Quantum histories thermodynamics

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Abstract

In this work we develop the new general approach to quantum theory, based on the algebraic QFT, Tomita–Takesaki modular theory and Isham–Linden–Sevridou continuous histories approach. We show that the temporal information about quantum system is described by the Kullback–Leibler entropy, what implies its relational, contextual and Bayesian character. We show also that the histories quantization of action functional plays the same role as Helmholtz energy in thermodynamics: it describes the behaviour of the system as superposition of internal dynamics and the interaction with an external observer (environment). We derive the quantum geometric phase, classical action functional and Lagrangean path-integral as the special cases of our approach. As an application, we use it to analyse the black hole thermodynamics and the infinitesimal Lorentz symmetry.

1 Introduction

In this work we present the new perspective on the quantum theory, which arises from the consideration of the quantum histories action operator as an analogue of the Helmholtz energy. In order to explain briefly what does it mean, let us recall first, that Helmholtz energy equation provides the description of the behaviour of the physical system which is in the interaction with environment [17]. On the other hand, the ordinary quantum mechanical or quantum field theoretical description assumes that systems are isolated, and no measurements are performed (or their impact may be neglected), so the behaviour of the quantum system is described perfectly by the unitary evolution (what actually means that quantum behaviour is concerned to be Markovian). However, this assumption is simply false. Every information that we have about quantum systems is obtained through the interaction of observer with these systems. The act of observation of quantum system breaks the unitarity of evolution, and unfortunately is not incorporated into the description of quantum systems provided by ordinary QM or QFT settings. Quantum histories programme [23], [31], [21], [33] was developed in order to describe the subsequent measurements of quantum system (performed by an external observer) on the equal footing with the internal unitary evolution of this system. This setting leads to a possibility of consideration of quantum systems as systems in interaction with the environment given by the observer. In thermodynamics such systems are modelled by the Helmholtz energy equation. We show that quantum histories action operator [59] (self-adjoint operator representing the histories-quantized classical action functional) provides an exact analogue of the Helmholtz energy, what in follows leads us to derivation of three laws of quantum histories thermodynamics, with the role of temperature played by the quantum Liouville operator. Introducing the quantum histories thermodynamics analogue of algebraic equilibrium Kubo–Martin–Schwinger state [27], we show that its Tomita–Takesaki [70], [69] modular automorphism describes the quantum geometric
phase [13], so it reflects the purely kinematical (non-Hamiltonian) observable properties of the quantum system. This leads us to the new description of the interaction in quantum field theory, provided by the integration of the Radon–Nikodym derivatives [28] of algebraic histories equilibrium states over the germs of these states. We recover the quantum geometric phase, classical action functional and the path-integral setting as the appropriate limits of our approach. By analysis of the Kullback–Leibler entropy [43] of the (germs of) algebraic states we show, that quantum histories probabilities are Bayesian, and that the classical action functional encodes the information about the Bayesian updating of the knowledge about the system which internal dynamics is unitary.

In the next section we recall and analyse the essential assumptions and elements of the structure of ordinary quantum theory: the character of the quantum measurement, the algebraic setting for quantum theory and the quantum geometric phase. In the third section we present the Isham–Linden–Savvidou–Anastopoulos histories projection operator approach to quantum histories [30] [32] [33] [34] [39] [60] [5] [3] [4]. In the fourth section we develop the quantum histories thermodynamics, which leads us to a new description of behaviour of quantum systems, and to a new perspective on infinitesimal Lorentz symmetry and Fulling–Unruh effect. Finally, we draw certain conclusions about black hole thermodynamics.

2 Ordinary quantum theory

2.1 Quantum description of the measurement

In every physical theory the events of measurement are represented by some statements. In quantum theory these are the Boolean-valued statements \( A \in \Delta \), which say that the measured value of the concrete quantity \( A \) lays in the subset \( \Delta \) of its possible values. Such description of the measurement reflects the concrete measurement techniques, based on filters and click-counters, which are designed to give the 'yes or no' outputs [46]. The quantum description of the measurements is provided by the probability assignment to statements of type \( A \in \Delta \). In order to give the quantitative meaning to this framework, an ordinary quantum theory introduces a certain mathematical structures, based on two constituents: the complex linear partial differential \( \text{Schrödinger equation} \)

\[
\frac{\partial}{\partial s} \psi = -iH\psi, \quad (1)
\]

and the inner product \( (\cdot,\cdot) : V^* \times V \to \mathbb{C} \) on the vector space \( V \) of solutions of (1), defined via the probabilistic measure\(^1\) \( \int dx \) on \( V \) as

\[
(\psi, \phi) := \int dx \bar{\psi}(x)\phi(x). \quad (2)
\]

The superposition principle is a direct consequence of linearity of (1). The inner product (2) turns the vector space \( V \) into unitary space and, after completion of this space in the norm generated by (2), into Hilbert space \( \mathcal{H} \). The concept of the Boolean-valued statement \( A \in \Delta \) is then mathematically implemented by the \textit{projection operators} \( P_{A,\Delta} \) acting on the elements of the Hilbert space. These operators are defined as such that satisfy \( P_{A,\Delta}^2 = P_{A,\Delta} \) and \( P_{A,\Delta}^* = P_{A,\Delta} \), what implies that they have only two eigenvalues: \( \{0,1\} \), corresponding to the spectrum \( \{\text{false}, \text{true}\} \) of the possible truth values of the statement \( A \in \Delta \). The probability of \( A \in \Delta \) is then defined, using the form (2), as

\[
p_{A\in\Delta}(\psi) := (\psi, P_{A,\Delta}\psi). \quad (3)
\]

\(^1\) The probabilistic measure is such measure that satisfies Kolmogorov axioms. For \( \dim V < \infty \) it is a Lebesgue additive measure. In the case when \( \dim V = \infty \) the Lebesgue measure cannot be properly defined and another measure (Gaussian) must be used.
This way the quantum description of measurement obtains the probabilistic character. Note that this does not imply that our measurements of observable physical quantities are probabilistic. The observable physical quantities are represented by the self-adjoint \((A^* = A)\) linear operators on the Hilbert space. They are selected among all operators on this space, because linearity implies preserving the superposition, and self-adjointness implies having only real eigenvalues. Due to spectral theorem, linear self-adjoint operators can be described in terms of the ‘sum’ (spectral integral of) projection operators corresponding to the elements of their spectrum. This way the measurements of \(P_{A,\Delta}\) enable the reconstruction of observable \(A\) represented by the linear self-adjoint operator with certain spectral projection.

Note that while on the algebraic level of observables and projection operators ‘there is no probability’, it is introduced by the inner product. In order to speak about a value of \(A \in \Delta\) of an observable \(A\) in a given measurement, that is, to theoretically estimate the concrete (measured) spectral projector, we have to specify (assume) an element of \(\mathcal{H}\) for which this operator is evaluated. This specification introduces the probabilistic character of the theoretically obtained results\(^2\). Moreover, the definition of probability of \(A \in \Delta\) given by (3) implies that the measurement breaks the continuous unitary evolution of \(\psi\) provided by the Schrödinger equation (1) and replaces \(\psi\) by the new element of the Hilbert space, given by

\[
\psi' = \frac{P_{A,\Delta}\psi}{(\psi, P_{A,\Delta}\psi)^{1/2}},
\]

where \(P_{A,\Delta}\) is a spectral projector of \(A\) on the subset \(\Delta\) of its spectrum which corresponds to the result obtained in the experiment. This means that in the quantum description of the system which is subjected to measurements there are two different time evolutions: unitary continuous (1) and nonunitary discrete (4). This seems to be quite mysterious on the first sight, but one can shed some light on this fact, considering the similarity between quantum mechanics and statistical classical mechanics. The latter theory is also established via introduction of probabilistic measure on the phase space of solutions of equations of motion and definition of an expectation values of observable quantities in terms of this measure. However, in quantum theory, unlike in statistical classical mechanics, the measurement of the observable of the system changes this system, what is reflected in the equation (4). If we interpret then the observables and projection operators as ‘probability-free’ properties of the system, and the inner product and expectation values as the ‘probability-involved’ properties of the measurement, then we have to consider these two time evolutions on the equal footing. The full description of the behaviour of the measured system depends then not only on its Hamiltonian dynamics, but also on the ‘observer’s’ inner product.

One should note, that the above definition of an observable in quantum physics implies, that observables do not refer to any ‘things’ or ‘physical objects’, but to certain properties of events. Haag [28] states it clearly saying that “position is not an attribute of an electron, it is an attribute of the ‘event’ i.e. of the interaction process between the electron and appropriately chosen measuring instrument. This means that the ontology of quantum theory is given by the events, while the observables, elements of \(\mathcal{H}\) and inner product are the epistemological entities providing the interface which serves to describe the events. The ‘most epistemic’ part of this interface is the inner product, because it provides the probabilistic estimate of the concrete quantities measured in experiment. On the other hand observables serve as a qualitative\(^3\) link between epistemology and ontology and are introduced to provide an idealization of the ‘inner’ properties of events (these which are independent from observer).

\(^2\)This state of affairs is reflected also on the algebraic level: an abstract \(C^\ast\)-algebra, independent from any concrete Hilbert space, has not enough projectors, and one cannot provide the spectral theorem for it. The latter has to be stated on the level of von Neumann algebra, which is closely tied with the concrete Hilbert space.

\(^3\)Note that due to the Kochen-Specker theorem [60], one cannot insist that quantum observables have quantitative character, because they generally fail to satisfy FUNC condition: \(f(v(A)) = v(f(A))\), for arbitrary von Neumann algebra \(A\), function \(f\), and evaluation \(v\colon A \to \mathbb{R}\).
2.2 Algebras of observables

The algebraic approach to quantum theory has shown that the essential elements of this theory are independent from the Hilbert space structure, and the latter may be reconstructed from these elements. The point of the departure of this approach is consideration of the abstract $C^*$-algebras of observables and positive normalized functionals on these algebras, called algebraic states, in the role of two fundamental entities of quantum theory [61], [62]. The guiding intention is to refer the abstract $C^*$-algebra with the possible measurable properties of the system under consideration, and to refer states with the concrete experimental arrangements (setups) of the measurement. This way the algebraic formulation strengthens the previous distinction between the observables and inner product. An abstract $C^*$-algebra is an algebra over the field $\mathbb{C}$ with an involution operation $^*$ defined as $(AB)^* := B^*A^*$, $(A+B)^* := A^* + B^*$, $(A^*)^* := A$. $(\lambda A)^* := \lambda A^*$ ($\lambda \in \mathbb{C}$) means complex conjugation of $\lambda \in \mathbb{C}$), equipped with a norm $|| \cdot ||$, which satisfies $||AB|| \leq ||A|| ||B||$ and $||A^*A|| = ||A||^2$. These both structures (involution and norm) reflect the genuine properties of the operator algebra $\mathcal{B}(\mathcal{H})$ of linear bounded observables on any Hilbert space $\mathcal{H}$ which serves as an example of the concrete $C^*$-algebra. On the other hand, the positivity $\omega(A^*A) \geq 0$ and norm $\omega(1) = 1$ of the algebraic state $\omega : A \rightarrow \mathbb{C}$ reflect the properties of the inner product $(\cdot, \cdot)$ on $\mathcal{H}$ (by $I$ we will always denote the unit element of a $C^*$-algebra). Every $C^*$-algebra and algebraic state enable the reconstruction of the Hilbert space through the Gelfand–Naimark–Segal (GNS) construction: for $C^*$-algebra $A$, elements $A, B \in A$ and algebraic state $\omega : A \rightarrow \mathbb{C}$, one defines the scalar product $(\cdot, \cdot)_\omega$ on $A$ as

$$\langle A, B \rangle_\omega := \omega(B^*A),$$

and the Gelfand ideal $I := \{ A \in A : \langle A, A \rangle_\omega = 0 \}$. The Hilbert space $\mathcal{H}_\omega$ is obtained by the completion of $A/I$ in the topology generated by $(\cdot, \cdot)_\omega$. The dense subspaces of $\mathcal{H}_\omega$ are then the equivalence classes of elements of $A$ modulo the ideal $I$. The state $\omega$ defines a representation $\pi_\omega$ of an abstract $C^*$-algebra $A$ in the space $\mathcal{B}(\mathcal{H}_\omega)$ of all bounded linear operators on $\mathcal{H}_\omega$ by the morphism

$$\eta_\omega : A \ni A \longmapsto [A]_\omega \in A/I$$

and

$$\pi_\omega(A) : \eta_\omega(B) \longmapsto \eta_\omega(AB).$$

This way the algebraic state $\omega$ leads to the reconstruction of the concrete Hilbert space from an abstract $C^*$-algebra and to representation of this algebra. Moreover, the algebraic state $\omega$ can be represented as the state $\Omega := \eta_\omega(1)$ in the Hilbert space $\mathcal{H}_\omega$. The vector $\Omega$ is cyclic, e.g. the set $\{ \bigcup_{A \in A} \pi_\omega(A) \Omega \}$ is dense in $\mathcal{H}_\omega$, and it satisfies the following property:

$$\forall A \in A \quad \omega(A) = \langle \Omega, \pi_\omega(A) \Omega \rangle_\omega.$$

This means that every physical function and expectation value is generated solely by a certain observable $A \in A$ and an algebraic functional $\omega$ on $A$. The cyclic character of $\Omega$ enables then the approximation of the state $\omega_\psi(A) = \langle \psi, \pi_\omega(A) \psi \rangle_\omega$ by $\omega(B^*AB)$ for some $B$, because $\psi$ can be approximated by $\pi_\omega(B)\Omega$. In such case we will also use the notation $\varphi = \eta_\omega(B)$.

