\gamma} \rho^{\gamma} & : \gamma \in [0, 1] \\
\log \rho & : \gamma = 0 \end{array} \right\} \in L_{1/\gamma}(\mathcal{B}(\mathcal{H}), \text{tr}_\mathcal{H})^+, \quad (103)$$

and [139]

$$\ell_\gamma^\mathsf{e} : N_+^+ \ni \omega \mapsto \ell_\gamma^\mathsf{e}(\omega) := \gamma^{-1} \Delta_{\omega, \psi}^{\gamma} \in L_{1/\gamma}(N, \psi) \quad \text{for } \gamma \in [0, 1], \quad (104)$$

use $\gamma$-powers of densities (Radon–Nikodým quotients) with respect to a fixed reference measure $\hat{\mu}$, trace $\text{tr}_\mathcal{H}$, or weight $\psi \in W_0(N)$, respectively. (For a semi-finite $N$ the embeddings (102) and (104) are the special cases of $\ell_\gamma^\mathsf{e}$ used in (87).) This restricts the generality of formulation. An important attempt to solve this problem in the commutative case was made by Zhu [265, 268], who considered the spaces of measures constructed through an equivalence relation based on $\gamma$-powers of Radon–Nikodým quotients, but without fixing any particular reference measure (hence, without passing to densities). However, his work remained unfinished and widely unknown, and it covered only the finite measures. The embeddings (100) solve these problems in the noncommutative case.

The most general quantum distance that that has been known so far to be a standard Brégnan distance that is monotone under coarse grainings is the \textit{Jenčová–Ojima $\gamma$-distance} [138, 139, 210]

$$D_\gamma(\omega, \phi) := \left\{ \begin{array}{ll} \gamma\omega(1) + (1-\gamma)\phi(1) - \gamma^\gamma \Delta_{\omega, \psi}^{\gamma} & : \omega \ll \phi \\
+\infty & : \text{otherwise} \end{array} \right\} \quad (105)$$

where $\gamma \in [0, 1], \psi \in W_0(N)$ is an arbitrary reference functional, $[\cdot, \cdot]_\psi$ is the Banach space duality pairing between the Araki–Masuda noncommutative $L_{1/\gamma}(N, \psi)$ and $L_{1/(1-\gamma)}(N, \psi)$ spaces (see [20] or [160]). However, (105) is not a canonical noncommutative generalisation of (93). The construction of (105) is dependent on the choice of fixed reference weight $\psi$, while (93) does not depend on any additional measure. (Nevertheless, the values taken by (105) are independent of the choice of $\psi$.) Using the Falcone–Takesaki theory we can make the reference-independent approach valid in all cases, including the infinite dimensional noncommutative one [158, 159].

\(^\dagger\)There exists Donald’s [95] characterisation of Donald’s distance, which coincides with $D_1|_{N_+^+}$ at least for injective $W^*$-algebras, as well as Petz’s [220] characterisation of $D_1|_{N_+^+}$ for injective $W^*$-algebras (however, see [26] for a discussion of a mistake in Petz’s proof).

\(^\ddagger\)The condition $\gamma \in [0, 1]$ in (102) and (103) can be replaced by $\gamma \in \mathbb{R} \setminus \{0\}$. However, for $\gamma \in \mathbb{R} \setminus [0, 1]$ the codomain of $\ell_\gamma$ is no longer given by the $L_{1/\gamma}$ space, so for some purposes we will consider this case separately.
Definition 4.1. Given $\gamma \in [0, 1]$, a **quantum $\gamma$-distance** is a map

$$D_\gamma : \mathcal{N}_+^* \times \mathcal{N}_+^* \ni (\omega, \phi) \mapsto D_\gamma(\omega, \phi) \in [0, \infty]$$

such that

$$D_\gamma(\omega, \phi) :=
\begin{cases}
\int \frac{1}{\gamma(1-\gamma)} (\gamma \omega + (1-\gamma)\phi - \omega^\gamma \phi^{1-\gamma}) & : \gamma \in [0, 1], \omega \ll \phi \\
\int \lim_{\tilde{\gamma} \to +\infty} \frac{1}{\tilde{\gamma}(1-\tilde{\gamma})} (\tilde{\gamma} \omega + (1-\tilde{\gamma})\phi - \omega^{\tilde{\gamma}} \phi^{1-\tilde{\gamma}}) & : \gamma \in \{0, 1\}, \omega \ll \phi \\
+\infty & : \text{otherwise},
\end{cases}$$

(106)

where the right limit, $\tilde{\gamma} \to +\gamma$, is considered for $\gamma = 0$, and the left limit, $\tilde{\gamma} \to -\gamma$, is considered for $\gamma = 1$.

**Remark 4.2.** Whenever required, the family (107) can be extended to the range $\gamma \in \mathbb{R}$ with the condition $\gamma \in [0, 1]$ replaced by $\gamma \in \mathbb{R} \setminus \{0, 1\}$, using the fact that (166) is well defined for any $\gamma > 0$, and defining $D_\gamma(\phi, \omega)$ for $\gamma < 0$ as $D_{-\gamma}(\omega, \phi)$.

**Proposition 4.3.** A quantum $\gamma$-distance (107) for $\gamma \in [0, 1]$ is an $f$-distance on $\mathcal{N}_+^*$ with $f$ given by (96).

**Proof.** Applying (96) for $\gamma \in [0, 1]$ to (3) for $\omega \ll \phi$ and using identity (166), we obtain

$$D_{\gamma}(\omega, \phi) = \langle \zeta_\pi(\phi), \left(\frac{1}{\gamma} + \frac{1}{1-\gamma} \Delta_{\omega,\phi} - \frac{1}{\gamma(1-\gamma)} \Delta_{\omega,\phi}^\gamma\right) \zeta_\pi(\phi) \rangle_{\mathcal{H}}$$

$$= \frac{1}{\gamma} \phi(\mathbb{I}) + \frac{1}{1-\gamma} \omega(\mathbb{I}) - \frac{1}{\gamma(1-\gamma)} \int \omega^\gamma \phi^{1-\gamma} = D_\gamma(\omega, \phi).$$

(108)

We have also used the identity $\Delta_{\omega,\phi}^{1/2} \xi_\pi(\phi) = \text{supp}(\phi) \xi_\pi(\phi)$, which holds for any $\phi, \omega \in \mathcal{N}_+^*$. Using $f_0(t) = \lim_{\gamma \to 1^+} f_\gamma(t)$ and $f_1(t) = \lim_{\gamma \to 1^-} f_\gamma(t)$ in (96), we obtain $D_{\gamma}(\omega, \phi) = D_\gamma(\omega, \phi)$ also for $\gamma \in \{0, 1\}$. \hfill $\square$

**Corollary 4.4.** From the above proof it follows that, for $\gamma \in \{0, 1\}$, (107) can be written explicitly as

$$D_0(\omega, \phi) = \langle \zeta_\pi(\phi), (-\log(\Delta_{\omega,\phi} + \Delta_{\omega,\phi}^0 - \mathbb{I}) \zeta_\pi(\phi)) \rangle_{\mathcal{H}}$$

$$= (\omega - \phi)(\mathbb{I}) - \langle \zeta_\pi(\phi), \log(\Delta_{\omega,\phi}) \zeta_\pi(\phi) \rangle_{\mathcal{H}}$$

(109)

and

$$D_1(\omega, \phi) = \langle \zeta_\pi(\phi), (\Delta_{\omega,\phi} \log(\Delta_{\omega,\phi}) - \Delta_{\omega,\phi}^0 + \mathbb{I}) \zeta_\pi(\phi) \rangle_{\mathcal{H}}$$

$$= (\phi - \omega)(\mathbb{I}) + \langle \zeta_\pi(\phi), (\Delta_{\omega,\phi} \log(\Delta_{\omega,\phi})) \zeta_\pi(\phi) \rangle_{\mathcal{H}}$$

$$= (\phi - \omega)(\mathbb{I}) + \langle \zeta_\pi(\omega), \log(\Delta_{\omega,\phi}) \zeta_\pi(\omega) \rangle_{\mathcal{H}}.$$ 

(110)

Hence,

$$\phi \ll \omega \ll \phi \implies D_\gamma(\omega, \phi) = D_{1-\gamma}(\phi, \omega) \quad \forall \gamma \in [0, 1],$$

(111)

$$D_\gamma(\omega, \phi) = D_\gamma(\phi, \omega) \iff \gamma = 0.5.$$ 

(112)