The restriction of the support of functions which are used to smear the operator valued distributions to the subsets $O$ of the base manifold $\mathcal{M}$ leads to consideration of the net of algebras $A(O)$ and states $\omega_O$ assigned to the structure of the region $O$ as primary object of interest of algebraic QFT [26]. In such situation the Reeh–Schlieder theorem [37] says that, for a given $A(O)$, $\omega_O$, Hilbert space $\mathcal{H}$ and a vacuum vector $\Omega \in \mathcal{H}$, the set of vectors $A(O)\Omega := \bigcup_{A \in A(O)} \pi_{\omega_O}(A)\Omega$ is dense in $\mathcal{H}$, hence $\Omega$ is cyclic and one can write

$$\forall \psi \in \mathcal{H} \quad \forall \epsilon > 0 \quad \exists A \in A(O) \quad \langle \psi - A\Omega, \psi - A\Omega \rangle_\omega < \epsilon.$$
can think about this result as an indication of a presence of non-zero nonlocal correlations, described as \( \omega(AB) - \omega(A)\omega(B) \neq 0 \) for \( A \in \mathcal{A}(O) \) and \( B \in \mathcal{B}(O') \), where \( O \) and \( O' \) are arbitrary distant from each other (for macroscopic distances, like Earth–Moon, these correlations appear to be very small \[72\]). This way the Reeh–Schlieder theorem shows the difference between the algebraic perspective on quantum theory, and the Hilbert-space-based one: the former enables separation between local and nonlocal aspects of the theory, as well as separation between the properties of the system and properties of an observer, while the latter merges all in the one frame. The separation of the structure of theory into algebras and algebraic states allows then to provide more precise analysis of certain physical situations. For example, if we consider a restriction of an algebra \( \mathcal{A}(M) \) and state \( \omega_M \), defined over Minkowski space-time manifold \( M \), to algebra \( \mathcal{A}(O) \) and state \( \omega_O \) over some submanifold \( O \subset M \), then, due to Reeh–Schlieder theorem, \( \omega_O \) will be generally a mixed state. Will show later in the section 4.7 that this leads straightly to the Fulling–Unruh effect \[19\], \[71\].

The algebraic approach to quantum theory is not only a certain mathematical reformulation of the existing Hilbert space formalism. Its importance is grounded at least by two aspects of the ordinary quantum field theory: the unitary inequivalent representations of algebra of observables and the Borchers class of quantum fields. In order to understand the second problem, note, that the significant part of the experimental confirmations of QFT are obtained through the evaluation of the expectation values of the so-called \( S \)-matrix and relating it with the data obtained in the collision experiments \[74\]. Borchers \[14\] has shown that two different quantum fields (operator valued distributions \[73\]) \( \Phi \) and \( \Phi' \) satisfying

\[
[\Psi(x), \Phi(y)] = 0 \quad \text{for space-like } x - y
\]

lead to the same \( S \)-matrix (this way the condition (5) defines a class of fields).\(^4\) So, the quantum field cannot be considered as direct description of the measurement, because no physical consequence of the theory depends on the choice of a specific system [of fields] within a Borchers class \[28\]. Hence, the algebras of observables generated by the polynomials of quantum fields appear to have more direct relation to experimental reality than the quantum fields itself. The similar conclusion may be drawn from the existence of the unitary inequivalent representations. Two quantum descriptions \( \{A_i\} \subset \mathcal{B}(\mathcal{H}) \) and \( \{A'_i\} \subset \mathcal{B}(\mathcal{H}') \) of a given system are considered to be equivalent only if the corresponding operators can be translated into each other by some unitary transformation \( U : \mathcal{H} \rightarrow \mathcal{H}' \) such that \( U^{-1}A_iU = A'_i \). However, it appears that for a given \( C^* \)-algebra there may exist many different representations which are unitary inequivalent (what can lead to the situation, where, for example, the same operator of the particle number may have different expectation values, depending on representation). This fact plays an essential role in the quantum field theory in curved space-time in general, and in particular in the Fulling–Unruh and Beckenstein–Hawking effects, where the accelerating observer measures the different temperature then the one who is in rest \[73\]. The conclusion is that the notion of an algebraic state embodies information not about the measured system, but about all experimental setup (preparation of the system and measuremental apparatus before obtaining the result of the measurement), and that the concrete Hilbert space, contrary to an abstract \( C^* \)-algebra, contains information about both: observer and observed. So, when one performs the GNS representation and moves into the realms of Hilbert spaces, the algebraic information about measured system is merged with the information about measuring system (and, possibly, their interaction). This perspective may be drawn even more sharply, by saying that while observables refer to the internal properties of the observed system, the algebraic state encodes the ‘external’ properties of observer.

We would like to note that the appearance of nonunitarity in (4) seems to be closely related with the one which is essential in the algebraic approach. Following Wald \[73\], one may replace (4) by its purely algebraic reformulation

\[
\omega' (\cdot) = \frac{\omega (P_{\Delta \pi} \cdot P_{\Delta \pi})}{\omega (P_{\Delta \pi})},
\]

and consider the Hilbert spaces \( \mathcal{H}_\omega \) and \( \mathcal{H}_{\omega'} \) generated respectively by \( \omega \) and \( \omega' \), with representations \( \pi_\omega \)

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\(^4\) More precise formulation of this condition would replace \( \Phi(x) \) and \( \Phi(y) \) by their smeared versions.
The geometric phase is then defined as
\[ e^{i\theta[\gamma]} := e^{i\int_{s}A} = e^{-\int_{s}i\langle\psi|d\psi|)}, \]
where \( \gamma \) is a closed path in \( P\mathcal{H} \). We will denote by \( \psi(\cdot) \) a path generated by family of vectors \( \psi(s) \in \mathcal{H} \). In a case of open paths \( \gamma = \psi(\cdot) \) in \( P\mathcal{H} \) it was shown \cite{1}, \cite{58} that the geometric phase (sometimes called the Aharonov–Anandan phase) is given by
\[ e^{i\theta[\gamma]} = e^{-\int_{s}i\sbf{d}s\langle\psi(s)|d\psi(s)|}\langle\psi_{s_{1}}|\psi_{s_{f}}\rangle}. \]

It can be described by the following pictures:

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5. Note that equation (6) means that in general every act of measurement introduces new algebraic state and the corresponding representation. Due to identification of physical events with the acts of measurement, we have to say, that every physical event defines its own state and representation (the question about the relation between events and points of space-time will be discussed later — now we will only assume that, for a given observer, events can be indexed by linearly ordered elements chosen from a partially ordered set). For a given sequence \( I \) of measurements of observables belonging to certain \( C^{*} \)-algebra \( A \), we can define the sequence \( \{\omega_{i}\}_{i \in I} \) of states and the corresponding sequence \( \{\pi_{\omega_{i}}\}_{i \in I} \) of representations. The change of an algebraic state (and representation) reflects the fact that the measurement is an act of dynamical interaction of the system with the measurement apparatus, which changes not only the geometry of observed system (reflected in (4) and (6)), but also the dynamics of the observing system, which is reflected by change of the kinematic structure of the Hilbert space. In ordinary quantum mechanics this effect was negligible, due to Stone-von Neumann theorem ensuring the unitary equivalence of all representations in finite-dimensional Hilbert space. However, in systems with infinite number of degrees of freedom, like in QFT, this theorem is not valid, and the change of representation caused by measurement has to be considered as the genuine property of the theory. Change of the algebraic state expresses then the change of the structure of knowledge of an observer caused by an interaction between system and observer. This should be understood as updating of the probability assignment \( \omega \) which is a result of obtaining of the new knowledge \( P_{\omega,\Delta} \).

6. The Hopf bundle is a \( U(1) \) principal bundle of the Hilbert space \( \mathcal{H} \) over the space of rays in \( \mathcal{H} \), given by the projective Hilbert space \( P\mathcal{H} \).
Actually the very similar structure appears in the case of Aharonov–Bohm effect [2], where the non-single-time observable is provided also by the holonomy of $U(1)$ connection (the four-vector electromagnetic potential $A_\mu$):

$$A[\Gamma] = \Psi^*(x)e^{-iq \int_\Gamma dx^\mu A_\mu(x)}\Psi(y),$$

where $q$ is an electric charge, and $\Gamma$ is a path in a space-time $M$. These phases describe the measurable quantum quantity which refers to an ordered sequence of measurements determined by the purely kinematical (geometric) elements of the theory. Consider now an initial vector $|\psi(s = 0)\rangle$ and the final vector $|\psi(s = \tau)\rangle$, equivalent to the initial one up to phase. The unitary time evolution $U(s)$ on $\mathcal{H}$, provided by the Schrödinger equation with the Hamiltonian $H(s)$,

$$U(\tau): |\psi(s = 0)\rangle \rightarrow |\psi(s = \tau)\rangle$$

generates a loop $\gamma$ on the space $\mathbb{P}\mathcal{H}$. The phase on the Hopf bundle is then transformed into:

$$e^{\int_0^\tau ds\langle \psi(s)| - \frac{d}{ds} - iH(s)|\psi(s)\rangle} = e^{-\int_0^\tau ds\langle \psi|d\psi\rangle - i\int_0^\tau ds\langle \psi(s)|H(s)|\psi(s)\rangle} = e^{i\theta[\gamma]}e^{-i\int_0^\tau ds\langle \psi(s)|H(s)|\psi(s)\rangle}. \quad (8)$$

The first term, given by the geometric phase, does not refers to dynamics, and reflects purely geometrical structure of the kinematical Hilbert space.
3 Quantum histories

3.1 Histories of propositions

Ordinary quantum mechanics refers to the measurements performed in a single instance of time, so actually it can describe the quantum effects only on three-dimensional slices of space-time. Historically, the appearance of time in quantum theory has begun from the Newtonian time used as an independent parameter in Schrödinger equation. This time was later changed to the Lorentzian time of Klein–Gordon and Dirac equations, leading to the Poincaré-invariant structure of QFT. However, the commutation relations and expectation values were still evaluated on three-dimensional time slices. A serious attempt to change this situation was performed by Griffiths [23], Omnes [51], [52], and Gell-Mann and Hartle [21], [22] by the development of the consistent histories approach. Isham and Linden [30], [32], [33], [34] have proposed a certain modification of this theory, called a history projection operator (HPO) approach, later developed by Savvidou and Anastopoulos [59], [60], [5], [3], [6]. We will consider here only this line of development of the general idea of quantum histories. The starting point of this approach is considered as a history α of measurements of quantum system resulting with a sequence of propositions \((α_1, α_2, \ldots, α_n)\) obtained in times \((t_1, t_2, \ldots, t_n)\). This history is given by the projection operator

\[ α := α_1 \otimes α_2 \otimes \ldots \otimes α_n, \]

acting in the Hilbert space \(V_n := \bigotimes_{t \in (t_1, \ldots, t_n)} \mathcal{H}_t := \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_n\), where \(α_t\) is a projection operator in the \(n\)-th copy of the Hilbert space \(\mathcal{H}_t := \mathcal{H}\) of the given quantum system.\(^5\) The information about the history of measurements and description of the dynamics of a given system is contained in the class operator on \(V_n\), defined as:

\[ C_α := U(t_0, t_1)α_1 U(t_1, t_2)α_2 \cdots U(t_{n-1}, t_n)α_n U(t_n, t_0), \]

where \(U(t_i, t_{i+1}) = e^{-i H(t_{i+1} - t_i)/\hbar}\) are unitary evolution operators from time \(t_i\) to \(t_{i+1}\), acting on the Hilbert space \(\mathcal{H}_{t_{i+1}}\). Class operator encodes then the information about measurement and unitary evolution. For a system specified by the Hamiltonian \(H\) with an initial state described by a density matrix \(ρ\), the probability of a history \(α\) is given by

\[ p(α; ρ, H) = \text{tr}(C_α^* ρ C_α). \]

Using this equation, for two given histories \(α\) and \(β\), one defines the decoherence functional \(d : \mathcal{B}(V_n) \times \mathcal{B}(V_n) \to \mathbb{C}\):

\[ d(α, β) := \text{tr}(C_α^* ρ C_β). \tag{9} \]

By definition, this functional depends on \(ρ\) as well as on \(H\). It gives a possibility to consider the pairs \((UP, D)\), where \(UP\) is a set of (abstract) histories, endowed with the structure of an orthoalgebra (with the operations \(¬\), \(\sqcup\) and \(\leq\) of negation, disjoint sum, and coarse-graining respectively), while \(D\) is a set of (abstract) decoherence functionals, defined as morphisms \(d : UP \times UP \to \mathbb{C}\) such that:

1. \(d(1, 1) = 1,\)
2. \(d(α, α) \geq 0 \quad \forall α \in UP,\)
3. \(d(α, β) = d(β, α)^∗ \quad ∀ α, β ∈ UP,\)
4. \(d(α \sqcup β, γ) = d(α, γ) + d(β, γ) \quad ∀ α, β, γ ∈ UP.\)

\(^5\)In all section 3 of the present paper we will use the standard notation of the quantum histories theory, denoting projection operators not by \(P_{α_i}\), but by \(α_i\).
3.2. Histories quantization

The equation (9) may be then reinterpreted as a particular representation of the above abstract definition. In such case operations $\neg$, $\sqcup$ and $\leq$ are represented as follows:

1. $\neg(\alpha_{t_1}, \ldots, \alpha_{t_n}) = (1 - \alpha_{t_1}, \ldots, 1 - \alpha_{t_n})$,
2. $(\alpha_{t_1}, \ldots, \alpha_{t_n}) \leq (\beta_{t_1}, \ldots, \beta_{t_n}) \iff \forall t_i \alpha_{t_i} \leq \beta_{t_i}$,
3. $(\alpha_{t_1}, \ldots, \alpha_{t_n}) \sqcup (\beta_{t_1}, \ldots, \beta_{t_n}) \iff \exists t_i$ such that the ranges of $\alpha_{t_i}$ and $\beta_{t_i}$ are orthogonal.