**Remark 4.5.** The special cases of the distance (107) are:

- the Jentzová–Ojima $\gamma$-distance (105) for $\gamma \in [0, 1]$, and for any choice of a reference weight $\psi \in \mathcal{W}_0(\mathcal{N})$, which determines the representation of the Falcone–Takesaki $L_p(\mathcal{N})$ space for every $p \in [1, \infty]$ provided by means of an isometric isomorphism with the Araki–Masuda $L_p(\mathcal{N}, \psi)$ space.
• the **Hasegawa γ-distance** [124]^{18}

\[
D_\gamma|_{\mathcal{N}^+_1}(\omega, \phi) = \frac{\tau(\rho_\omega - \rho_\phi \gamma)}{\gamma(1 - \gamma)} = \frac{\tau(\rho_\omega - \Delta^\gamma_{\omega,\phi})}{\gamma(1 - \gamma)} = \frac{1}{\gamma(1 - \gamma)} - \tau(\ell_\gamma(\rho_\omega)\ell_\gamma(\rho_\phi)),
\]

for \(\gamma \in [0, 1]\), \(\omega \ll \phi\), semi-finite \(\mathcal{N}\), and \(\rho_\phi, \rho_\omega \in L_1(\mathcal{N}, \tau)^+\) defined as the non-commutative Radon–Nikodym densities of \(\omega, \phi \in \mathcal{N}^+_1\) with respect to a faithful normal semi-finite trace \(\tau\) on \(\mathcal{N}\), i.e. \(\phi(\cdot) = \tau(\rho_\phi \cdot)\) and \(\omega(\cdot) = \tau(\rho_\omega \cdot)\). The map \(\ell_\gamma : L_1(\mathcal{N}, \tau)^+ \ni \phi \mapsto \gamma^{-1}\rho_\phi \in L_1(\mathcal{N}, \tau)\) is a straightforward generalisation of (103) and a special case of (100). If \(\mathcal{N}\) is a type I factor, then the standard representation of \(\mathcal{N}\) on \(\mathcal{H}\) is isomorphic to \(\mathfrak{B}(\mathcal{H})\) as a von Neumann algebra and \(\tau(\cdot) = tr(\cdot)\), where \(tr\) is a standard normalised \((\tau(1) = 1)\) trace on \(\mathfrak{B}(\mathcal{H})\).

• the **Araki distance** [16, 18, 19]

\[
D_1|_{\mathcal{N}^+_1}(\omega, \phi) = \left\{ \begin{array}{ll}
-\langle \xi_\pi(\omega), \log(\Delta_{\phi,\omega})\xi_\pi(\omega) \rangle_\mathcal{H} & : \omega \ll \phi \\
+\infty & : \text{otherwise,}
\end{array} \right.
\]

which is a Kosaki–Petz \(f\)-distance with an operator convex function \(f(\lambda) = -\log \lambda\). The alternative definitions generalising WGKL distance to \(\mathcal{N}^+_1\), given by Uhlmann [249] and by Pusz and Woronowicz [224], were shown to be equal to (114) in [129] and [95], respectively. If \(D_1|_{\mathcal{N}^+_1}(\omega, \phi) < \infty\), then (114) takes the form [217, 219]

\[
D_1|_{\mathcal{N}^+_1}(\omega, \phi) = \left\{ \begin{array}{ll}
\lim_{t \to +0} \int_0^\infty \xi(\phi : [\omega]_\mathcal{H}, \mathbb{I}) & : \omega \ll \phi \\
+\infty & : \text{otherwise.}
\end{array} \right.
\]

For a semi-finite \(\mathcal{N}\), normal faithful semi-finite trace \(\tau\) on \(\mathcal{N}\) and \(\rho_\phi\) and \(\rho_\omega\) defined as in (113), the Araki distance (114) turns to the **Umegaki distance** [250, 251] (cf. also [16, 17])

\[
D_1|_{\mathcal{N}^+_1}(\omega, \phi) = \tau(\rho_\omega(\log \rho_\omega - \log \rho_\phi)) = \tau\left(\rho^{1/2}_\omega(\log \Delta_{\omega,\phi})\rho^{1/2}_\phi\right)
\]

\[
= \int_0^1 d\lambda \tau\left(\rho_\omega - \frac{1}{\rho_\phi + \lambda}(\rho_\omega - \rho_\phi) - \frac{1}{\rho_\omega + \lambda}\right)
\]

if \(\omega \ll \phi\), and \(D_1|_{\mathcal{N}^+_1}(\omega, \phi) = +\infty\) otherwise.

• the **Liese–Vajda γ-distance** (93) for \(\gamma \in [0, 1]\), and commutative \(\mathcal{N}\), such that \(\mathcal{N} = L_\infty(\mathcal{A})\).^{19}

• the **Amari–Cressie–Read γ-distance**\(^{20}\) [11, 72, 12, 176, 225]

\[
D_\gamma|_{L_1(\mathcal{A})^+_2}(\omega, \phi) = \left\{ \begin{array}{ll}
\frac{1}{1 - \gamma} \int \left(\frac{\mu_\omega - \nu_\phi}{\mu_\phi}\right)^\gamma & : \mu_\omega \ll \nu_\phi \\
+\infty & : \text{otherwise,}
\end{array} \right.
\]

for \(\gamma \in [0, 1]\), commutative \(\mathcal{N}\), and normalised measures \(\nu_\phi\) and \(\mu_\omega\) (\(\int \mu_\omega = 1 = \int \nu_\phi\)) on the mcβ-algebra \(\mathcal{A}\) associated with \(\mathcal{N}\) by means of \(L_\infty(\mathcal{A}) = \mathcal{N}\). Consider the

\[\footnote{Here we have generalised the original definition given by Hasegawa in a way analogous to Umegaki’s definition of \(D_1|_{\mathcal{N}^+_1}\) given in [250] and (116).}^{18}
\]

\[\footnote{Under extension of the domain of \(\gamma\) to \(\mathbb{R}\), the \(D_{\gamma=2}\) distance is a Csiszár–Morimoto \(f\)-distance with \(f(\lambda) = \frac{1}{2}(\lambda - 1)^2\), and coincides, up to multiplication by 2, with the \(\chi^2\) distance, \(D_{\gamma=2}(\omega, \phi) = 2\chi^2(\omega, \phi)\).}^{19}
\]

\[\footnote{The family (117) can be considered for \(\gamma \in [0, 1]\) replaced by \(\gamma \in \mathbb{R} \setminus \{0, 1\}\). This family corresponds bijectively, but is not equal, to the \(\gamma\)-distance families of Chernoff [66], Kraft [164], Rényi [227, 228], Pérez [215], Havrdra–Chávrat [126], Linhard–Nielson [178], and Tsallis [248] (for a review with calculations, see e.g. [73]). One should note that, in particular, the Bhattacharyya, Chernoff, and Rényi distances do not belong to the class of the Csiszár–Morimoto \(f\)-distances [176].}^{20}\]
Kakutani–Hellinger distance [149, 127, 185, 186][21], defined as a square root of the Csiszár–Morimoto $f$-distance with $f(\lambda) = (1 - \sqrt{\lambda})^2$,
\[
D_{KH}(\omega, \phi) := \sqrt{\int (\sqrt{\omega} - \sqrt{\phi})^2}.
\] (118)

The case $\gamma = 1/2$ satisfies
\[
D_{KH}\big|_{L_1(A)^\dagger} (\omega, \phi) = \sqrt{\frac{1}{4} D_{1/2}\big|_{L_1(A)^\dagger} (\omega, \phi)}
\] (119)
and allows to define the Bhattacharyya distance [41, 40, 42, 43]
\[
D_B\big|_{L_1(A)^\dagger} (\omega, \phi) := 4 - 4 D_{1/2}\big|_{L_1(A)^\dagger} (\omega, \phi).
\] (120)

If a representation of $A$ in terms of some $(\mathcal{X}, \mathcal{U}(\mathcal{X}), \tilde{v})$ is given, with $\tilde{\mu}_\omega \ll \tilde{v}$ and $\tilde{\nu}_\phi \ll \tilde{v}$ such that $p_\omega := \tilde{\mu}_\omega / \tilde{v}$ and $q_\phi := \tilde{\nu}_\phi / \tilde{v}$, then (119) and (120) imply
\[
D_{KH}\big|_{L_1(A)^\dagger} (\omega, \phi) = \sqrt{\frac{1}{2} \int \tilde{v} \left( \frac{\sqrt{p_\omega}}{\sqrt{q_\phi}} - \sqrt{q_\phi} \right)^2} = \sqrt{1 - \int \tilde{v} \sqrt{p_\omega q_\phi}}
= \frac{1}{\sqrt{2}} \sqrt{\left\| \sqrt{p_\omega} - \sqrt{q_\phi} \right\|_{L_2(\mathcal{X}, \mathcal{U}(\mathcal{X}), \tilde{v})}} = \sqrt{1 - D_B\big|_{L_1(A)^\dagger} (\omega, \phi)}.
\] (121)

- the WGKL distance (99), for commutative $\mathcal{N} = L_\infty(A)$, and $\int \mu_\omega = 1 = \int \nu_\phi$. (For an explicit derivation of the WGKL distance from the Araki distance for $\mathcal{N} = L_\infty(\mathcal{X}, \mathcal{U}(\mathcal{X}), \tilde{v})$ see [129].)

**Proposition 4.6.** If $\gamma \in [0, 1]$, then quantum $\gamma$-distance (107) is an abstract Brègman distance (66), a standard Brègman distance (69), and a quantum Brègman–Legendre distance (80), with a dual coordinate system $(\ell_\gamma, \ell_{1-\gamma})$ given by (100), with a convex proper function
\[
\Psi_\gamma : L_{1/\gamma}(\mathcal{N}) \ni x \mapsto \Psi_\gamma(x) := \frac{1}{1 - \gamma} \int (\gamma x)^{1/\gamma} = \frac{\int \gamma x^{1/\gamma}}{1 - \gamma} \in [0, +\infty[,
\] (122)
with a dualiser
\[
L_{\Psi_\gamma} := \ell_{1-\gamma} \circ \ell_\gamma^{-1} : L_{1/\gamma}(\mathcal{N}) \ni \frac{1}{1 - \gamma} u |\phi|^{1-\gamma} \mapsto \frac{1}{1 - \gamma} u |\phi|^{1-\gamma} \in L_{1/(1-\gamma)}(\mathcal{N}),
\] (123)
and with a Brègman functional, in the sense of $(B_D)$ and $(B_k)$,
\[
D_{\Psi_\gamma}(x, y) = \Psi_\gamma(x) + \Psi_{1-\gamma}(L_{\Psi_\gamma}(y)) - \rho \left[ x, L_{\Psi_\gamma}(y) \right]_{L_{1/\gamma}(\mathcal{N}) \times L_{1/(1-\gamma)}(\mathcal{N})}.
\] (124)

Furthermore, under restriction to $L_{1/\gamma}(\mathcal{N}, \tau)$ for a semi-finite $\mathcal{N}$ and a faithful normal semi-finite trace $\tau$ on $\mathcal{N}$, (107) becomes (up to a multiplicative constant) a special case of a quantum Brègman–Orlicz distance (87), with $D_\gamma = \frac{1}{\gamma(1-\gamma)} D_{B_{\gamma\tau}}$ where $B_{\gamma}(x) = \tau(x^{1/\gamma})$.

**Proof.** Our method of proof will be based on the approach of [139] (which in turn used some of the ideas introduced in [117]).

The embeddings $\ell_\gamma$ defined by (100) allow to construct the real valued functional on $\mathcal{N}_\gamma^+$ using the duality (161),
\[
\mathcal{N}_\gamma^+ \times \mathcal{N}_\gamma^+ \ni (\omega, \phi) \mapsto \int \ell_\gamma(\omega) \ell_{1-\gamma}(\phi) = \left[ \ell_\gamma(\omega), \ell_{1-\gamma}(\phi) \right]_{L_{1/\gamma}(\mathcal{N}) \times L_{1/(1-\gamma)}(\mathcal{N})} \in \mathbb{R}.
\] (125)

---

[21] As pointed in [174], the reference to [127] is traditional, but quite irrelevant. The referenced paper contains only the integrals of the form $\int \frac{d\mu_1}{d\nu_1}$ for $\mu_1 \ll \nu_1$ and $\mu_2 \ll \nu_2$. 27
In these terms, \( D_\gamma \) defined in (107) for \( \gamma \in [0,1] \) is equal to
\[
D_\gamma(\omega, \phi) = \int \left( \frac{\omega}{1-\gamma} + \frac{\phi}{\gamma} - \ell_\gamma(\omega)\ell_{1-\gamma}(\phi) \right) = \frac{\omega(1)}{1-\gamma} + \frac{\phi(1)}{\gamma} - \left[ \ell_\gamma(\omega), \ell_{1-\gamma}(\phi) \right]_\gamma, \tag{126}
\]
where we have simplified the notation by \( [\cdot, \cdot]_\gamma := [\cdot, \cdot]_{L_1/(1-\gamma)(\mathcal{N})} \).

We begin by proving that that a function \( L_{\Psi_\gamma} \) is a homeomorphism in the corresponding norm topologies. Its bijectivity follows from the bijectivity of \( \ell_\gamma \). For \( \phi \in \mathcal{N} \), denote its unique polar decomposition as \( |\phi|(\cdot u) \). From (160) it follows that
\[
\|u|\phi|\|_{1/\gamma} = (|\phi|(\|)|)^\gamma, \tag{127}
\]
so
\[
\gamma x \|_{1/\gamma} := \|(1-\gamma)L_{\Psi_\gamma}(x)\|_{1/(1-\gamma)} = \int |\phi| = \int |\phi|^\gamma |\phi|^\gamma \text{supp}(\phi) = \int u|\phi|^\gamma |\phi|^\gamma u^* = \int u|\phi|^\gamma (u|\phi|^\gamma)^* = \gamma(1-\gamma) \left[ x, (L_{\Psi_\gamma}(x))^* \right]_\gamma. \tag{128}
\]

According to [83], if a Banach space \( X \) is Gâteaux differentiable except 0 \( \in X \), then
\[
\left[ [y, \mathfrak{D}^G \|]\right]_{X \times X^*} = \text{re} \left[ [y, v_x/\|v_x\|]\right]_{X \times X^*}, \tag{129}
\]
where \( v_x/\|v_x\| \) denotes a unique point on a unit sphere in \( X^* \) such that
\[
\left[ [x, v_x/\|v_x\|]\right]_{X \times X^*} = \|x\|_X. \tag{130}
\]
If \( X \) is also uniformly convex and \( \|\cdot\|_X \) is Fréchet differentiable, then a map
\[
F_v: \left\{ \begin{array}{c} X \setminus \{0\} \ni x \mapsto \|x\|_X v_x/\|v_x\| \in X^* \setminus \{0\} \\
X \ni 0 \mapsto 0 \in X^* \end{array} \right. \tag{131}
\]
is a homeomorphism in the norm topologies of \( X \) and \( X^* \).

The function
\[
v_\gamma(x) := \|\gamma x\|_{1/\gamma}^{1-\gamma} (1-\gamma)(L_{\Psi_\gamma}(x))^* \tag{132}
\]
satisfies
\[
\left[ x, v_\gamma(x) \right]_\gamma = \|\gamma x\|_{1/\gamma}^{1-\gamma} (1-\gamma) \left[ x, (L_{\Psi_\gamma}(x))^* \right]_\gamma = \|\gamma x\|_{1/\gamma}^{1-\gamma} (1-\gamma)\|\gamma x\|_{1/\gamma}^{1-\gamma} (1-\gamma)^{-1} = \|x\|_{1/\gamma}. \tag{133}
\]

From (128) it follows that \( L_{\Psi_\gamma}(x) \) is continuous at 0. From uniform convexity and uniform Fréchet differentiability of \( L_{1/\gamma}(\mathcal{N}) \) for \( \gamma \in [0,1] \) it follows that \( v_\gamma(x) = v_x/\|v_x\| \) for \( X = L_{1/\gamma}(\mathcal{N}) \), so for \( x \in L_{1/\gamma}(\mathcal{N}) \setminus \{0\} \) the function \( F_{v_\gamma} \) reads
\[
F_{v_\gamma}(x) = \|x\|_{1/\gamma} v_\gamma(x) = (1-\gamma)\gamma^{1-\gamma} \|x\|_{1/\gamma}^{2-\gamma} (L_{\Psi_\gamma}(x))^*, \tag{134}
\]
which implies that \( L_{\Psi_\gamma} \) is also a homeomorphism.

Next, we will prove that \( \Psi_\gamma \) is Fréchet differentiable, with
\[
(\mathfrak{D}^F x_{\Psi_\gamma})(y) = \text{re} \left[ [y, L_{\Psi_\gamma}(x)] \right]_\gamma \quad \forall x \in L_{1/\gamma}(\mathcal{N}) \tag{135}
\]
and
\[
\Psi_\gamma(x) + \Psi_{1-\gamma}(L_{\Psi_\gamma}(x)) - \text{re} \left[ [x, L_{\Psi_\gamma}(x)] \right]_\gamma = 0 \quad \forall x \in L_{1/\gamma}(\mathcal{N}). \tag{136}
\]
From the uniform Fréchet differentiability of \( L_{1/\gamma}(\mathcal{N}) \) it follows that \( \| \cdot \|_{1/\gamma} \) is Fréchet differentiable at any \( x \in L_{1/\gamma}(\mathcal{N}) \setminus \{ 0 \} \), and

\[
(\mathcal{D}_x^F \| \cdot \|_{1/\gamma})(y) = \text{re} \left[ y, v_{\gamma} \right]_{\gamma} \quad \forall y \in L_{1/\gamma}(\mathcal{N}), \tag{137}
\]

so

\[
(\mathcal{D}_x^F \Psi_\gamma)(y) = \left( \mathcal{D}_x^F \left( \frac{1}{1-\gamma} \| \gamma x \|_{1/\gamma}^{1/\gamma} \right) \right)(y) = \left( \frac{1}{1-\gamma} \| \gamma x \|_{1/\gamma}^{1-1/\gamma} \mathcal{D}_x^F \| \cdot \|_{1/\gamma} \right)(y) = \text{re} \left[ y, \frac{1}{1-\gamma} \| \gamma x \|_{1/\gamma}^{1-1/\gamma} (1-\gamma)(L_{\Psi_\gamma}(x))^* \right]_{\gamma} = \text{re} \left[ y, L_{\Psi_\gamma}(x) \right]_{\gamma}. \tag{138}
\]

The function \( \| \gamma x \|_{1/\gamma} \) is also Fréchet differentiable at \( x = 0 \), which implies

\[
(\mathcal{D}_0^F \Psi_\gamma)(y) = 0 = \text{re} \left[ y, L_{\Psi_\gamma}(0) \right]_{\gamma}. \tag{139}
\]