In order to exclude unphysical predictions [18], one has to consider only the values of the decoherence functional between histories which are consistent ($d(\alpha, \beta) = 0$) within an exhaustive ($\bigcup_{\alpha_k \in \mathcal{C}} \alpha_k = 1$) and exclusive ($\alpha \beta = \delta_{\alpha \beta}$) subset $\mathcal{C}$ of $\mathcal{UP}$. For such histories the value $d(\alpha, \alpha)$ is interpreted as a probability $p(\alpha)$ of a history $\alpha$, while the value $d(\alpha, \beta)$ is interpreted as the amplitude of interference between histories $\alpha$ and $\beta$. This means that the propositions in quantum histories theory are contextual (with the context given by the consistent set $\mathcal{C} \subset \mathcal{UP}$). Actually, this is a genuine property of all so-called ‘realist’ approaches to quantum theory.\footnote{Another contextual realist interpretations of quantum theory are provided by the Everett many-worlds interpretation, Isham–Butterfield special prequel topos-theoretical approach to the Kochen–Specker theorem, and non-local Bohmian hidden variables theories.}

3.2 Histories quantization

The ordinary Heisenberg commutation relations in $\mathcal{H}$, given by $[q, p] = i\hbar$, $[q, q] = 0$, $[p, p] = 0$, are in the quantum histories framework extended to the relations:

$$
[q_{t_i}, q_{t_j}] = 0, \quad [p_{t_i}, p_{t_j}] = 0, \quad [q_{t_i}, p_{t_j}] = i\hbar \delta_{ij},
$$

where the operators $A_{t_i}$ for different events $t_i$ of measurement (moments of time) are considered in the Schrödinger picture. In order to formulate the extension of this formalism to the case of continuous time $t$, Isham and Linden have changed the above relations into the form of the so-called history algebra:

$$
[q_f, q_g] = 0, \\
[p_f, p_g] = 0, \\
[q_f, p_g] = i\hbar \tau \int_{-\infty}^{+\infty} dt f(t)g(t),
$$

where $f$ and $g$ are smearing test functions belonging to $L^2(\mathbb{R}, dt)$, and $\tau$ is a constant with dimensions of time. Using the similarity of histories quantization with one-dimensional quantum field theory, they have shown that these commutation relations may be represented on the Hilbert history space $\mathcal{Y} := \bigotimes_t \mathcal{H}_t := \bigotimes_t L^2(\mathbb{R}, dt)$ with a ‘continuous tensor product’ $\bigotimes_{t \in \mathbb{R}}$, build as the bosonic Fock space $\mathcal{F}[\mathcal{H}] = \mathcal{F}[L^2(\mathbb{R}, dt)]$ with the measurement of history in $n$ times $(t_1, \ldots, t_n)$ described as the $n$-particle state $\mathcal{Y}$ [33], [34], [59]. Consequently, in the histories quantization of a classical field, instead of the canonical variables $(q, p)$ one considers the canonical field variables $(\phi, \pi)$, and extends the ordinary space-like canonical commutation relations of QFT with the time dimension handled by histories algebra [33], [34], [59]:

$$
[\phi_{t_1}(\vec{x}_1), \phi_{t_2}(\vec{x}_2)] = 0, \quad [\pi_{t_1}(\vec{x}_1), \pi_{t_2}(\vec{x}_2)] = 0, \quad [\phi_{t_1}(\vec{x}_1), \pi_{t_2}(\vec{x}_2)] = i\hbar \delta(t_1 - t_2)\delta^3(\vec{x}_1 - \vec{x}_2).
$$

Hence, the histories quantization of three-dimensional field theory is actually three-plus-one-dimensional canonical quantum field theory. Later Anastopoulos [3] has shown, using the coherent states representation, that the construction of the continuous history Hilbert space may be provided also for non-quadric...
Hamiltonians. For a given Hamiltonian $H_t$ of a system, the self-adjoint histories Hamiltonian operator in Schrödinger picture is defined as

$$H_\kappa := \int_{-\infty}^{+\infty} dt \kappa(t) H_t,$$

where $\kappa(t)$ is a test function which smears $H_t$ in time. The histories algebra generates then the commutation relations with Hamiltonian. In the case of the histories quantization of harmonic oscillator $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$ one obtains [59]:

$$[H_\kappa, q_f] = -\frac{i\hbar}{m} p_{\kappa f}, \quad [H_\kappa, p_f] = i\hbar \omega^2 q_{\kappa f}, \quad [H_\kappa, H_{\kappa'}] = 0.$$

The crucial property of the histories approach, discovered by Savvidou [59], is the existence of the self-adjoint quantum action operator, acting on $\mathcal{V}$ and defined as

$$S_{\lambda,\kappa} := \int_{-\infty}^{+\infty} dt (\lambda(t)p_t \dot{q}_t - \kappa(t) H_t),$$

by an analogy to a classical action functional

$$S_{cl} = \int_{-\infty}^{+\infty} dt (p(t) \dot{q}(t) - H(t)),$$

where dot (in both cases) denotes the differentiation $\frac{\partial}{\partial s}$ with regard to time $s$ of the evolution generated by Hamiltonian, that is (in the quantum theory)

$$q_t \equiv q_{H,t}(s) := e^{isH_t/\hbar} q_t e^{-isH_t/\hbar},$$

$$\dot{q}_t \equiv \frac{\partial}{\partial s} q_{H,t}(s).$$

Savvidou has shown that the Liouville operator

$$V := \int_{-\infty}^{+\infty} dt (p_t \dot{q}_t)$$

also belongs to $\mathfrak{B}(\mathcal{V})$. Hence, for $\lambda = 1$, one may describe the action operator as

$$S_\kappa = V - H_\kappa = \int_{-\infty}^{+\infty} dt p_t \dot{q}_t - \int_{-\infty}^{+\infty} dt \kappa(t) H_t,$$

with the following commutation relations:

$$[S_\kappa, H_{\kappa'}] = i\hbar H_{\kappa'}, \quad [S_\kappa, V] = -i\hbar H_\kappa, \quad [V, H_\kappa] = -i\hbar H_\kappa.$$

One should note that for $\kappa(t) = 1$ the histories quantum theory reduces to an ordinary quantum theory, what (for $H := \int_{-\infty}^{+\infty} dt H_t$) is reflected in the commutators: $[V, H] = 0, \quad [V, S] = 0$. Using the fact that $V$ is defined as Fock space representation of the history algebra, one can consider the ‘annihilation operator’ $b_t$ defined from the Hamiltonian $H_t = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$ as

$$b_t := \sqrt{\frac{m\omega}{2\hbar}} q_t + i \sqrt{\frac{1}{2m\omega\hbar}} p_t.$$

The operator $V$ acts on $b_t$ in the following way [59]:

$$e^{i\tau V/\hbar} b_{f(t)} e^{-i\tau V/\hbar} = b_{f(t+\tau)}.$$
Moving to the Heisenberg picture, we may compare the action of $V$, $H$, and $S$:

\[
e^{-i\tau V/\hbar} b_{t,s} e^{i\tau V/\hbar} = b_{t+\tau,s},
\]

\[
e^{-i\tau H_{t}/\hbar} b_{t,s} e^{i\tau H_{t}/\hbar} = b_{t,s+\tau},
\]

\[
e^{-i\tau S/\hbar} b_{t,s} e^{i\tau S/\hbar} = b_{t+\tau,s+\tau}.
\]

This means that $V$ transforms $b_t$ from time $t$, related with the Hilbert space $H_t$, to time $t+\tau$, related with the Hilbert space $H_{t+\tau}$ (strictly speaking, $V$ transforms the support of the operator valued distribution).

It is important to note that this is, by definition, purely kinematical operation, which does not depend on Hamiltonian. On the other hand, $H_t$ generates the unitary evolution of the system in the single space $H_t \subset V$. The action operator joins together these two types of transformation. This may be illustrated by the following picture:

As notes Savvidou [59], it may be concluded, that these two operators ($V$ and $H_{\kappa}$) are generators of two different types of time evolution: the nonunitary ‘wave-packet reduction’ performed in subsequent measurements, and the ordinary Hamiltonian evolution. She has also noticed that, while the ‘inner clock’ of every process is given by its Hamiltonian $H = H_t$, the reparametrisation of time $s$ depends on the function $\kappa(t)$ used in the definition of $H_{\kappa}$, so, for a given quantum process, its $\kappa(t)$ has to be fixed. This leads us to consideration of the close relationship between the Isham–Linden–Savvidou approach and the ‘theory of uncertainty of time’ proposed by Guts [24]. The starting point of the later programme is an observation that every physical measurement is performed by an observer which is contained in a finite volume of space-time. So, every real clock $f$ enables one to assign to every event $t$ only some extended time-epoch $\tau$. It is common in physics to idealize the description of an event $t$ as a point $\tau$ of the timeline $\mathbb{R}$, so its epoch is precisely localized. However, generally, one should consider physical phenomena (appearances of physical processes) as given by distributions over the mathematical space which is used to model time. Guts develops then the stochastic theory of phenomena, based on the Kolmogorov probability theory, with density of probability of time-epoch $\tau$ given by the function $f_{\tau}(t)$. This suggest us to interpret physical quantum process, measured in the history $\alpha$, as a distribution $\kappa(t)$ over the partially ordered space of measurement acts, with the Hamiltonian given by (11). From this point of view on quantum histories, $H_t$ describes the Hamiltonian of the physical phenomena $\kappa$ which, when considered as process, evolves in the ‘internal’ time $s$, but also undergoes changes in the subsequent measurements, partially ordered by the ‘external’ time $t$. The function $\kappa(t)$ describes the distribution of the process, given by $H = H_t$, over the ‘probabilistic’ quantum histories Hilbert space $V$. The physical phenomena is then described not only by the specification of its Hamilton and Liouville operators, but also by the function $\kappa(t)$, which describes the statistical significance (appearance) of the given process in concrete measurements. In particular, there is no appearance of the given physical process outside of the $supp(\kappa)$. It is worth to note that the HPO formalism works as well for the case when the time $t$...
is discrete (in such case the integration in (10), (11) and (12) has to be replaced by sum). This means that the crucial property of the external time is not its continuity, but partial order, which expresses the main aspect of the structure of set of observations (events), while the continuity and differentiability is a secondary property which may be lost in some cases (contrary to the case of internal time, where ordering plays no role, while differentiability and continuity is crucial). Actually, due to the quantum Zeno [48] and anti-Zeno [8] effects, the consideration of continuity in the set of measurements seems to be prohibited, and, as has been argued by Anastopoulos [6], the continuum-time quantum histories should be considered rather as a limit of genuinely discrete theory. In such case one should think about equations (11), (12) and (13) in terms of Ito integral instead of Lebesgue measure (we will later discuss this stochastic process aspect of quantum histories in more details). In any case, continuous or discrete, the change of the external time describes the movement between the acts of measurement and is related with an observer, contrary to the change of internal time, which describes the unitary movement of the internal dynamics of the process and is independent from the acts of observation. The spaces of the external and internal time may be in some cases joined together and may have very similar properties, like in the continuous version of the HPO approach, but generally there is no reason why they should have the same structure (e.g. of the Hilbert space), because they describe very different aspects of the notion of ‘time’. The dynamics of quantum field, given by its Hamiltonian and internal time $s$, has no preferred direction, is described in terms of differential calculus and is not related with an observation. On the other hand, the external time has a priori no relation with differentiation, and does not reflect the ‘dynamics’ of a field, but is a tool for ordering the measurements and kinematical rearranging $\mathcal{H}_t \to \mathcal{H}_t'$ of the Hilbert space assigned with particular measurement performed by an observer. This leads to the new perspective on the meaning of action operator $(S, \kappa)$, as the description of the behaviour of quantum phenomena which relates the processes $(H, s)$ of the system and the measurements $(V, t)$ of the observer. Only a specification of the concrete action functional (together with ‘phenomena function’ $\kappa$) enables one to joint these two notions of time into one structure.