This gives (135). The equation (136) follows as straightforward calculation. Note that (136) is just \( \bar{D}_{\Psi_\gamma}(x,y) = 0 \) for \( \bar{D}_{\Psi_\gamma} \) given by (124). From the fact that (124) satisfies (B_4), it follows that \( \bar{D}_{\Psi_\gamma}(x,y) \geq 0 \). Moreover, from Fréchet differentiability and continuity of \( \Psi_{1-\gamma} \) on all \( L_{1/(1-\gamma)}(\mathcal{N}) \) and reflexivity of \( L_{1/\gamma}(\mathcal{N}) \) spaces it follows that \( \Psi_{\gamma} \) is essentially strictly convex, hence, due to (30), \( \bar{D}_{\Psi_\gamma}(x,y) = 0 \iff x = y \). This implies that the equation (136) is a unique solution of the variational problem

\[
\Psi_{1-\gamma}(L_{\Psi_\gamma}(x)) = \sup_{y \in L_{1/\gamma}(\mathcal{N})} \left\{ \text{re} \left[ y, L_{\Psi_\gamma}(x) \right]_{\gamma} - \Psi_\gamma(y) \right\}, \tag{140}
\]

because

\[
y \neq x \implies \Psi_\gamma(y) + \Psi_{1-\gamma}(L_{\Psi_\gamma}(x)) - \text{re} \left[ y, L_{\Psi_\gamma}(x) \right]_{\gamma} > 0, \tag{141}
\]

\[
\Psi_{1-\gamma}(L_{\Psi_\gamma}(x)) > \text{re} \left[ y, L_{\Psi_\gamma}(x) \right]_{\gamma} - \Psi_\gamma(y). \tag{142}
\]

Comparing (140) with (9), we see that

\[
\Psi_{1-\gamma} = \Psi_\gamma^F, \tag{143}
\]

with respect to the duality \( [\cdot, \cdot]_\gamma \).

If \( X \) is a Banach space and \( f : X \to \mathbb{R} \) is norm continuous and convex function, then \( f \) is Gâteaux differentiable iff \( \partial f(x) = \{ * \} \) \( \forall x \in X \). The norm continuity and Fréchet differentiability of \( \Psi_\gamma \) on \( L_{1/\gamma}(\mathcal{N}) \) implies that

\[
\partial \Psi_\gamma(x) = \{ * \} = \mathcal{D}_x^F \Psi_\gamma, \tag{144}
\]

so

\[
L_{\Psi_\gamma}(y) \in \partial \Psi_\gamma(x) \iff x = y \ \forall x, y \in \text{efd}(\partial \Psi_\gamma). \tag{145}
\]

Hence, \( (L_{1/\gamma}(\mathcal{N}), L_{1/(1-\gamma)}(\mathcal{N}), L_{\Psi_\gamma}) \) is an abstract Legendre transform, and \( D_{\Psi_\gamma}(\omega, \phi) \) is an abstract Brègman distance of the form

\[
D_{\Psi_\gamma}(\omega, \phi) = \Psi_{\gamma}(\ell_\gamma(\omega)) + \Psi_{1-\gamma}(\ell_{1-\gamma}(\phi)) - \ell_{1-\gamma}(\omega, \ell_{1-\gamma}(\phi))_{L_{1/(1-\gamma)}(\mathcal{N})} \tag{146}
\]

with \( \Psi_{\gamma}(\ell_\gamma(\omega)) = \frac{1}{1-\gamma} \omega(\|) \).

The proof of compatibility with the definition of the Brègman–Orlicz distance follows straight by a direct calculation. \( \square \)
Proposition 4.7. If \( \gamma \in [0,1] \), then \( D_\gamma(\omega, \phi) \) satisfies the generalised cosine equation

\[
D_\gamma(\omega, \phi) + D_\gamma(\phi, \psi) = D_\gamma(\omega, \psi) + \int \left( \ell_\gamma(\omega) - \ell_\gamma(\phi) \right) \left( \ell_{1-\gamma}(\psi) - \ell_{1-\gamma}(\phi) \right). \tag{147}
\]

In finite dimensional setting (147) holds also for \( \gamma \in \{0,1\} \), with \( \ell_\gamma \) given by (103).

Proof. Straightforward calculation based on equations (126) and (72).

\[\square\]

Corollary 4.8. The equation (147) is an instance of the ‘standard’ generalised cosine equation (73) applied to \( D_\omega \), given by (124), while the equation (111) follows from the ‘representation-index duality’ equation\(^{22}\)

\[
\tilde{D}_{\Psi_\gamma}(x, y) = \tilde{D}_{\Psi_{1-\gamma}}(L_{\Psi_\gamma}(y), L_{\Psi_\gamma}(x)), \tag{148}
\]

where \( x, y \in L_{1/\gamma}(\mathcal{N}) \). For \( \gamma = 1/2 \) the \( L_{1/\gamma}(\mathcal{N}) \) space becomes a Hilbert space \( \mathcal{H} \), the generalised Brègman functional \( \tilde{D}_{\Psi_\gamma} \) becomes the norm distance on it,

\[
\tilde{D}_{\Psi_{1/2}}(x, y) = \| x - y \|^2_\mathcal{H}/2, \tag{149}
\]

so the generalised cosine equation for \( \tilde{D}_{\Psi_\gamma} \) turns to the cosine equation in Hilbert space \( \mathcal{H} \),

\[
\| x - y \|^2_\mathcal{H} + \| y - z \|^2_\mathcal{H} = \| x - z \|^2_\mathcal{H} + 2 \langle x - y, z - y \rangle_\mathcal{H}. \tag{150}
\]

Remark 4.9. From the fact that (107) is an \( f \)-distance it follows that it has the following properties [156, 218, 209, 139]:

1) \( D_\gamma(\omega, \phi) \geq D_\gamma(T_\omega(\omega), T_\omega(\phi)) \),

2) \( D_\gamma \) is jointly convex on \( \mathcal{N}_*^+ \times \mathcal{N}_*^+ \),

3) for \( \gamma \in [0,1] \), \( D_\gamma \) is lower semi-continuous on \( \mathcal{N}_*^+ \times \mathcal{N}_*^+ \) endowed with the product of norm topologies, while for \( \gamma \in \{0,1\} \) it is also lower semi-continuous on \( \mathcal{N}_*^+ \times \mathcal{N}_*^+ \) endowed with the product of weak-* topologies.

Remark 4.10. The family (107) provides a canonical infinite dimensional noncommutative generalisation of the family (93) of Liese–Vajda \( \gamma \)-distances, and generalises the family (105) of Jenčová–Ojima \( \gamma \)-distances in terms of canonical noncommutative \( L_{1/\gamma}(\mathcal{N}) \) spaces. These properties, considered together with Propositions 4.3 and 4.6 suggest a quantum analogue of Amari’s [13] characterisation of the Liese–Vajda \( \gamma \)-distances. Amari’s characterisation holds for \( \gamma \in \mathbb{R} \). On the other hand, Hasegawa [125] proved that (96), when extended with the range of \( \gamma \) to \( \mathbb{R} \), is operator convex only for \( \gamma \in [-1,2] \). This leads us to:

Conjecture 4.11. The family \( D_\gamma \) with \( \gamma \in [-1,2] \) is the unique (up to a multiplicative constant) family of quantum distances \( D \) on \( \mathcal{N}_*^+ \) that satisfies the conditions:

(\text{strong version}): \[s1\) \( D(\omega, \phi) \geq D(T_\omega(\omega), T_\omega(\phi)) \quad \forall \omega, \phi \in \mathcal{N}_*^+ \quad \forall T_\omega \in \text{Mark}_*(\mathcal{N}_*^+) \),

\(s2\) \( D \) is representable in the form (69),

\(s3\) \( \exists C \subseteq \mathcal{N}_*^+ \quad \forall (\phi, \psi) \in K \times C \)

\[
\Psi K D(\psi) = \{ \ast \} \quad \Rightarrow \quad D(\phi, \psi) = D(\phi, \Psi K D(\psi)) + D(\Psi K D(\psi), \psi), \tag{151}
\]

for every \( K \subseteq C \subseteq \mathcal{N}_*^+ \) such that \( \ell_\Psi(K) \) is affine, where \( \ell_\Psi \) is as in (69), and \( \Psi K D(\psi) := \arg \inf_{\phi \in K} \{ D(\phi, \psi) \} \).

\[^{22}\text{The finite dimensional commutative version of the equation (148), with a dualiser given by gradient, was discussed in [263].}\]
apply also in this case. The corresponding results for projections for theorems on projections for the abstract Brègman functional. Our proof follows a bit different path, relying on the general theory of Brègman projections.