### 3.3 Histories and geometric phase

Now, following Anastopoulos and Savvidou [5], one can analyze the geometric phase from the perspective of quantum histories. Consider a system with a Hamiltonian $H = 0$. For a given history $\alpha$ of the measurements $(\alpha_0, \alpha_1, \ldots, \alpha_n)$, where $\alpha_i = |\psi_{t_i}\rangle\langle\psi_{t_i}|$ is a projection operator, the trace of the class operator is

$$trC_{\alpha} = \langle\psi_t|\psi_{s_n}\rangle \langle\psi_{t_2}|\psi_{t_1}\rangle \cdots \langle\psi_{t_n}|\psi_{t_{n-1}}\rangle.$$  

Assuming that $\delta t := \sup_i \{|t_i - t_{i-1}|\} \sim O(\frac{1}{n^2})$, one can approximate $|\psi_{t_i}\rangle$ by the path $\psi(\cdot)$ on $\mathbb{P}\mathcal{H}$, and for large $n$ this gives

$$\log trC_{\alpha} = \log \langle\psi_{s_0}|\psi_{s_n}\rangle = \sum_{i=1}^{n} \log \langle\psi_{s_i}|\psi_{s_{i-1}}\rangle = \log \langle\psi_{s_0}|\psi_{s_n}\rangle - \sum_{i=1}^{n} \langle\psi_{s_i}|\psi_{s_i} - \psi_{s_{i-1}}\rangle + O(\frac{1}{n^2}),$$

hence

$$\lim_{\delta t \to 0} \log trC_{\alpha} = \log \langle\psi_{s_0}|\psi_{s_n}\rangle - \int_{s_0}^{s_n} ds \langle\psi(s)|d\psi(s)\rangle,$$

where the last term is a Stieltjes (or, generally, Ito) integral. Comparing this result with Aharonov–Anandan equation (7), we see that for any path which enables the definition of Ito integral, the trace of a class operator is equal to geometric phase:

$$trC_{\alpha} = e^{i\theta(\psi(\cdot))}. \quad (14)$$

We may say, that for a given history $\alpha$, its (observable!) geometric phase is defined by the trace of a class operator. Observing that $C_{\alpha}$ is used in the definition (9) of the decoherence functional, we may
3.3. Histories and geometric phase

rewrite it in terms of the geometric phase into form:

\[ d(\alpha,\beta) = \langle \psi(t_0) | p_{\alpha} | \phi(t_0) \rangle \langle \psi(t_n) | \phi(t_n) \rangle e^{-\int_{t_0}^{t_n} ds \langle \psi(s) | d\psi(s) \rangle - \int_{t_0}^{t_n} ds \langle \phi(s) | d\phi(s) \rangle}. \]

For a system with dynamics given by the non-zero Hamiltonian \( H(s) \), we use the equation (8), and the decoherence functional becomes [3]:

\[ d(\alpha,\beta) = \langle \psi(t_0) | p_{\alpha} | \phi(t_0) \rangle \langle \psi(t_n) | \phi(t_n) \rangle e^{i \int_{t_0}^{t_n} ds \langle \psi(s) | H(s) | \psi(s) \rangle}, \tag{15} \]

where

\[ \langle S^*_n[\psi()] \rangle := \int_{t_0}^{t_n} ds \langle \psi(s) | i \frac{\partial}{\partial s} H(s) | \psi(s) \rangle = i \int_{t_0}^{t_n} ds \langle \psi(s) | d\psi(s) \rangle - \int_{t_0}^{t_n} ds \langle \psi(s) | H(s) | \psi(s) \rangle. \tag{16} \]

Now, by summing over all ‘fine-grained’ paths \( \psi() \) and \( \phi() \) which are compatible with the given histories \( \alpha \) and \( \beta \) respectively, we may reconstruct full decoherence functional as:

\[ d(\alpha,\beta) = \sum_{\psi() \in \alpha} \sum_{\phi() \in \beta} d(\alpha,\beta). \tag{17} \]

Anastopoulos and Savvidou conclude, that the knowledge of the geometric phase – for a set of histories and of the automorphism that implements the dynamics – is sufficient to fully reconstruct the decoherence functional – and hence all the probabilistic content of the theory [5].

Using these results of Anastopoulos and Savvidou we would like to provide now an algebraic reformulation of the histories description of the geometric phase. In order to compare the equation (16) with the equation (13), consider the Schrödinger representation: \( p_t = -i \frac{\partial}{\partial x} \) and \( \dot{x}_t = \frac{\partial}{\partial x} x_t. \) Then \( V_t = p_t \dot{x}_t = -i \frac{\partial}{\partial x} \), hence the equation (16) may be written in the form

\[ \langle S^*_n[\psi()] \rangle = - \int_{t_0}^{t_n} ds \langle \psi(s) | V_t + H(s) | \psi(s) \rangle. \tag{18} \]

This allows us to generalize the definition of the geometric phase \( e^{i \theta[\gamma]} \) from

\[ \theta[\gamma] := i \int_{\gamma} ds \langle \psi(s) | d\psi(s) \rangle, \quad d = \frac{\partial}{\partial s}; \]

to

\[ \theta[\gamma] := - \int_{\gamma} ds \langle \psi(s) | V_t | \psi(s) \rangle, \tag{19} \]

which may be approximated via \( \psi() \approx | \psi_t \rangle \in \gamma \) as

\[ \theta[\gamma] = - \int_{\gamma} ds \langle \psi_t | V_t | \psi_t \rangle. \tag{20} \]

This definition actually describes how the non-Hamiltonian part of the action \( S \) acts on the measurable properties of the system. Reformulating this description into algebraic terms, we obtain:

\[ \theta[\gamma] = - \int_{\gamma} ds \omega_t(A^* V_t A), \quad \psi_t = \eta_\omega(A), \quad t \in \gamma. \tag{21} \]

where integration is provided in terms of Ito integral and \( \omega_t \) is iteratively given by (6) as:

\[ \omega_{t_k}(A^* V_t A) = \frac{\omega_{t_{k-1}}(P_{B,\Delta_{t_k}}^* A^* V_t A P_{B,\Delta_{t_k}})}{\omega_{t_{k-1}}(P_{B,\Delta_{t_k}})}, \tag{22} \]
with discretisation \( t_k \in \{t_0, t_n\} = \gamma \) describing the sequence of measurements. If we assume now that the observer measures constantly the fixed subspace \( \Delta \) of the spectrum of the fixed observable \( B \) then the iterative description (22) leads to expressions

\[
\theta[\gamma] = \lim_{n/\gamma \to \infty, t_n - t_0 = \text{const}} \sum_{i=1}^{n} \omega_i(A^* V_i A)(s_i - s_{i-1}) = \\
= \lim_{n/\gamma \to \infty, t_n - t_0 = \text{const}} \sum_{i=1}^{n} \omega_0(P_{\Delta_i} \cdots P_{\Delta_1} A^* V_i A P_{\Delta_1} \cdots P_{\Delta_i}) = \int_{t_0}^{t_n} ds \omega_0(P(s) A^* V_i A P(s)) \frac{\omega_0(P(s) P(s-1))}{\omega_0(P(s))},
\]

and

\[
e^{i\theta[\gamma]} = \lim_{n/\gamma \to \infty, t_n - t_0 = \text{const}} \prod_{i=1}^{n} e^{-i \omega_0(A^* V_i A)(s_i - s_{i-1})} = e^{-i \int_{t_0}^{t_n} ds \omega_0(P(s) A^* V_i A P(s)) \frac{\omega_0(P(s))}{\omega_0(P(s)) P(s-1))}.
\]

where \( P_{\Delta_i} := P_{B,\Delta_i} \), and \( P(s) = P_{\Delta_1} \cdots P_{\Delta_i} \), hence

\[
\begin{align*}
P(s)|\psi\rangle &= |\psi\rangle & \iff & \forall t \ P_{\Delta_t}|\psi\rangle = |\psi\rangle \\
P(s)|\psi\rangle &= 0 & \iff & \exists t \ P_{\Delta_t}|\psi\rangle = 0.
\end{align*}
\]

The equation (24) provides an algebraic description of geometric phase for any given quantum system, so it may be used in order to find experimentally its Liouville operator. This description is performed in terms of the (known) initial state \( \omega_0(A^* \cdot A) \) and the results of repeated measurements of any algebraic observable \( B \) of the system. Of course, in any real experiment this limit can be only approximated, by means of the \( t_i - t_{i-1} := \delta t > 0 \) given by some real clock. One of the most natural choices for such clock may be provided by the Hamiltonian of the field under consideration, if its phenomena function \( \kappa \) is known (fixed). Note that the limit in (23) and (24) closely resembles the properties of the thermodynamical limit. This actually suggests the new perspective on the quantum histories theory, which we will develop in what follows.

4 General quantum theory

In this section we will develop a general setting of quantum theory, based on the histories and algebraic approaches. Our main tool will be the modular theory, while the main ansatz will be consideration of Liouville operator as a quantum histories analogue of temperature. The first subsection will serve as an introduction and motivation. In the second one we will introduce the modular theory in the quantum histories thermodynamic context. In the third section we will provide the rigorous development of the new framework, deriving the quantum geometric phase, Lagrangean path integral and classical action functional as special cases. Next we will show that this setting has natural interpretation in terms of Bayesian contextual and relational description of probability, based on the Kullback-Leibler entropy. This will enable us to reconstruct the setting of the first subsection, and also to obtain new insight on the infinitesimal Lorentz symmetry and the Minkowski–Wick trick, which transforms Lorentzian quantum integrals into Euclidean stochastic ones. We will finish this section with the discussion of the relation between Lagrangean and algebraic quantum field theory and our approach, and the meaning of the modular Hamiltonian (through the analysis of the Fulling–Unruh effect).
4.1 Quantum histories thermodynamics

We would like to propose now the certain conclusions which arise from the quantum histories formalism. The equation

\[ S_\kappa = \int_{-\infty}^{+\infty} dt V_t - H_\kappa \]

may be written in the differential form

\[ dS_\kappa = V_t dt - dH_\kappa \]

or

\[ ds'_\kappa = dH_\kappa - V dt \] \hspace{1cm} (25)

for \( S'_\kappa := -S_\kappa \) and \( V_t dt \approx V dt \). Note that the last equation resembles the form of the thermodynamical Helmholtz equation:

\[ dF = dE - T dS \] \hspace{1cm} (26)

where \( F \) is 'free' Helmholtz energy, \( E \) is 'internal' energy obtained from the Hamiltonian of a system, \( T \) is temperature and \( S \) is an entropy. It is striking that the equation (25) resembles actually not only the form of (26). The physical meanings of their corresponding elements are completely analogous:

- \( H_\kappa \) and \( E \) describe the purely dynamic and energetic aspect of the behaviour of the system,
- \( t \) and \( S \) describe the purely kinematic and informational (structural) aspect of the behaviour of the system,
- \( V \) and \( T \) parametrize the magnitude of kinematic changes of the inner structure of the system,
- \( S'_\kappa \) and \( F \) describe the overall behaviour of the observed system, and the most probable physical states are those which extremalize the value of these functionals.

What is the source of this analogy? Helmholtz equation is used in thermodynamics in order to describe the systems for which the environment (observer) acts as thermostat (and it is the special case of the Gibbs equation in the case of constant volume). On the other hand, the systems described solely by the Hamiltonian cannot undergo any changes caused by the environment (observer). So, if we want to describe the effects caused in the system due to interaction with the observer, we have to introduce the non-Hamiltonian terms, which lead to the Liouville operator in action in classical mechanics and quantum histories, and, on the other hand, to the entropic part \( -T dS \) of the Helmholtz equation. Note that in the classical mechanics and classical field theory it is assumed that the measurements do not disturb the determinism of the Hamiltonian time evolution of the system, hence that they can be a priori absolutely precise. In order to describe the effects of the non-Hamiltonian interaction with observer (modelled mostly as an 'environment'), the classical mechanics (and field theory) has to be equipped with some probabilistic notion, which leads to statistical classical mechanics (and field theory). It should be stressed that the ontological part of statistical mechanics is given solely by classical mechanics, and only its epistemic part is described by the probability theory. The probability of finding the system in the state \([p, p + d^n p] \times [q, q + d^n q]\) is defined to be given by the measure (distribution) \( \rho(p, q)d^n pd^n q \) over the \((2n)\)-dimensional phase space. The number of states in the phase space is uncountably infinite, so, in order to define functions of state, we have to divide it into countable cells, evaluated by the measure. This selection of cells is arbitrary, so the measure is defined up to additive constant (the same is true in the case of the Lebesgue measure in the Hilbert space). The evolution of the behaviour of the system is described by the Liouville equation:

\[ 0 = \frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \{H, \rho\} =: \frac{\partial\rho}{\partial t} + V \rho, \] \hspace{1cm} (27)
where \( H \) is the (classical) Hamiltonian, while \( V \) is the (classical) Liouville operator. This equation is fundamental for all statistical classical mechanics. One should note that

\[
V = 0 \iff \{ H, \rho \} = 0,
\]

that is, the behaviour of the system is described purely by Hamiltonian if and only if Liouville operator is equal to zero. This means that the Liouville operator is equivalent to non-Hamiltonian (non-dynamical) part of the behaviour of the system. This part of the behaviour is introduced by the interaction with an observer.

In the quantum mechanics an interaction with observer plays also the fundamental role: even if we know absolutely all states of the system, the measurement introduces an interaction, which has to be included in the description of the behaviour of measured properties. In the quantum histories framework, the non-zero value of the (quantum) Liouville operator provides the transition \( t \to t' \), and the reorganisation of the kinematic structure of the Hilbert space \( \mathcal{H} = \mathcal{H}_t = \mathcal{H}_{t'} \). For \( V_i = 0 \) the physical picture resulting from quantum histories equation (25) is the same as in the case of zero temperature \( T \) in the Helmholtz equation: the behaviour of the system is governed totally by its dynamics, without any “informational” reorganisations of the inner (kinematic) structure. This leads us to consideration of the most intriguing aspect of this analogy — between the external time \( t \) and the thermodynamic entropy \( S \).

In the classical thermodynamics the state functions (such as entropy) are defined as such macroscopic properties of the system, which depend only on the actual state of the system and do not depend on its history (so, this notion actually encodes the Markov property of the behaviour of the process). In quantum histories the parameter \( t \) describes only the actual state and it does not depend on any history — so it can be considered as state function. It also satisfies the thermodynamical postulate for entropy (known under the name of the second law of thermodynamics), that the transition between states \( A \) and \( B \) of a system should be possible only when

\[
S(B) \geq S(A).
\]

In quantum histories this postulate is encoded from the beginning in the partial order structure of the set of parameters \( t \):

\[
t_{i+1} \geq t_i.
\]  \hspace{1cm} (28)

On the other hand, from the perspective of the statistical classical mechanics, the thermodynamic entropy \( S \) is a measure of the number of microscopic configurations which enable the given microscopical state. Hence, if we are going to interpret \( t \) as thermodynamic entropy, the growth of its values should correspond to the growth of the number of microscopic configurations which enable the given macroscopic state. Note then that in every transition \( t \to t' \), the Liouville \( V \) generates the non-Hamiltonian evolution of the Hilbert space. However, \( V \) is not an observable in any particular measurement \( t \) (because it is self-adjoint on \( V \) but not on \( \mathcal{H}_t \) ), so we do not have an information in which of possible configurations the Hilbert space actually is, thus, in every transition \( t \to t' \) the number of possible configurations which allow the given history of measurements, grows. Hence, every measurement belonging to a given history \( \alpha = (\alpha_{t_0}, \ldots, \alpha_{t_n}) \) leads to the growth of this lack of information and to growth of the uncertainty about the inner structure of the system, despite obtaining even the maximal knowledge which is accessible. However, all this happens only for non-zero ‘quantum temperature’ of Liouville operator! This certifies the physical meaning of the analogy between (25) and (26), enabling the interpretation of (25) and (28) as, respectively, the first and the second law of quantum histories thermodynamics, and leads us straight to the striking conjecture, which might be called the third principle of quantum histories thermodynamics:

\[
\lim_{(V)\to \infty} t = t_p \cdot \log g_0,
\]  \hspace{1cm} (29)

where \( t_p \) is a Planck time, while \( g_0 \) is the degeneration number of possible microscopic realisations of the ground energy state of the system. This equation means, that quantum histories naturally imply
4.1. Quantum histories thermodynamics

The quantisation of time iff the quantum histories analogue of the ground energy state of the system under consideration is degenerated. The equation (29) should be understood as follows: in the limit of no measurements performed by an external observer, there is present a growth of uncertainty about the system from the perspective of external observer caused by the non-trivial structure of the lowest energy state of the system or there is trivial 'eternal dynamics' without any entropy, in the case when the lowest energy state is non-degenerate \((g_0 = 1)\). Hence, the complete list of laws of quantum histories thermodynamics is then following:

- **Zeroth**: For a system in quantum histories thermal equilibrium the Liouville operator must be the same for all measurements in a given history.

- **First**: \(d S'_\kappa = d H_\kappa - V_t dt\).

- **Second**: \(t_{i+1} \geq t_i\), with equality obtained only for reversible processes (with no measurements).

- **Third**: \(\lim_{\langle V \rangle \to 0} t = t_p \log g_0\).

Coming back to the Helmholtz equation (25) and (26), recall that it is obtained under the consideration of the existence of an observer (thermostat) which fixes the value of \(V_t (T)\) or varies it with small changes. But what will happen if we will change rapidly the value of temperature? In the statistical mechanics this situation is described by expressing the statistical Helmholtz ensemble in terms of the Kubo–Martin–Schwinger (KMS) condition put on the algebraic state used for GNS representation of the C*-algebra of statistical observables (we will discuss this issue in the next subsection). The essential change of temperature means the essential change of thermostat. The situations of different temperatures \(T\) are described by representations generated by the different corresponding KMS algebraic states \(\omega_T\).

In quantum histories the change of \(V_t\) means actually the change of the kinematical transformation \(V_t : \mathcal{H} \rightarrow \mathcal{H}_t, \ t < t'\),

what implies the change of the expectation values of the observables over \(\mathcal{H}\). From the viewpoint of the algebraic approach, the observables remain unchanged after the change of \(V_t\), what implies that \(V_t\) can enter only the definition of the scalar product \((-,-)\) on \(\mathcal{H}\) and the algebraic state \(\omega\) on algebra of observables \(\mathcal{A}\). So, the change of the 'quantum histories temperature of observer' \(V_t\) induces the change of the GNS state \(\omega_{V_t}\) and the representation \(\pi_{\omega_{V_t}}\), analogically to the change of the KMS state \(\omega_T\) induced by the change of the temperature \(T\) of environment in the statistical mechanics. This means that algebraic state \(\omega\) is dependent on the kinematical, non-Hamiltonian contents of the theory, and the change of this state describes the change of the Liouvillean part of the behaviour of the system. On the other hand, *-algebras of observables are independent on the structure of the inner product and express only the dynamical, Hamiltonian, contents of the theory. With the help of the algebraic approach, one can use the Helmholtz equation also in the situation when \(T\) changes. Hence, the equation (25) may be used also when \(V_t\) depends on \(t\), but for every change of \(V_t\) one has to change the GNS representation of the algebra of observables, using the fact that GNS state \(\omega\) depends on \(V_t\). We can join now the above conclusions together in the one table, presenting which elements of the discussed theories describe the internal (dynamical) properties of the system, with evolution governed by Hamiltonian dynamics and parametrized by the internal time \(s\), and which describe the external (kinematical) properties, appearing as a result of measurement performed by an observer, with their change governed by partially ordered sequence of measurements, parametrized by the external time parameter \(t\).
So, the quantum action operator provides a thermodynamic description of the behaviour of the quantum systems observed by an external observer, and its expectation value on a history space gives the classical action functional. On the other hand, the decoherence functional enables one to provide a stochastic description of the kinematic part of the theory, related with the change introduced by an observer (through the subsequent measurements) on an equal footing with the dynamical part, given by the Hamiltonian. This leads to the question, can quantum histories be considered as an equilibrium ‘thermoodynamical limit’ of the stochastic description in terms of the decoherence functional? It is easy to see that continuous quantum histories have a form of the thermodynamic limit of the statistical theory of discrete quantum histories (see eq. (24)), but such statement is too general to be useful, and it does not give us tools for the description of relations between detailed statistical and thermodynamical quantum histories.

One should also note that there is qualitative difference between the first two cells and the last two cells in the last row in the above table: while the former are genuinely algebraic in their nature, the latter are referring to expectation values on a given Hilbert space. This suggests that there should be a possibility to describe the behaviour of quantum histories thermodynamics using only algebraic tools. In the next sections we will investigate this possibility in details.

4.2 Quantum histories modular automorphisms

We will show first that some new insight can be obtained with the help of Tomita–Takesaki modular theory [70], [69]. Our approach is motivated by the Kubo–Martin–Schwinger (KMS) theory [42], [44] in the algebraic setting [27], which characterizes the equilibrium thermodynamical algebraic states $\omega_{eq}$ as...
such for which the KMS condition
\[ \omega_\beta((\alpha_t A)B) = \omega_\beta(B\alpha_{t+i\beta} A) \] (30)
is satisfied, and relates it with the certain structural properties of algebras of observables (here and from now on \( \alpha_t \) will denote the automorphism of the \( C^* \)-algebra of observables).

The starting point of the Tomita–Takesaki modular theory is consideration of von Neumann algebra \( \mathcal{R} \) acting on a Hilbert space and possessing a separating and cyclic vector \( \Omega \) (which corresponds to algebraic state \( \omega \) on \( C^* \)-algebra which generates \( \mathcal{R} \)). One then defines a conjugate linear operator \( \mathcal{S} \), acting on a dense subspace of \( \mathcal{H} \) generated by action of \( \mathcal{R} \) on \( \Omega \), as:
\[ \mathcal{S}A\Omega = A^*\Omega. \]

Its closure (denoted usually by the same letter) has a unique polar decomposition \( \mathcal{S} = J\Delta^{1/2} \), with the following properties:
\[ J\Delta^{1/2}\Lambda\Omega = A^*\Omega, \quad \Delta\Omega = \Omega, \quad J\Omega = \Omega, \quad J = J^{-1} = J^*, \quad J\Delta J = \Delta^{-1}, \quad \Delta = \mathcal{S}^*\mathcal{S}. \]

The Tomita–Takesaki theorem states that for such situation there exists a (strongly continuous and unitary) group \( \sigma_\tau \) of automorphisms of \( \mathcal{R} \) generated by \( \omega \), called the group of modular automorphisms, given by
\[ \sigma_\tau A := \Delta^{i\tau} A\Delta^{-i\tau} =: U(\tau)A^*(\tau). \]

Consequently, \( \Delta \) is called a modular operator, and it depends on algebra of observables \( \mathcal{R} \) (or its \( C^* \)-algebraic origin) and of the state \( \omega \) (represented by a cyclic vector \( \Omega \)). It satisfies \( U(\tau)\Omega = \Delta^{i\tau}\Omega = \Omega \). One then defines the modular Hamiltonian by
\[ e^{-K} := \Delta, \]
which satisfies \( K^* = K \) and \( K\Omega = 0 \). Hence, we can write:
\[ U(\tau) = (e^{-K})^{i\tau} = e^{-iK\tau}. \] (31)

Coming back to KMS condition (30), it can be reexpressed in the language of modular automorphisms as [28]:
\[ \omega_\beta(\sigma_\tau A)B) = \langle \Omega|AU^*(\tau)B|\Omega \rangle = \ldots = \langle \Omega|Be^{-K(1-i\tau)}A|\Omega \rangle = \langle \Omega|B\sigma_{-\tau} A|\Omega \rangle = \omega_\beta(B\sigma_{-\tau} A). \]

This actually means that an equilibrium [KMS] state with an inverse temperature \( \beta \) may be characterized as a faithful state over the observable algebra whose modular automorphism group \( \sigma_\tau \) (as a group) is the time translation group, the parameter \( \tau \) being related to time \( s \) by \( s = -\beta \tau \) [28]. Hence, we will take \( \tau = -s\beta^{-1} \) and put it into (31):
\[ U(s) = e^{isK\beta^{-1}}. \]

In thermodynamics \( \beta = (k_B T)^{-1} \), hence \( \beta^{-1} = k_B T \). Then:
\[ U(s) = e^{ik_s Ks T}. \]

In quantum histories thermodynamics we replace temperature \( T \) by Liouville operator \( V_t \), and replace \( k_B \) by \( \ell_p \) (see (29)), so (in the natural units) we obtain:
\[ U(s) = e^{iKs V_{t_p}}. \] (32)

\(^9\text{We have changed the letter denoting the time in order to obtain consistency with notation used in the present paper.}\)
Note that while $V_t = -V_t^{\ast}$ is not self-adjoint, the operator $U(s)$ in (32) is self-adjoint. The most simple choice of the modular Hamiltonian is the unit matrix $K = I$, and then

$$U(s) = e^{isV_t},$$

which implies that, for $|\psi\rangle = [C]_{\omega V_t} = \eta_{\omega V_t}(C)$,

$$(U(\tau)|\psi\rangle|\psi\rangle) = \omega_{V_t}(\sigma^{\ast}_\tau(C^\ast)C) = (\Omega|C^\ast e^{-isV_t}C|\Omega) = (\psi|e^{-isV_t}|\psi\rangle) = \omega_{\psi}(e^{-isV_t}).$$

In the case when $\omega_{\psi}$ is a homomorphism of $C^\ast$-algebras, we have

$$\omega_{\psi}(e^{-isV_t}) = e^{-is\omega(V_t)}.$$ \hspace{1cm} (35)

Comparing this equation with (19), we have to conclude that the modular automorphism of algebra of observables satisfying quantum histories thermodynamics equilibrium condition describes the geometrical phase, which encodes purely kinematical properties of the quantum system, contained in the Liouvillean part of the action. Modular automorphisms in the equilibrium setting describe the change along the dynamically equivalent solutions, which represents the kinematical part of the behaviour of the system and observer. This means that we can upgrade our previous table into the following form:

<table>
<thead>
<tr>
<th>theory</th>
<th>internal dynamics</th>
<th>external kinematics</th>
<th>general behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hamiltonian $H$</td>
<td>Hamiltonian operator $H$ internal time $s$</td>
<td>Liouvillean $V_t$ external time $t$</td>
<td>action operator $S$ phenomena function $\kappa$</td>
</tr>
<tr>
<td>classical</td>
<td>Hamiltonian $H$</td>
<td>Liouvillean $V_t$</td>
<td></td>
</tr>
<tr>
<td>quantum histories thermodynamics</td>
<td>Hamiltonian $H$ internal time $s$, algebraic statistics</td>
<td>dynamic automorphism $e^{-isH}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>kinetic automorphism $e^{\kappa V_t}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>histories space $\mathcal{V}$</td>
<td></td>
</tr>
<tr>
<td>quantum histories</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>algebraic statistics</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that the time parameter $\tau$ of the KMS modular automorphism has to be translated in terms of an inner time $s$, because it refers to the given algebra of observables and the given state. It describes the automorphisms of an algebra appearing due to representation of this algebra in a given state (of observation). The ‘evolution’ $\sigma_\tau A = U(\tau)AU^{\ast}(\tau)$ does not change the expectation values of observables, so it does not say anything about the dynamics and it characterizes completely (due to Tomita–Takesaki theorem) purely kinematical properties of von Neumann algebra. Thus, it describes the development of the kinematical degrees of freedom generated by the representation. It does not reflect the dynamics, because dynamics is defined as the automorphism of an algebra of observables which is independent of state (in other words – the dynamics of a system is a change of the system which happens independently from the fact of observing or non-observing), but it is dependent on the Liouville operator, which is the non-Hamiltonian (non-dynamical) part of the action. So, for a given algebra of observables and given state, we have the rule to describe the dynamical (Hamiltonian) and kinematical (Liouvillean) parts of the action, and, consequently the full quantum behaviour of the system. The choice of the modular Hamiltonian equal to identity of the algebra enables the consideration of evolution of states in the Hilbert space caused by purely kinematical part of the action. This description is equivalent to the description of the kinematical evolution of the states in the Hilbert space in terms of the geometric phase.