Proposition 4.12. 1) if \( y \in L_{1/\gamma}(N) \) and \( K \subseteq L_{1/\gamma}(N) \) is nonempty, weakly closed, convex, then:

i) \( \Psi_{K}^{\gamma}(y) := \arg \inf_{x \in K} \{ D_{\gamma}(x, y) \}\),

ii) \( D_{\gamma}(x, y) \geq D_{\gamma}(x, \Psi_{K}^{\gamma}(y)) + D_{\gamma}(\Psi_{K}^{\gamma}(y), y) \forall x \in K \),

and, equivalently,

\[
\text{re} \left[ \left[ x - \Psi_{K}^{\gamma}(y), L_{\gamma}(y) - L_{\gamma}(\Psi_{K}^{\gamma}(y)) \right] \right]_{L_{1/\gamma}(N) \times L_{1/(1-\gamma)}(N)} \leq 0 \forall x \in K. \tag{153}
\]

iv) the equality in (152) and (153) holds if \( K \) is additionally a vector subspace of \( L_{1/\gamma}(N) \),

2) if \( \psi \in N_{\gamma}^{+} \) and \( C \subseteq N_{\gamma}^{+} \) is nonempty, \( \ell_{\gamma}(C) \subseteq L_{1/\gamma}(N) \) is convex, and \( C \) is closed in the topology induced by \( \ell_{\gamma} \) from the weak topology of \( L_{1/\gamma}(N) \), then

i) \( \Psi_{C}^{D_{\gamma}}(\psi) := \arg \inf_{\phi \in C} \{ D_{\gamma}(\phi, \psi) \}\),

ii) if \( \ell_{\gamma}(C) \) is a vector subspace of \( L_{1/\gamma}(N) \), then the generalised pythagorean equation holds:

\[
D_{\gamma}(\omega, \psi) = D_{\gamma}(\omega, \Psi_{C}^{D_{\gamma}}(\psi)) + D_{\gamma}(\Psi_{C}^{D_{\gamma}}(\psi), \psi) \forall \omega \in C. \tag{154}
\]

Proof. Because \( D_{\gamma} \) given by (124) is a Brègman functional in the sense of (B4), the theorems (P1) on existence, uniqueness and properties of Brègman projections for definitions (B3) and (B4) provided in Section 2 apply also in this case. The corresponding results for \( D_{\gamma} \) follow from the fact that it is a reflexive quantum Brègman distance, so the Proposition 3.2 applies. More specifically, this can be obtained by an extension of \( D_{\gamma} \) to \( D_{\gamma}^{\hat{\gamma}} \), defined on the whole space \( N_{\gamma} \) by replacing the term \( \ell_{\gamma}(\omega), \ell_{1-\gamma}(\phi) \) in (126) by \( \left[ \ell_{\gamma}(\omega), \ell_{1-\gamma}(\phi) \right]_{\gamma} \). Because \( \ell_{\gamma} \) are homeomorphisms (hence, bijections) between Banach spaces \( N_{\gamma} \) and \( L_{1/\gamma}(N) \), the theorems on existence, uniqueness, and pythagorean theorem for projections for \( D_{\gamma} \) on \( L_{1/\gamma}(N) \) can be translated in terms of topology induced by \( \ell_{\gamma} \) on \( N_{\gamma} \), turning them into the corresponding theorems on projections for \( D_{\gamma} \). The results for \( D_{\gamma} \) follow then by the restriction of domain of \( D_{\gamma}^{\hat{\gamma}} \) to \( N_{\gamma}^{+} \).

Most of the conditions for (P1) were already verified: \( L_{1/\gamma}(N) \) is reflexive, \( \Psi_{\gamma} \) is lower semi-continuous, Gâteaux differentiable, essentially Gâteaux differentiable and essentially strictly convex on \( \text{efd}(\Psi_{\gamma}) = L_{1/\gamma}(N) \). The strict convexity of \( \Psi_{\gamma} \) follows from Gâteaux differentiability of \( \Psi_{1-\gamma} \). Finally,

\[
\lim_{|x|_{1/\gamma} \to +\infty} \frac{\Psi_{\gamma}(x)}{|x|_{1/\gamma}} = \frac{\gamma^{1-\gamma}}{1 - \gamma} \frac{\gamma^{1-\gamma}}{|x|_{1/\gamma}} = +\infty \forall x \in K. \tag{155}
\]

\[\square\]

\[23\] In [221] Petz claims (without proof) this uniqueness property for \( D_{1} \) for normalised states.
Remark 4.13. Jenčová [139] proved also that, under the same assumptions as in 1) and 2) above, respectively:

1.ii) $y \mapsto \mathcal{P}^{\Psi}_{K}(y)$ is a continuous function from $L_{1/\gamma}(\mathcal{N}, \phi)$ with its norm topology to $K$ with the relative weak topology,

2.ii) $\psi \mapsto \mathcal{P}^{D_{\gamma}}_{C}(\psi)$ is a continuous function from $\mathcal{N}^{+}$ with the topology induced by $\tilde{\ell}_{\gamma}$ from the norm topology of $L_{1/\gamma}(\mathcal{N}, \phi)$ to $C$ with the relative topology induced by $\tilde{\ell}_{\gamma}^{-1}$ from the weak topology of $L_{1/\gamma}(\mathcal{N}, \phi)$.

By the isometric isomorphism of the Araki–Masuda $L_{p}(\mathcal{N}, \phi)$ spaces and the Falcone–Takesaki $L_{p}(\mathcal{N})$ spaces, these results hold also in our case. This provides a topological specification of the stability of the behaviour of $D_{\gamma}$ projections, $\gamma \in [0, 1[$, under the change of initial state $\psi$.

Definition 4.14. Let $A$ be a JBW-algebra and $\tau$ a faithful normal semi-finite trace on $A$ (see Appendix 2 for all notions of nonassociative integration that we use here). For any $\phi \in A^{+}$, let $h_{\phi}$ denote its unique nonassociative Radon–Nikodým quotient, determined by $\tau(h_{\phi}x) = \phi(x)$ $\forall x \in A$ [23]. We define a nonassociative $\gamma$-distance as a map $D_{\gamma} : A_{+}^{+} \times A_{+}^{+} \to [0, \infty]$ such that

$$D_{\gamma}(\omega, \phi) = \frac{1}{1 - \gamma} \omega(I) + \frac{1}{\gamma} \phi(I) - \frac{1}{\gamma(1 - \gamma)} \tau(h_{\phi}^{\gamma}h_{\phi}^{1 - \gamma})$$

(156)

for $\omega \ll \phi$ and $D_{\gamma}(\omega, \phi) = \infty$ otherwise.

Corollary 4.15. Because the proofs of Propositions 4.6 and 4.12 do not depend on the associative structure of $\mathcal{N}$, only on Fréchet and Gâteaux differentiability, as well as on reflexivity, of $L_{1/\gamma}(\mathcal{N})$ spaces, these properties hold also for (156). The same is true for Proposition 4.7 and Corollary 4.8.

5 Brègman nonexpansive $\tilde{\ell}$-operations and nonlinear resource theories

Let $X$ be a Banach space, and $\Psi : X \to [-\infty, +\infty]$ be proper, convex, lower semi-continuous with $\text{int}(\text{efd}(\Psi)) \neq \emptyset$, and let $\tilde{D}_{\Psi}$ be given by (B2). Let $\emptyset \neq M \subseteq \text{int}(\text{efd}(\Psi))$, then the function $T : M \to \text{int}(\text{efd}(\Psi))$ will be called [55, 34, 46]:

(i) Brègman completely nonexpansive iff $\tilde{D}_{\Psi}(T(x), T(y)) \leq \tilde{D}_{\Psi}(x, y) \forall x, y \in M$;

(ii) Brègman quasi-nonexpansive iff $\tilde{D}_{\Psi}(x, T(y)) \leq \tilde{D}_{\Psi}(x, y) \forall x, y \in \text{Fix}(T) \times M$, where $\text{Fix}(T) := \{x \in M \mid T(x) = x\} \neq \emptyset$;

(iii) Brègman firmly quasi-nonexpansive iff $\tilde{D}_{\Psi}(p, T(x)) + \tilde{D}_{\Psi}(T(x), x) \leq \tilde{D}_{\Psi}(p, x) \forall (x, p) \in K \times \text{Fix}(T)$;

(iv) Brègman firmly nonexpansive iff $\tilde{D}_{\Psi}(T(x), T(y)) + \tilde{D}_{\Psi}(T(y), T(x)) + \tilde{D}_{\Psi}(T(x), x) + \tilde{D}_{\Psi}(T(y), y) \leq \tilde{D}_{\Psi}(T(x), y) + \tilde{D}_{\Psi}(T(y), x)$.

The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) hold [183]. For Legendre $\Psi$ and closed convex $M$ if $T$ satisfies (ii) then $\text{Fix}(T)$ is closed and convex [226, 183]. If $\Psi$ is Gâteaux differentiable on $\text{int}(\text{efd}(\Psi))$ while $C$ is closed and convex, then $\mathcal{P}^{\tilde{D}_{\Psi}}_{C}$ satisfies (iv) [34]. A restriction of a domain and codomain of Brègman nonexpansive operators to closed convex sets is considered explicitly in [226].