### 4.3 The new description of quantum behaviour

The results of the previous subsection lead us straight to formulate now the general algebraic description of the behaviour of observed quantum systems, which will incorporate the kinematical Liouvillean part of the description (related closely with the geometric phase) on the equal footing with the dynamical Hamiltonian one. Our motivation is to replace the decoherence functional by another, genuinely algebraic, object. We will not assume here anything from the quantum histories thermodynamics propositions of the section 4.1, except of the ansatz $\tau = -s\beta^{-1} = -sV_t$. Note that the description (34) of the geometric phase based on the Tomita–Takesaki automorphism refers only to the single algebraic state $\omega$. 

4.3. The new description of quantum behaviour

However, in general, we would like to consider the change of states (and of the Liouville operator) in the subsequent measurements. Fortunately, we may provide such description with the help of the relative modular operators [28]:

$$J_{V,\Omega}(\Delta_{V,\Omega})^{1/2}A|\Omega\rangle := A^*|\Omega'\rangle,$$

where $|\Omega\rangle$ and $|\Omega'\rangle$ are cyclic separating vector representatives of the faithful normal states $\omega$ and $\omega'$ on the Hilbert spaces $\mathcal{H}_\omega$ and $\mathcal{H}_{\omega'}$, respectively. These operators allow to define the unitary Radon–Nikodym derivatives

$$\frac{D\omega'}{D\omega}(\tau) := U(\tau)U^*(\tau) := \Delta_{\Omega,\Omega'}^{\uparrow\tau}\Delta_{\Omega',\Omega}^{-\uparrow\tau},$$

satisfying the cocycle conditions:

- Initial: $\frac{D\omega_1}{D\omega_2}(0) = \mathbb{I}$, chain: $\frac{D\omega_1}{D\omega_2}(\tau)\frac{D\omega_2}{D\omega_3}(\tau) = \frac{D\omega_1}{D\omega_3}(\tau)$,
- Intertwining: $\frac{D\omega_1}{D\omega_2}(\tau)(\sigma^w_\tau A) = (\sigma^w_\tau A)\frac{D\omega_1}{D\omega_2}(\tau)$, identity: $\frac{D\omega_1}{D\omega_2}(\tau_1 + \tau_2) = \frac{D\omega_1}{D\omega_2}(\tau_1)(\sigma^w_\tau\frac{D\omega_1}{D\omega_2}(\tau_1)).$

The inner automorphism of the algebra of observables is described through the action of $Ad_{\sigma^w_\tau(A)}$ as [67]:

$$\sigma^w_\tau(A) = U(\tau)\sigma^w(A)U^*(\tau).$$

Using the quantum histories thermodynamics ansatz $\tau = -s\beta^{-1} = -sV_t$, we can write the above equation as

$$\sigma^w_\tau(A) = e^{isV_t}\sigma^w(A)e^{-isV_t}.$$

This equation allows us to provide the complete description of the ‘kinematical evolution’ of the quantum system with the Liouville operator $V_t$. Note, that we introduce here the parameter $t \in \gamma$ only as an element of the path in the partially ordered set, which is used to index the changes of an algebraic state (interpreted as subsequent measurements). We do not assume here any interpretation of $t$ as time, nor any concrete relation between $t$ and $s$.

We want now to consider the expectation values of type $\langle \psi|A|\psi\rangle$, however, with the state which evolves kinematically. Let us remind first, that in the algebraic picture we consider

$$\langle A \rangle_{\psi} = \langle \psi|A|\psi\rangle = \langle \Omega|C^*AC|\Omega\rangle = \omega(C^*AC)$$

for $|\psi\rangle = \eta,(C)$. Presence of the kinematic modular automorphism evolution leads to consideration of $\omega(\sigma^w_\tau(A)B)$ instead of $\omega(AB)$, what implies that

$$\langle A \rangle_{\psi,\sigma^w_\tau} = \omega(\sigma^w_\tau(C^*AC)) = \omega(C^*AU^*(\tau)C).$$

Consider now the class of states $\{\omega_{\gamma}\}_{\gamma \in \mathbb{I}}$ and the changes between them described through the Radon–Nikodym cocycles. We can define the expectation value of an observable $A$ on the path $\gamma$ of measurements, beginning in $t_0$ and ending in $t_n$, as:

$$\langle A \rangle_{\psi,\gamma} := \langle A \rangle_{\omega_{\gamma}} := \int_{t_0}^{t_n} D\omega_t'(\frac{D\omega_t}{D\omega_t'}(A)), \quad (36)$$

where the integration is provided in terms of Stieltjes (one can think about this process as integration over the germs of subsequent algebraic states $\omega_t'$). Under the assumption of fixed state $\omega$ for all $t$ on $\gamma$, we obtain

$$\langle A \rangle_{\psi,\gamma} = \int D\omega_t'(\frac{D\omega_t}{D\omega_t'}(C^*AC)) = \int D\omega_t'(\sigma^w_\omega(C^*AC)) = \omega(\sigma^w(\tau)C) = \langle \psi|Ae^{-is\int dsV_t}|\psi\rangle, \quad (37)$$
where the operator $e^{-i \int ds K_V}$ was obtained through the limit
\[ \prod U^*(dt) = \prod e^{iK_{dt}} = e^{-i \int ds K_V}. \]

If we assume (i) the statistical independence between an observable $A$ and the quantum geometric phase, and (ii) that $\omega_{\psi,t}$ are homomorphisms of $C^*$-algebras, than this result may be presented as:
\[ \langle A \rangle_\psi|_\gamma = \langle A \rangle e^{i\theta(\gamma)}. \]  

Now consider a quantum system with a given Hamiltonian internal evolution
\[ a^H_s(A) = e^{-i \int ds H(s)} A e^{i \int ds H(s)}. \]

Using the definition (36), we write
\[ \langle a^H_s(A) \rangle_\gamma = \int_{t_0}^{t_\gamma} D\omega_t^s (\frac{D\omega_t^s}{D\omega_t^s}(a^H_s(A))). \]

Under the assumption of fixed state $\omega_t$ for all $t$ on $\gamma$ and the ‘adiabatic condition’ $e^{i \int ds H(s)} C|\Omega\rangle = Ce^{i \int ds H(s)}|\Omega\rangle$, we obtain
\[ \langle C | \sigma_s^V (a^H_s(A)C) \rangle_\gamma = \langle \Omega | C^* e^{i \int ds (K_V-H(s))} AC | \Omega \rangle = \langle C | e^{iS_s} A | C \rangle. \]

For $C = \mathbb{I}$, $K = I$, and fixed algebraic state $\omega_s$ for all $t \in \gamma$, this equation, integrated over all possible $\gamma$ with Gaussian measure over the space of paths, provides the expectation value of an observable $A$ described in terms of the Lorentzian path-integral approach to quantum theory:
\[ \langle a^H_s(A) \rangle = \int d\mu(\gamma) \langle a^H_s(A) \rangle_\Omega |_\gamma = \langle \Omega | \int d\mu(\gamma) e^{iS_s(\gamma)} A(\gamma) | \Omega \rangle. \]  

This means that the path-integral Lagrangean method derives from our approach in the concrete case, when we assume several simplifying conditions, and ask a question not about a given observer, but a Gaussian average over all possible observers. Consider now the Kulback–Leibler divergence [43] of the Radon–Nikodym cocycle, given by
\[ S_{KL}(\omega' || \omega) := - \int D\omega' \log \frac{D\omega}{D\omega'}. \]

Direct calculation shows that, in the simple case of fixed state $\omega_t$ for all $t \in \gamma$,
\[ S_{KL}(\omega' || \omega)(a^H_s(\mathbb{I}))_\psi |_\gamma = -\langle C | \log(\sigma_s^V (a^H_s(\mathbb{I})C)) \rangle = (\langle C | -iS_s | C \rangle = \langle -iS_s | \psi \rangle |_\gamma, \]
which means that the classical action functional is equal to the Kulback–Leibler divergence of unitary evolution operator (in the Lorentzian picture, and modulo the $-i$ factor, to be discussed in the section 4.5). This result is striking, and now we will show why it is.

### 4.4 Relational and contextual setting of quantum theory

Recall that the algebraic state $\omega$ should be interpreted as (epistemic) probabilistic measure on (to some extent ontological) $*$-algebra of observables. It provides a definition and enables the calculation of an expectation value (probability) of an observable:
\[ p(\omega)[A] = \omega(A). \]
However, if we consider the change of algebraic state caused by the external time evolution of observer in subsequent events of measurement, we have to use the relative probability

\[ p(\omega||\omega')[A] = \int D\omega' \frac{D\omega}{D\omega'}(A). \]

So, due to the fact that \( \omega \) and \( \omega' \) are the information measures, the Kullback-Leibler divergence measures the entropy of the change of information measure given by the Radon-Nikodym derivative \( \frac{D\omega'}{D\omega} \). Consider now two definitions of informational entropy:

- (discrete) Shannon entropy [65]: \( S_{Sh}(p) := -\sum_i p_i \log p_i \),
- (continuous) Kullback-Leibler entropy [43]: \( S_{KL}(p||q) := \int dp \log \frac{p(x)}{q(x)} \).

Shannon entropy is a special case of the Kullback-Leibler entropy, obtained under the assumptions of discretisation of probability distributions, and the equal a priori probability, which implies the uniform distribution of \( q(x) \). The last assumption means that the Shannon entropy describes the ‘contextless’ approach to information. However, if we will not assume the equal a priori probabilities, and consequently use the Kullback-Leibler entropy instead of Shannon, we can describe information as contextual property – that is, dependent on the realms of reference. In physics this reference is always given by the preparation of state, prior to the measurement. This critique of Shannon entropy equally applies to its single-measurement quantum mechanical analogue given by the von Neumann entropy \( tr_H(\rho \log \rho) \), also based on the assumption of equal a priori probability (which holds only for unitary evolution without measurements). Single-time quantum mechanics has Markov property: the probability distribution of the future states of the process does not depend on the past states, but only on a current one. But if we consider the multi-time histories perspective on quantum theory, the Markovian property is lost. This is reflected in the shift from Shannon to Kullback-Leibler description of the information about the system. Stating this from the another point of view, note, that due to Liouville theorem in a case of Hamiltonian dynamics (that is, with \( V = 0 \)), the volume of the subspace of the classical phase space is constant along the evolution trajectory, so the informational Shannon entropy has to be constant. However, in the case of system in interaction with an observer (that is, when \( V \neq 0 \)), the act of observation (external interaction) introduces the essential change in the value of the entropy. In the quantum setting this situation reappears in the fact that the Hamiltonian evolution is totally deterministic and von Neumann entropy is constant. The indeterminity and the change of von Neumann entropy due to the essential change of the state is introduced only by the act of measurement.

The Kullback-Leibler entropy satisfies the following properties:

1. \( S_{KL}(p||q) \geq 0 \),
2. \( S_{KL}(p||q) = 0 \iff p = q \),
3. \( S_{KL}(p||q) \neq D_{KL}(q||p) \),
4. \( S_{KL}(p||q) = D_{cr}(p, q) - D_{Sh}(p) \),

where \( S_{cr}(p, q) \) is called the cross-entropy of \( p \) and \( q \). These properties allow to understand the Kullback-Leibler divergence as relational entropy. One may think of it as a general description of change of information during the relational time evolution of informational measures, which (contrary to Shannon entropy) is well defined in discrete as well as continuous cases and describes the actual knowledge as dependent on the previous one. Moreover, contrary to Shannon entropy, Kullback-Leibler entropy is invariant under parameter transformations. This is important, because it happens often that one measures not the observable \( A \), but some function of it, like \( A^2 \) or \( \cos A \) (later we will see, that this property
is crucial when one considers the problem of renormalisation). Other notions of information theory, like self and mutual information as well as conditional entropy may be defined as special application or cases of the Kullback-Leibler entropy.

From the perspective of the Bayesian approach to statistics, the Kullback-Leibler entropy provides the natural measure of the input of information obtained by updating the knowledge from a priori probability distribution \( p(x|I) \) to a posteriori probability distribution \( p(x|y) \) due to discovery of the fact \( y \). This updating is described by the Bayes theorem

\[
p(x|y) = p(y|x) \frac{p(x|I)}{p(y|I)}. \tag{43}
\]

The input of (useful) information about \( x \) obtained through the discovery of \( y \) is given by

\[
S_{KL}(p(x|y)||p(x|I)) = \sum_y p(x|y) \log \frac{p(x|y)}{p(x|I)}. \tag{44}
\]

Comparing (44) with (41) we see that our algebraic description (36) of the expectation value of the observable, obtained through the histories generalization of the algebraic approach to quantum theory, describes the Bayesian change of knowledge of the observer. In particular, the classical action functional describes the Bayesian updating of the knowledge about unitary quantum evolution.

Edwin Jaynes has shown [37] that the Gaussian Kolmogorov theory of probabilistic inference is only an approximation of exact methods provided by the Bayesian probabilistic inference. Our histories generalisation of the algebraic framework for quantum theory shows that quantum probabilities are genuinely Bayesian. The Gaussian sampling used in the Euclidean path-integral approach is only a best ‘uniform distribution of a priori ignorance’ approximation to Bayesian description of relational and contextual change of the knowledge of observer, described in the quantum histories thermodynamics setting by the Kullback-Leibler entropy of algebraic states. Jaynes has also shown [35] that the equilibrium thermodynamical description of the systems may be recovered from the Shannon entropy by using the principle of maximisation of Shannon entropy. If we consider now the relation between the Shannon entropy and Kullback-Leibler entropy given by \( S_{KL}(p||q) = -S_{Sh}(p) + S_{eq}(p,q) \), we may postulate the principle of minimalisation of the Kullback-Leibler entropy (called minimum discrimination information principle) as the rule of recovering of the most probable equilibrium states. The principle of minimalisation of \( S_{KL} \) means that we minimize our estimate of information gain which we have learned in subsequent measurements. For our purposes we can restate this principle in form: observer choses such trajectory \( \gamma \) that minimizes his Kullback-Leibler entropy.

4.5 Infinitesimal Lorentz symmetry

4.5.1 Quantum histories thermodynamics entropy

Consider now Jaynes’ definition [35] of a thermodynamical entropy based on the Shannon information entropy [65]:

\[
S_{\text{termo}} := k_B S_{Sh}. \tag{45}
\]

The quantum histories thermodynamics ansatz \( k_B T \sim t_p V_t / \hbar \) leads us, in the scope of the above considerations, to the definition of the quantum histories thermodynamic entropy:

\[
S_{\text{qht}} := t_p S_{KL}. \tag{46}
\]

In the simple case of fixed algebraic state \( \omega_t \) along the path \( \gamma = [t_0, t_n] \) and a unital modular Hamiltonian, we have

\[
S_{\text{qht}}(\psi|\omega_{t_\gamma}, \gamma) = t_p S_{KL}(\psi|\omega_{t_\gamma}, \gamma) = -it_p \int_{\gamma} d\sigma \hbar^{-1} \langle C|V_t|C\rangle t_{\sigma}, \quad \psi = \eta_{\omega_t}(C).
\]
Due to the fact that $S_{KL}$ is a number without units, the quantum histories thermodynamic entropy has units of time, what enables us to write it as

$$S_{\text{qht}}(\Gamma)\psi|_{\omega_{\gamma}} = -i \int_{\gamma} ds \left( \frac{t_p}{\hbar} (C|V_t|C)_{t \in \gamma} \right) = \int_{\gamma} dt \left( \frac{t_p}{\hbar} V_t \right)_{t \in \gamma}. \quad (47)$$

This equation provides the relation between the external and internal time:

$$dt = -ids \iff s = it, \quad (48)$$

and the general description of the geometric phase derived from quantum histories thermodynamics ansatz

$$e^{i\theta(\gamma)} = e^{S_{\text{qht}}(\Gamma)\psi|_{\omega_{\gamma}}} = e^{t_p S_{KL}(\Gamma)\psi|_{\omega_{\gamma}}}, \quad (49)$$

which, contrary to the equations (35) and (38), is independent from any extra assumptions. The principle of minimal Kullback–Leibler entropy implies that the most probable are the paths $\gamma$ that minimize the value of $\int_{\gamma} dt \left( \frac{t_p}{\hbar} V_t \right)_{t \in \gamma}$. Note now that in the section 4.3 we have defined

$$S_s := \int ds (K_{V_t} - H).$$

If we will define, in order to obtain consistency with (13) and (25),

$$S_t := \int dt (K_{V_t} - H),$$

then

$$iS_s = -S_t, \quad (50)$$

and we can write (42) as

$$S_{KL}(\omega'||\omega) = (e^H(\Gamma))\psi|_{\gamma} = \langle S_t \rangle \psi|_{\gamma}. \quad (51)$$

This way, using the Kullback–Leibler informational entropy and relative modular automorphism of algebraic states satisfying the KMS equilibrium condition with the $\beta^{-1} = t_p V_t / \hbar$ ansatz, we have recovered not only the geometric phase, but also the expectation value of the quantum histories action operator in the external-time picture. It means that the classical action functional represents the Bayesian updating of knowledge about the unitary quantum evolution of the system. Moreover, the principle of minimization of the Kullback–Leibler entropy says us that the paths which minimize the value of the classical action functional are the most probable ones for the quantum histories thermodynamics equilibrium. Note that we have used here the Kullback–Leibler entropy in two different ways: in order to establish the notion of quantum histories thermodynamical entropy in the units of time (by evaluating $S_{KL}$ on the unit operator) and in order to recover the classical action operator (by evaluating $S_{KL}$ on the unitary Hamiltonian evolution of the unit operator). The first application concerns only the inference about the kinematical properties of the system, while the second refers to updating of the information about full behaviour of the system and observer. Now we are able to restore the quantum histories thermodynamics propositions of the section 4.1. However, with one important difference: the quantum histories thermodynamical entropy is now given not by $\int dt$ only, but it is rescaled by the real numerical factor $\langle \frac{t_p}{\hbar} V_t \rangle$, which is constant for fixed value of $V_t$. We will discuss this difference more carefully in the section 4.5.4.

### 4.5.2 Quantum Liouville equation

The classical probability measure $\rho_{cl}$ on the phase space (used to found statistical mechanics on the ontological base of classical mechanics) is purely epistemical construct – it does not describe the internal properties (structure) of the system under observation (like the internal Hamiltonian dynamics), but
the lack of the knowledge, which is a property of an observer. Consequently, the thermodynamical state functions (defined through the thermodynamic limit of certain statistical quantities build upon the Liouville equation on $\rho_{cl}$) refer to the state of knowledge of observer, and not only to the state of a system. In particular, the thermodynamical entropy (45) reflects only the state of knowledge of an observer. Consequently, $V_t$ has to be understood as a quantum histories temperature of observer. The action operator joins together the energetic internal dynamics of the system and the entropic external dynamics of the observer, providing the full description of the behaviour of the system. Its expectation value, the classical action functional, is equal to the Kullback-Leibler informational entropy of knowledge about unitary evolution.

These considerations suggest that the Liouville equation (27) for the probability measure $\rho_{cl}$

$$\frac{\partial}{\partial t} \rho_{cl} = - V \rho_{cl}$$

has its quantum analogue, given, in the case of fixed representation for all $t \in \gamma$, by:

$$\frac{\partial}{\partial t} \rho_t = - [KV_t, \rho_t],$$

where $t$ is the external time, $V_t$ is the quantum Liouville operator, $K$ is the modular Hamiltonian, while $\rho_t$ is the representative of the changing algebraic state $\omega_t \in \gamma$. If the representative of $\omega_t$ is a pure state, then (53) can be written in the form

$$\frac{\partial}{\partial t} \psi = - KV_t \psi.$$  (54)

This equation describes the kinematical evolution of the system which is caused by the change of the quantum and relativistic frame of reference of observer during subsequent measurements. The Liouville operator $V_t$ encodes the change of the Hilbert space structure along the external time trajectory of the system, while the modular Hamiltonian describes the change of the orientation of the orthonormal tetrad (Lorentzian rotations and boosts)\(^\text{10}\). What is the actual content of the operator $V_t$ in the case of Schrödinger representation? $V_t$ is defined as $p_t \dot{q}_t$, where $\dot{q}_t = \frac{\partial}{\partial s} q_t$ and $s$ denotes the internal time. In the given representation (and for $\hbar = 1$) we have $p_t = - i \frac{\partial}{\partial q_t}$ and $q_t = q_t$, so $V_t = - i \frac{\partial}{\partial s}$. The equation (54) implies that, for $K = \mathbb{I}$,

$$\frac{\partial}{\partial t} = i \frac{\partial}{\partial s} = \frac{\partial}{\partial(-is)},$$

Hence, in the Schrödinger representation the relation between the external and internal time reads:

$$s = it,$$  (56)

what confirms the ‘thermodynamic’ result (48) on the ‘stochastic’ grounds. This means that (54) may be understood as the external time version of the equation (34).

The quantum histories formalism leads us then to consideration of the ‘purely stochastical’ quantum analogue (54) of the Liouville equation on the equal footing with the ‘purely dynamical’ Schrödinger equation (1)

$$\frac{\partial}{\partial s} \psi = -iH \psi,$$  (57)

which only if taken together lead to the predictive power of quantum theory\(^\text{11}\) (and, as we will see in the next subsection, to the explanation of the appearance of the infinitesimal Lorentz symmetry in quantum

\(^{10}\)We will explain such interpretation of the role of modular Hamiltonian in the section 4.7.

\(^{11}\)This is contrary to the attempts of [90] and [4], who tended to interpret Schrödinger equation as a quantum analogue of the Fokker-Planck equation. However, we have to note that, despite contrary views, we were actually inspired by this approach.
4.5. Infinitesimal Lorentz symmetry

field theory and general relativity). Recall that in the quantum mechanics the stochastic aspect of the theory is introduced by the measurement and not by the Schrödinger equation. The equation (54), generated by the Liouville operator acting between Hilbert spaces of two subsequent measurements, encodes the full information about this probabilistic contents of quantum epistemology. But why one needs the Hilbert spaces of two measurements? The answer is already contained in the Dirac’s bra-ket formalism: to obtain any expectation value one has first to prepare the state, and to measure it later. The operations of preparing and measuring are causally ordered, hence are different events, and lead to two different Hilbert spaces. The equation (54) provides a translation between these spaces (one may say that this equation provides a change of context). So, the general description of the quantum behaviour is provided by the action of quantum Liouville (54) and Schrödinger (57) equations taken together:

$$\frac{\partial}{\partial s} \psi(s) = i(KV_t - H)\psi(s),$$

(58)

what leads to solution

$$\psi(s) = e^{i\int ds(KV_t - H)}\psi(0) = e^{iS_s}\psi(0).$$

For the general algebraic states this description is given by the equation (39):

$$\langle \sigma_{s_{\gamma}}^{\gamma}(\alpha^H(A)) \psi \rangle_{\gamma} = e^{iS_s}A \psi.$$  

One can understand the Hamiltonian component $\alpha^H$ of the measured behaviour as internal evolution of the system, and the Liouvillian component $\sigma_{s_{\gamma}}^{\gamma}$ as external evolution of the observer.

We should note that it is commonly assumed that the quantum version of the classical Liouville equation (27) is given by the von Neumann master equation

$$\frac{\partial}{\partial s} \rho = -i[H, \rho],$$

(59)

where $\rho$ is a density matrix (this equation is a special case of the Heisenberg evolution equation $\frac{\partial}{\partial s} A = -i[H, A]$). However, as observed by Chris Isham [31], this law of time evolution of a density matrix is correct only if we assume that the time evolution preserves the decomposition of $\rho$ into the affine sum of the projection operators of type $|\psi\rangle\langle\psi|$. Such definition of change in time is not capable to describe the ‘reduction of the state vector’ (or density matrix) which happens at the time of measurement (and which may be in general described by the change of algebraic state $\omega$ on the algebra of observables or its representatives in a given fixed representation). This means that in general we have to consider the equation (53) on the equal footing with the von Neumann equation. One should note that these equations have different nature. Equation (53) refers to the external time and describes the change of states in the subsequent measurements (hence has a stochastic character), while the equation (59) refers to the internal time and describes the evolution of state in a Hilbert space of single measurement (hence has a dynamical character).