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24Except of the Brègman–Orlicz part of Proposition 4.6 (while we have introduced abstract nonassociative Orlicz spaces, their reflexivity under global $\Delta_{2}$ condition imposed on $T$ and $\Psi_{X}$ is currently only a conjecture).
Assuming that $X$ is a reflexive Banach space, let $\Psi : X \to ]-\infty, \infty]$ be Legendre, and let $Z \subseteq \text{int} (\text{efd}(\Psi))$. Given a set $U$, let $\tilde{\ell} : U \to Z$ be a bijection, hence $D_\Psi := D_\Psi(\tilde{\ell}(\cdot), \tilde{\ell}(\cdot))$ is a distance on $Z$. Consider a subset $\emptyset \neq W \subseteq U$, a bijection $\tilde{T} : \tilde{\ell}(W) \to Z$. Then $\tilde{T} := \tilde{\ell}^{-1} \circ T \circ \tilde{\ell} : W \to U$ is (in general) a nonlinear map, which we will call an $\ell$-operation. In principle, if $U$ is a convex set, then an $\ell$-operation on $U$ belongs to a larger class of nonlinear transmitters, introduced by Mielnik [188, 189] (see also [121]). However, our intent is to use $\ell$-embeddings for the purpose of the local operationalisation (see [161] for a wider discussion), and thus it is more natural to consider the condition that $Z$ is convex (or, equivalently, that $U$ is $\ell$-convex), so that $T$ can be seen as a (generic) nonlinear transmitter, while $\tilde{T}$ is its nonlinear local representation. The classes (i)-(iv) of Brègman nonexpansive functions on $X$ determine the corresponding classes of Brègman nonexpansive $\ell$-operations $\tilde{T}$ on $U$. This opens the doors for defining various categories $\text{Oper}_\ell(\ell, \Psi)$ of subsets of $U$ for which the corresponding collections of $\ell$-operations $\tilde{T}$ have well-behaved associative composition (with '$\cdot$' denoting a range of possible options). In this sense, the category $\text{Cvx}_\ell(\ell, \Psi)$ and its subcategories serve for us as toy models for a wider family of nonlinear categories of Brègman nonexpansive $\ell$-operations between $\ell$-convex $\ell$-closed sets.

If $U$ is taken to be subset of a positive cone generated by the base of the base norm space $Y$, then the above framework provides a basis to: (a) introduce relative Brègman entropies $(-D_\Psi)$ on $Y$ together with further infinitesimal information geometric structures (hessian manifolds equipped with a pair of flat and dually paired affine connections) associated with their Taylor expansion; (b) introduce inequivalent families of convex operational theories over $Y$, each associated with a specific choice of the pair $(\ell, \Psi)$; (c) study the nonlinear Brègman nonexpansive $\ell$-operations within the context of otherwise linear setting of $Y$ (e.g. to model eavesdropping operations introduced by nonlinear data processing). Regarding (b), let $\overline{\text{conv}}(Q)$ denote a weak closure of a convex hull of a subset $Q$ of $X$. For any $W \subseteq U$, if $K := \overline{\text{conv}}(\tilde{\ell}(W))$ is bounded (but not necessarily closed) in the norm of $X$, then it forms a base (normalised state space) of the base norm space, canonically associated to it by means of the Kadison–Semadeni theorem [146, 147, 236]. It may happen that $K$ is beyond the range of bijectivity of $\tilde{\ell}$, so it cannot be pulled back to an $\ell$-compact $\ell$-convex subset of $U$. In such case, it remains to analyse it directly in terms of weak topology of $X$, and to quantify its information content by means of $\tilde{D}_\Psi$.

Whether or not $\tilde{\ell}^{-1}$ can be applied, this leads to an interesting observation that a subset of positive states of an original convex operational theory (and, more generally, any nonlinear convex set theory in the Mielnik sense [188, 121, 189, 190]) may be considered as a generating set for a variety of inequivalent convex operational theories. From the fact that any set of experimentally obtained quantities is always finite, hence it determines an infinite set only up to a finite precision, there follows a question how to discriminate between different possible families of operational $(\ell, \Psi)$-theories. The choice of a particular linearising space $X$ together with the choice of $\ell$-embedding determines a context (frame) within which the finite sets of data are subjected to an idealised (optionally, compact) infinitary closure. Furthermore, the standard setting of a convex operational approach does not take into account the structure of preferred morphisms (admitting, in principle, all positive functions that are compatible with an a priori postulated state space). Yet, as stressed in [84, 190], the choice of the preferred type of dynamics should be considered as underlying the choice of the structure of the state space. The nonlinear behaviour of $\ell$-operations imposes nontrivial constraints on their composability, implying in turn the limitations on the allowed type of state spaces that are compatible with these morphisms. The choice of a geometric structure on $X$ (a particular Brègman function $\Psi$) allows to control this behaviour, as shown by various theorems on Brègman nonexpansive functions (in general), and our toy example $\text{Cvx}_\ell(\ell, \Psi)$ (in particular). Thus, the Brègman–Legendre divergences, together with $\ell$-embeddings and families of Brègman nonexpansive operations as morphisms, offers a shift of convex operational theories from the absolute terms of sets of
conditional probability evaluations to the relational terms of categories of statistical inference procedures, implying the shift of corresponding “operational semantics” from functional (e.g. as in [179]) to functorial. (See [162] for a further discussion.)

In what follows, we will show how Brègman nonexpansive \( \ell \)-operations induce corresponding nonlinear resource theories. Let \( Y \) be a base normed space with a base \( K \). Distilling the underlying structure of quantum [38, 37, 132, 90, 133, 69], postquantum [243], and abstract [111, 88, 87] resource theories, we define a **resource theory of states** as a triple \( (P,Q,R) \), where \( P \) is a submonoid of linear endomorphisms of \( K \), \( Q \) is a nonempty subset of \( K \) satisfying

\[
Q := \{ \phi \in K \mid \forall \psi \in K \ \exists p \in P \ p(\psi) = \phi \},
\]

while \( R \) is a set of maps \( r : K \rightarrow \mathbb{R}^+ \) satisfying

\[
(r \circ p)(\phi) \leq r(\phi) \ \forall \phi \in K.
\]

The elements of the triple \( (P,Q,R) \) are called, respectively, **free operations**, **free states** [132], and **resource monotones** [252]. In principle, one can define a (postquantum) **nonlinear resource theory** (of states) as a triple \( (P,Q,R) \) satisfying all of above conditions except of linearity of the elements of \( P \). As it turns out, the Brègman–Legendre distances provide a nontrivial class of such theories. Let \( X \) be a reflexive Banach space, and let \( \ell : K \rightarrow Z \subseteq X^+ \) be a bijection. Consider a family \( \mathcal{T} \) of \( \ell \)-operations on \( K \), determined by a monoid \( \mathcal{T} \) of functions on \( X \) and satisfying \( K_{\mathcal{T}} := \{ \phi \in K \mid \forall \psi \in K \ \exists f \in \mathcal{T} \ f(\psi) = \phi \} \neq \emptyset \). Let \( D_\Psi \) be a Brègman–Legendre distance on \( K \), determined by a Brègman functional \( D_\Psi \) on \( X \) and the map \( \ell \), according to (80) with \( U = K \). If each \( T \in \mathcal{T} \) is Brègman completely nonexpansive with respect to \( D_\Psi \) and \( K_{\mathcal{T}} \) is \( \ell \)-closed \( \ell \)-convex (resp., if each \( T \in \mathcal{T} \) is Brègman quasi-nonexpansive with respect to \( D_\Psi \) and \( \bigcap_{T \in \mathcal{T}} \text{Fix}(T) =: \text{Fix}(\mathcal{T}) \neq \emptyset \)) then the triple

\[
\text{BNRT}_{(i)}(\mathcal{T}) := \left( \mathcal{T}, K_{\mathcal{T}}, \left\{ \inf_{\phi \in K_{\mathcal{T}}} \{ D_\Psi(\phi, \cdot) \} \right\} \right)
\]

(resp., \( \text{BNRT}_{(ii)}(\mathcal{T}) := \left( \mathcal{T}, K_{\mathcal{T}}, \bigcup_{\phi \in K_{\mathcal{T}}} D_\Psi(\phi, \cdot) \right) \)) is a nonlinear resource theory.

The study of sets \( \bigcap_{T \in \mathcal{T}} \text{Fix}(T) \) is one of the important topics of the research on Brègman monotone operations, and various results on its properties are available, see e.g. [34, 184] and references therein. Dependently on the choice of the family \( \mathcal{T} \), one can construct more or less restricted Brègman nonexpansive resource theories. In particular, for any fixed choice of \( L \in \text{Ob}(\text{Cvx}(\ell, \Psi)) \), let \( \mathcal{T} \) be given by the family of all \( D_\Psi \)-projections onto \( \ell \)-closed \( \ell \)-convex sets containing \( L \). It satisfies the definition of \( \text{BNRT}_{(ii)}(\mathcal{T}) \) with \( K_{\mathcal{T}} = L \) and the set of resource monotones given by \( R = \bigcup_{\phi \in L} \{ D_\Psi(\phi, \cdot) \} \). We will denote this resource theory as \( \text{BNRT}_L(\mathcal{P}^{D_\Psi}) \).

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25While some resource theories of states may be derived by using a priori choice of a tensor product on \( Y \), we view the tensor product structure as an additional property, which should be, in principle, considered after the particular triple \( (P,Q,R) \) is constructed. Even under the restriction to the regime of linear operations, the availability of a wide range of possible tensor norms [205, 261] that can be associated with a given base norm space \( Y \) calls, in general, for ex post analysis and justification of the possibility of choosing a particular tensor product structure \( S \) that is compatible with already constructed resource theory, complementing it into a symmetrically monoidal resource theory \( (S,P,Q,R) \). Thus, our approach here differs from the view of [71, 69], while being in line with the perspective expressed explicitly in [88, 87] and present implicitly in [243]. See [212] for an explicit example of a physically sound situation when the compatible tensor product structure is not available, and [191] for a discussion of tensor structure in the context of nonlinear postquantum theories.
Appendix 1: Noncommutative $L_p(\mathcal{N})$ spaces

In this work we use the terminology and notation which is in agreement with the exposition of noncommutative integration theory given in [160]. We refer to that text for the detailed discussion and references. Here we will just recall briefly few key notions and facts used in the current paper.

A $W^*$-algebra is defined as such $C^*$-algebra $\mathcal{N}$ that has a Banach predual $\mathcal{N}_*$ [233]. If a predual of $C^*$-algebra exists then it is unique. We will denote $\mathcal{N}_0 := \{\phi \in \mathcal{N}_* \mid \phi(x^*x) \geq 0 \ \forall \ x \in \mathcal{N}\}$, $\mathcal{N}_+ := \{\phi \in \mathcal{N}_0 \mid \phi(x^*x) = 0 \ \Rightarrow \ x = 0 \ \forall \ x \in \mathcal{N}\}$, $\mathcal{N}_a := \{\phi \in \mathcal{N}_+ \mid \|\phi\| = 1\}$, $\mathcal{N}_{sa} := \{x \in \mathcal{N} \mid x^* = x\}$, $\mathcal{N}^+ := \{x \in \mathcal{N} \mid \exists y \in \mathcal{N} \ x = y^*y\}$. A space of all semi-finite normal weights on $\mathcal{N}$ is denoted $W_0(\mathcal{N})$. We will call a boolean algebra $\mathcal{A}$ an mcb-algebra iff it is Dedekind–MacNeille complete and allows for a semi-finite strictly positive countably additive measure [234]. Every commutative $W^*$-algebra $\mathcal{N}$ is isometrically isomorphic and $*$-isomorphic to $L_\infty(\mathcal{A})$ space, where $\mathcal{A}$ is an mcb-algebra constructed as the lattice of projections of $\mathcal{N}$ (in such case $L_\infty(\mathcal{A})_* \cong L_1(\mathcal{A})$ is an isometric isomorphism and a Riesz isomorphism). More generally, there holds an equivalence between the categories of: commutative $W^*$-algebras with $*$-homomorphisms, mcb-algebras with order continuous boolean homomorphisms, and localisable measure spaces with complete morphisms. Let $\tau \in W_0(\mathcal{N})$ be a semi-finite trace. A closed densely defined linear operator $x : \text{dom}(x) \rightarrow \mathcal{H}$, with $\text{dom}(x) \subseteq \mathcal{H}$, is called $\tau$-measurable iff $\exists \lambda > 0 \ \tau(P_{[\lambda, +\infty]}x) < \infty$, where $P_{[\lambda, +\infty]}$ is a spectral measure of $|x|$. The space of all $\tau$-measurable operators affiliated with the GNS representation $\pi_\tau(\mathcal{N})$ will be denoted by $\mathcal{M}(\mathcal{N}, \tau)$. For $x, y \in \mathcal{M}(\mathcal{N}, \tau)$ the closures of the algebraic sum $x + y$ and algebraic product $xy$ (denoted with the abuse of notation by the same symbol) belong to $\mathcal{M}(\mathcal{N}, \tau)$.

Falcone and Takesaki [103] have constructed a family of noncommutative $L_p(\mathcal{N})$ spaces that are canonically associated to every $W^*$-algebra. The key feature is a construction of a semi-finite von Neumann algebra $\tilde{\mathcal{N}}$ and a faithful normal semi-finite trace $\tilde{\tau} : \tilde{\mathcal{N}} \rightarrow [0, \infty]$ that are uniquely defined for any $W^*$-algebra $\mathcal{N}$, with no dependence of an additional weight or state on $\mathcal{N}$. Using these objects, a topological $*$-algebra $\mathcal{M}(\tilde{\mathcal{N}}, \tilde{\tau})$ of $\tilde{\tau}$-measurable operators is defined. It is equipped with a grade function $\text{grd} : \mathcal{M}(\tilde{\mathcal{N}}, \tilde{\tau}) \rightarrow \mathbb{C}$ satisfying $\text{grd}(x^*) = (\text{grd}(x))^*$, $\text{grd}(|x|) = \text{re}(\text{grd}(x)) = \frac{1}{2}(\text{grd}(x) + \text{grd}(x)^*)$, $\text{grd}(xy) = \text{grd}(x) + \text{grd}(y)$, $\text{re}(\text{grd}(x)) \geq 0 \Rightarrow |x|^{1/\text{re}(\text{grd}(x))} \in \mathcal{N}_+$, where $xy$ is the closure of $xy$. The canonical integral $f : \mathcal{M}(\tilde{\mathcal{N}}, \tilde{\tau}) \rightarrow \mathbb{C}$ satisfies $\int : L_1(\mathcal{N}) \ni \phi \mapsto \int \phi = \phi(\|\|) \in \mathbb{C}$. The spaces $L_p(\mathcal{N})$ for $p \in \mathbb{C} \setminus \{0\}$ are defined as the spaces of $\tilde{\tau}$-measurable operators of grade $1/p$ affiliated with $\tilde{\mathcal{N}}$. The norms $\|\|_p$ for $p \in \mathbb{C}$ and $\text{re}(p) \geq 1$ read

$$\|\|_p : L_p(\mathcal{N}) \ni x \mapsto \|x\|_p := \left(\int |x|^{\text{re}(p)}\right)^{1/\text{re}(p)} \in \mathbb{R}^+,$$

and turn $L_p(\mathcal{N})$ into Banach spaces, with their Banach duals given by $L_q(\mathcal{N})$ spaces with $\frac{1}{p} + \frac{1}{q} = 1$. The space $L_\infty(\mathcal{N})$ is defined as $\mathcal{N}$, and an isometric isomorphism $\mathcal{N}_* \cong L_1(\mathcal{N})$ holds. The Banach space duality between $L_p(\mathcal{N})$ and $L_q(\mathcal{N})$ for $1/p + 1/q = 1$ and $p \in \{\lambda \in \mathbb{C} \mid \text{re}(\lambda) > 0\}$ reads

$$L_p(\mathcal{N}) \times L_q(\mathcal{N}) \ni (x, y) \mapsto [x, y]_{\mathcal{N}} := \int xy \in \mathbb{C}.$$
The space $L_2(N)$ is a Hilbert space with respect to the inner product

$$L_2(N) \times L_2(N) \ni (x_1, x_2) \mapsto \langle x_1, x_2 \rangle_{L_2(N)} := \int x_1^* x_2 \, d\mathcal{C}. \quad (162)$$

If $\{x_i\}_{i=1}^n \subseteq \mathcal{M}(\mathcal{N}, \mathfrak{T})$, $\sum_{i=1}^n \text{grd}(x_i) = r \leq 1$ and $\text{re}(\text{grd}(x_i)) \geq 0 \forall i \in \{1, \ldots, n\}$, then the noncommutative analogue of the Rogers–Hölder inequality holds [157],

$$|x_1 \cdots x_n|_{1/r} \leq \|x_1\|_{1/\text{re}(\text{grd}(x_1))} \cdots \|x_n\|_{1/\text{re}(\text{grd}(x_n))}. \quad (163)$$

The stronger condition $\sum_{i=1}^n \text{grd}(x_i) = 1$ implies that $x_1 \cdots x_n \in L_1(N)$, and in such case

$$\int x_1 \cdots x_n = \int x_1 x_2 \cdots x_{n-1}. \quad (164)$$

This suggests to use the notation $y = x \phi^{\text{grd}(y)} = x \phi^\gamma$ with $(x, \phi) \in N \times W_0(N)$ for a generic element $y$ of the space $L_{1/\gamma}(N)$ with $\text{re}(\gamma) \in [0, 1]$; with boundary cases given by $x \in L_{\infty}(N) = N$ and $\phi \in L_1(N) \cong N$. For the negative powers of weights, $\phi^{-p}$ for $p > 0$, there are no corresponding $L_{-p}(N)$ spaces. However, as shown in [237], the right and left multiplications, $\mathcal{R}(\phi^{-p})$ and $\mathcal{L}(\phi^{-p})$, for $\phi \in W_0(N)$ are well defined$^{26}$ and satisfy $\mathcal{R}(\phi^{-p}) = (\mathcal{R}(\phi^p))^{-1}$, $\mathcal{L}(\phi^{-p}) = (\mathcal{L}(\phi^p))^{-1}$, $\mathcal{R}(\phi^{-p})\mathcal{R}(\phi^p) = \mathbb{I}$, as well as

$$\Delta_{\phi, \psi}^{1/p} = \mathcal{R}(\phi^{-1/p})\mathcal{L}(\psi^{1/p}), \quad (165)$$

where $\psi \in W(N)$. This gives

$$\int \psi^\gamma \phi^{1-\gamma} = \int \psi^\gamma \phi^{-\gamma} \phi = \int (\mathcal{R}(\phi^{-\gamma})\mathcal{L}(\psi^\gamma)\mathbb{I}) \phi = \left(\xi_\pi(\phi), \Delta_{\psi, \phi}^\gamma, \xi_\pi(\phi)\right)_{\mathcal{H}} \quad (166)$$

for any standard representation $(\mathcal{H}, \pi, J, \mathcal{H}^2)$. The equation (166) holds also when $\phi, \psi \in N^+_s$ and $\psi \preceq \phi$, because in such case $\phi$ is faithful on $N_{\text{supp}(\phi)}$ and this algebra contains the support of $\phi$.

**Appendix 2: Nonassociative $L_p(A, \tau)$ spaces**

The integration theory on nonassociative JBW-algebras is little known (and not covered in [160]), so we will give a brief account of it below, focusing only on the properties required for the purposes of our paper.

A **JB-algebra** is defined [9, 241] as a real Jordan algebra [142, 143, 144] $A$ (with respect to a nonassociative product $\circ : A \times A \to A$) that is a real Banach space with a norm $\|\cdot\|$ satisfying $\|x^2\| = |x|^2$ and $\|x^2 - y^2\| \leq \max\{|x|^2, |y|^2\}$ $\forall x, y \in A$ (if $A$ contains a unit $\mathbb{I}$, then this inequality is equivalent with $\|x^2\| \leq \|x^2 + y^2\|$), where $x^2 := x \circ x$. If a JB-algebra $A$ is a Banach dual of some Banach space (denoted $A_*$), then it is called a **JB-algebra**. Every JB-algebra contains a (unique) unit [98]. The set $A^+ := \{x^2 \mid x \in A\}$ is a closed convex cone, generating $A$ (i.e., $A = A^+ - A^+$). If, given $x \in A^+$, $\exists! y \in A^+$ such that $y^2 = x$, then one denotes $y \equiv \sqrt{x}$. For any $x, y \in A$, $|x| := \sqrt{x^2} \in A^+$, $L_{xy} := x \circ y$, $U_x := 2(L_x)^2 - L_x^2$. If $A$ is a JB-algebra, then each $x \in A$ has a unique decomposition

$^{26}$More precisely, let the adjective ‘strong’ refer to the topological closure of some algebraic operation in $\mathcal{M}(N, \mathfrak{T})$. For any $\lambda \geq 0$, $t > 0$, $\phi \in N^*_t$, the map $\mathcal{R}(\phi^\gamma) : L_{1/\gamma}(N) \to L_{1/\gamma(1+t)}(N)$, defined as a strong composition with $\phi^\gamma$ from right, is everywhere defined, bounded, and injective with dense range. Moreover, the maps $\mathcal{R}(\phi^{-1})$ and $\mathcal{R}(\phi^t)$ have the same range and agree (from this it follows that they are equal). The map $\mathcal{R}(\phi^{-1})$ is closed, and is understood as a strong product, defined only when the closure is $\mathfrak{T}$-measurable. The same holds for $\mathcal{R}$ replaced by $\mathcal{L}$. If $\phi \in N^*_t$ is replaced by $\phi \in W_0(N)$, then all those properties hold except that $\mathcal{R}(\phi^t)$ and $\mathcal{L}(\phi^t)$ are no longer everywhere defined or bounded.
$x = s|x|$, where $s^2 = \mathbb{I}$. For any C$^*$-algebra (resp., W$^*$-algebra) $\mathcal{N}$, the Banach space $\mathcal{N}^{sa}$ is a JB$^*$-algebra (resp. JBW$^*$-algebra) with respect to $L_{xy} := \frac{1}{2}(xy - yx)$ \forall x, y \in \mathcal{N}^{sa}$ (the case for $\mathcal{N} = \mathcal{B}(\mathcal{H})$ was first considered in [254]). An example of a JB$^*$-algebra that is not a self-adjoint part of any C$^*$-algebra was constructed in [145]. The notions of a projection in JB$^*$-algebra and of a support of an element $\phi \in A^+_1$ for a JBW$^*$-algebra $A$ are the same as in C$^*$-W$^*$-algebraic case.

A trace on a JB$^*$-algebra $A$ is a map $\tau : A^+ \to [0, \infty]$ such that [246, 22] $\tau(x + y) = \tau(x) + \tau(y)$, $\tau(\lambda x) = \lambda \tau(x)$ (with $0 \cdot \infty \equiv 0$), $\tau(U_s x) = \tau(x)$ \forall x, y \in A^+$ \forall $\lambda \geq 0$ \forall $s \in A$ such that $s^2 = \mathbb{I}$. Given a trace $\tau$ on $A$, we define $A^+_\tau := \{x \in A^+ | \tau(x) < \infty\}$, $A_\tau := A^+_\tau - A^+_\tau$, and call $\tau$: faithful if $\tau(x) = 0 \Rightarrow x = 0$; semi-finite iff $\tau(x) = \sup\{\tau(y) | y \in A^+, |y| \leq x\}$ (or, equivalently, if $A_\tau$ is weakly dense in $A$); normal iff $A$ is a JBW$^*$-algebra and $\tau(x) \to \tau(x)$ for every increasing net $x_\alpha \to x$ with $x, x_\alpha \in A^+$. Equivalently, a trace on a JB$^*$-algebra $A$ can be defined as a map $\tau : A^+ \to [0, \infty]$ such that $A_\tau$ is an ideal of $A$ and $\tau(x \circ (y \circ z)) = \tau((x \circ y) \circ z)$ \forall $x \in A_{\tau}$ \forall $y, z \in A$ [136, 137, 151] (see [214] for a discussion of alternative equivalent definitions). Every faithful normal semi-finite trace $\tau$ on a JBW$^*$-algebra $A$ can be extended by linearity from $A^+ \to A_\tau$. In what follows, we will consider only such pairs $(A, \tau)$. Given $x \in A$, $p \in [0, \infty]$, the maps $\|x\|_p$ defined by $\|x\|_p := \tau(|x|^p)^{1/p} \in [0, \infty]$ and $\|x\|_\infty := \|x\|$ are norms on $A_\tau$. The nonassociative $L_p$ space is defined as a Banach space $L_p(A, \tau) := \overline{A^+_\tau}^{\|\cdot\|_p}$ for $p \in [1, \infty]$ and $L_\infty(A) := A [1, 2, 134, 135]$. $L_2(A, \tau)$ is self-dual with respect to Banach space duality, and hence it is a Hilbert space [22, 39, 134]. The map $A_{\tau} \ni x \mapsto \tau(x \cdot)$ extends to an isometric order- isomorphism $L_1(A, \tau) \cong A_{\tau} [22, 151, 134]$, which implies $(L_1(\gamma(A, \tau))^* \cong L_1(1(\gamma(A, \tau))$ \forall $\gamma \in [0,1] [2, 134]$. The bijection $A^+_\tau \ni \tau(h_{\phi} \cdot) \in L_1(A, \tau)^+$ determines a nonassociative Radon–Nikodým quotient $h_{\phi}$ of $\phi$ with respect to $\tau$ [151, 2, 134, 23]. Furthermore, for $p \in [1, \infty]$, $L_p(A, \tau)$ is uniformly convex and Gâteaux differentiable, with norms $\|\cdot\|_p$ Fréchet differentiable everywhere except at 0 [135]. If $\tau_1$ and $\tau_2$ are two faithful normal semi-finite traces on a JBW$^*$-algebra $A$, then $L_p(A, \tau_1)$ is isometrically isomorphic to $L_p(A, \tau_2)$ [24]. If $A = \mathcal{N}^{sa}$ for some semi-finite $W^*$-algebra $\mathcal{N}$, with corresponding traces $\tau_A$ on $A$ and $\tau_{\mathcal{N}}$ on $\mathcal{N}$, then the nonassociative $L_p(A, \tau_A)$ space is isometrically isomorphic to a noncommutative $L_p(\mathcal{N}, \tau_{\mathcal{N}})$ space [24]. The class of nonassociative $L_p$ spaces is larger (from the Banach space theory perspective) than the class of noncommutative $L_p$ spaces (which in turn is larger than the class of commutative $L_p$ spaces [150]): for $p \neq 2$, given $A_1 = \mathcal{N}_1^{sa}$ and $A_2 = \mathcal{N}_2^{sa}$ for $W^*$-algebras $\mathcal{N}_1$ and $\mathcal{N}_2$, $L_p(A_1, \tau)$ is isometrically isomorphic to $L_p(A_2, \tau)$ iff $A_1$ is normal Jordan isomorphic to $A_2$ [24].

References


27The case of $p \in \{1, 2\}$ was considered earlier in [39] and in [22] for finite $\tau$; $p = 1$ case with $\tau$ replaced by a normal state was independently constructed in [102]; a generalisation to $\tau$ given by a weight followed in [151, 2, 23] for $p = 1$ and in [247] for $p \in \{1, 2\}$. The concept of a weight on a JB$^*$-algebra was introduced in [151, 134] and further studied in [247, 3].


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Quantum resource theories

Individual ergodic theorems in noncommutative Orlicz spaces


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