4.5.3 The Minkowski–Wick trick

The equation $s = it$ explains what is the physical nature of the ‘Minkowski–Wick trick’ $\tau \rightarrow i\tau$, which is widely used in the Lagrangian approach to QFT, but was never justified physically. Due to Osterwalder–Schrader theorem [53], [54], the Minkowski–Wick trick transformed $T$-functions $T_n(\tau_1, \bar{x}_1; \ldots; \tau_n, \bar{x}_n)$, which appear from the expansion of the Feynman Lorentzian path-integral, are equivalent with the Schwinger functions $S_n(\tau_1, \bar{x}_1; \ldots; \tau_n, \bar{x}_n)$, appearing from the expansion of the Euclidean path-integral:

$$S_n(\tau_1, \bar{x}_1; \ldots; \tau_n, \bar{x}_n) = T_n(i\tau_1, \bar{x}_1; \ldots; i\tau_n, \bar{x}_n).$$

As a consequence, the description in terms of the Lorentzian path-integral

$$\int d\mu(\gamma, \phi) e^{i\int d\tau L_t(\gamma, \phi)} A(\gamma, \phi)$$

(60)
is transformed to the equivalent description in terms of the Euclidean path-integral

$$\int d\mu^e(\gamma, \phi)e^{-\int dt L^e(\gamma, \phi)}A(\gamma, \phi),$$

(61)

and the results of Euclidean path-integrals are mathematically consistent with the results of Lorentzian path-integrals. However, the physical meaning of this procedure was lacking (‘what the imaginary time means?’). From our perspective, this ‘trick’ in fact provides a transition $s = it$ from the internal time $s$ to the external one $t$. So, the mystery dissapears, and we can write:

$$S_n(t_1, \vec{x}_1; \ldots; t_n, \vec{x}_n) = T_n(it_1, \vec{x}_1; \ldots; it_n, \vec{x}_n) = T_n(s_1, \vec{x}_1; \ldots; s_n, \vec{x}_n)$$

and

$$e^{\frac{i}{\hbar}\int ds L^e(\gamma, \phi)} = e^{-\int dt L^e(\gamma, \phi)}$$

Note also that, while the conservation of the time-dependent Noether charges is present in Minkowski space-time, it is absent in the Euclidean space-time [45]. So, the conservation laws for physical quantities lose their derivation in the Euclidean domain. However, if we consider path-integrals in the Euclidean domain as stochastic description in the external time provided by an external observer, we do not have to be worried that the infinitesimal internal symmetry which implies conservation of the internal properties of the system is absent outside the system. In the presence of the $s = it$ equation, the local (or infinitesimal) conservation of Noether charges in Lorentzian framework suffices to provide the physical sense to quantities evaluated through the Euclidean stochastic integrals. This enables us to call the external time $t$ as stochastic time and the internal one as quantum time.

Using the equation $s = it$, we may explain now also the Lorentzian nature of the Minkowski metric of the special-relativistic space-time,

$$g_{\mu\nu}x^\mu x^\nu = -(cds)^2 + (d\vec{x})^2,$$

as a result of the equality $t = -is$ and the external description in terms of the Euclidean metric

$$g_{\mu\nu}x^\mu x^\nu = (cdt)^2 + (d\vec{x})^2.$$ 

This is striking, because it implies that the Lorentz symmetry is an intrinsic property of the dynamics of the system, obeyed by a $\gamma$-algebra of observables represented in a given single measurement, but it is not the property of the external time $t$! This explains how to understand the fact that the algebraic observables of QFT, representing the single-time measurements, are Lorentz-invariant and that the same is true for GR, but only infinitesimally. This situation becomes self-consistent if we attribute Lorentz-invariance only to observables in an internal time, while interpreting the curved-geometry structure of GR as an external and kinematical property.

### 4.5.4 The third law

Now we would like to turn back to the question of geometric (kinematic) properties of the Hilbert space. The inner product on $\mathcal{H}$ induces on $\mathcal{P}\mathcal{H}$ not only a connection $i\langle \psi|d\psi\rangle = -\langle \psi|V_t|\psi\rangle$, but also a metric, given by

$$ds^2 = (d\psi|d\bar{\psi}) - (\langle \psi|d\bar{\psi}\rangle)^2 = (\psi|V^2_t|\psi) - (\langle \psi|V_t|\psi\rangle)^2 = (\langle \psi|V_t - (V_t)|\psi\rangle)^2 = (\Delta V_t)^2,$$

what in coordinate-dependent form may be written as $g_{ij}(\psi)|\psi\rangle\langle \psi|\psi\rangle$ (the space of rays $\mathcal{P}\mathcal{H}$ may be considered as the Kähler manifold). Anastopoulos and Savvidou have shown [6] that when one will take under consideration also the metric on $\mathcal{P}\mathcal{H}$, then the decoherence functional for histories of coherent states is given by

$$d(\alpha, \alpha) = e^{-\int_0^t d\tau (\psi|d\psi\rangle - \frac{1}{\hbar}\int_0^t d\phi \gamma_i(\psi)|\psi\rangle\langle \psi|\psi\rangle - i\int_0^t ds \mathcal{H}),}$$
where $\nu^{-1}$ is a constant with dimensions of (internal) time. For $H = 0$ and $\nu^{-1} \to 0$ this expression gives the quantum geometric phase. The analogous result was obtained earlier by Klauder [39], who showed that the coherent state propagator is given by

$$
\langle \psi|e^{-iHt}|\psi'\rangle = \lim_{\nu^{-1} \to 0} \int D\psi'() e^{i\nu s} - \int s d\psi'() - \frac{1}{2\nu} \int s dG_{\nu}() \psi^i() - \int s dH.
$$

We will now assume that $\nu^{-1} = it_p$. Then, for $H = 0$, one can write

$$
d(\alpha, \alpha) = e^{-i\int d\psi(-1\langle\psi|V_i|\psi\rangle + \frac{t_p}{2\hbar^2}(V_i)^2)}.
$$

(62)

If we consider now

$$
S_{qht} = \int dS_{qht} = t_p \log(e^{-i\int d\psi(-1\langle\psi|V_i|\psi\rangle + \frac{t_p}{2\hbar^2}(V_i)^2)}) = -t_p \int d\psi(-1\langle\psi|V_i|\psi\rangle + \frac{t_p}{2\hbar^2}(V_i)^2),
$$

then we obtain

$$
\lim_{\langle V_i \rangle \to 0} S_{qht} = t_p \cdot \lim_{\langle V_i \rangle \to 0} \frac{t_p}{2\hbar^2} \int d\psi(-1\langle\psi|V_i|\psi\rangle + \frac{t_p}{2\hbar^2}(V_i)^2) = t_p \cdot \log g_0,
$$

(63)

where

$$
g_0 := \exp(-\frac{it_p}{2\hbar^2} \int d\psi(V_i^2)) = \exp\left(\frac{1}{2it_p} \int d\psi(-1\langle\psi|V_i|\psi\rangle + \frac{t_p}{2\hbar^2}(V_i)^2)\right).
$$

(64)

This proves our conjecture (29). However, it remains still an open question, how to describe this result using algebraic formalism. One possible way is to extend the quantum Liouville equation (54) into the Fokker-Planck form, which after integration and taking the expectation value would lead to (62). The extra term would be play the role of heat diffusion, with the Planck time $t_p$ (transformed into internal time units via $s = it$) playing the role of diffusion coefficient (a similar interpretation was given in [6]). This actually suggests consideration of some Wiener process which may serve as a source of this description. Let us take then two stochastic processes $u(\tau)$ and $X_\tau$, described by the Ito stochastic differential equation

$$
dX_\tau = u(\tau)dt + dW_\tau,
$$

where $dW_\tau$ is a differential of Brownian motion (the white noise). The change of probability measure $\mu$ induced by the change of time $\tau$ may be described in terms of Radon-Nikodym derivatives in the following way [68]:

$$
\frac{d\mu'}{d\mu} = \exp\left(\int_{\gamma} dX_\tau u(\tau) - \frac{1}{2} \int_{\gamma} d\tau ||u(\tau)||^2\right).
$$

(65)

If we substiute now

$$
dX_\tau = dt/t_p, \quad u(\tau) = V_i t_p/\hbar, \quad ||u(\tau)||^2 = (V_i^2)/t_p^2/\hbar^2, \quad \tau = -t/t_p,
$$

what means that

$$
-\frac{t_p}{\hbar} V_i = 1 + \frac{dW_\tau}{d\tau},
$$

(66)

then we obtain

$$
\frac{d\mu'}{d\mu} = \exp\left(\frac{1}{\hbar} \int_{\gamma} dt V_i + \frac{t_p}{2\hbar^2} \int_{\gamma} dt (V_i^2)\right).
$$

This means that

$$
S_{qht} = t_p \log \frac{d\mu'}{d\mu} = \int_{\gamma} dt \frac{t_p}{\hbar} \left(V_i + \frac{t_p}{2\hbar} V_i^2\right),
$$

hence, for $dt = -ids$,

$$
\lim_{\langle V_i \rangle \to 0} S_{qht} = t_p \log g_0, \quad g_0 = e^{-\frac{it_p}{2\hbar^2} \int_{\gamma} ds(V_i^2)},
$$

(67)
what agrees with the result (63-64). One should note that \((\Delta V_t)^2\) represents the uncertainty of the observer about the value of an observable quantity \(V_t\), but it does not mean that \(V_t\) has to fluctuate! So, the description (66) of \(V_t\) in terms of the Wiener process driven by the white noise does not refer to some ontological reality, but only to uncertainty of observer’s description about the actual value of \(V_t\), which is caused by the fact that \(V_t\) is not an observable in any Hilbert space of single-time measurement. This recovers our arguments from the section 4.1 about the nature of quantum histories thermodynamical entropy; with the only difference, that this role is played not by the stochastic time \(t\) but by the entropic time \(S_{qht}\), which is dependent on the observer’s path \(\gamma\). This means that while the quantum time \(s\) is continuous, the discretisation appears on the grounds of the external time and for specified observer. This discretisation of the external entropic time is the result of presence of the metric on the quantum phase space (space of rays), induced by the inner product on \(H\), which is generated by the algebraic state of an observer. The equation (65) shows that crucial for the discretisation of \(S_{qht}\), is the path along which observer measures the changes of algebraic states, described in terms of Radon-Nikodym derivatives. Hence, the discretisation of external time reflects the structure of the sheaf of germs of algebraic states which is selected by an observer in subsequent measurements. This means that the relation between \(t\) and \(S_{qht}\) essentially depends on the trajectory \(\gamma\) and the properties of observer encoded in the selected sheaf of germs of algebraic states \(\omega_{t,\gamma}\). But the discretisation of an external time of observation does not imply that the internal time of dynamics of process is quantized. In other words, the fact that one cannot measure the time \(t\) less than \(t_p \log q_0\) does not implies that the process cannot undergo its own changes in \(s < it_p \log q_0\). This implies that, for \(q_0 > 1\), the quantum Zeno and anti-Zeno paradoxes cannot appear: there is always some internal non-measured amount of dynamics present ‘between’ the measurements.

The difference between two external times: the ‘stochastic’ time \(t\) and the ‘entropic’ time \(S_{qht}\) is very similar to the difference between coordinate and the proper time in GR. Let us first notice that the Lorentz time \(dx^{(0)}\) of GR is the internal time \(ds\) of quantum histories thermodynamics: both are used in order to describe the infinitesimally Lorentz-invariant properties, and to serve as an internal infinitesimal clock of the process. In GR there is a concrete fixed rule of transformation of description of time from the internal Lorentzian terms \(dx^{(0)}\) to external space-time coordinate terms \(dx^0\):

\[
\begin{align*}
 dx^{(0)} &= h_\mu^{(0)} dx^\mu = h_0^{(0)} dx^0 + h_i^{(0)} dx^i,
\end{align*}
\]

where \(h_\mu^{(0)}\) is an orthonormal Lorentz tetrad field, \(\mu\) is a space-time coordinate index, and \((\alpha)\) is Lorentzian coordinate index. This is analogous to the rule

\[
ds = idt.
\]

However, there is also another notion of an external time in general relativity, which depends on the properties of observer. This is a proper time \(dt\), which can be expressed in terms of the coordinate-external or Lorentz-internal times only after fixing the trajectory on the manifold \(M\) and fixing the gravitational gauge. In general quantum theory the analogous role is played by the external-entropic time \(dS_{qht}\), because in order to specify its value in terms of external-stochastic or internal-Lorentzian time, one has to fix the trajectory \(\gamma\) on the Kähler manifold \(\mathbb{P}H\) and the ‘quantum gauge’, given by the selection of the sheaf of germs of algebraic states \(\omega_{t,\gamma}\) on that trajectory. Fixing of the gauge, both in general relativity and general quantum theory, provides the specification of the properties of observer. This way the external times \(dt\) and \(dS_{qht}\) have to be considered as the ‘time of observer’, while the external times \(dx^0\) and \(dt\) only as an abstract translation from the description in the inner terms of process to the external terms of all possible (unspecified) observers. In effect, the equation

\[
dS_{qht} = \frac{t_p}{\hbar} (\langle V_i \rangle + g_{ij} \dot{\psi}^i \dot{\psi}^j) dt
\]

12Roughly speaking, quantum Zeno paradox [18] states, that when one continuously measures if the system has performed a transition to another state, it never performs this transition. The quantum anti-Zeno paradox [19] states, that when one continuously measures if the system follows some given path, it always follows this path.
4.6 Quantum behaviour vs quantum dynamics

should be considered as quantum analogue of the general-relativistic equation translating between $d\tau$ and $dx^0$, like, for example,

$$d\tau = \frac{1}{c} \left( \sqrt{-g_{00}} dx^0 + \sqrt{-g_{0i}} dx^i \right).$$

Hence, the relation between $S_{qht}$ and $s$ is different for every trajectory of sheaf of algebraic states, and depends on Liouvillean. The Liouville operator determines then, in every point of the space-time, the observed ‘deformation’ of the Wick rotation and the ‘deviation’ of the metric from its Minkowski form. One can still choose a Schrödinger representation in every selected point (and can describe the metric in Minkowski form), but in order to describe the full space-time quantum behaviour of the system in terms of the fixed observer, one has to relate the measurements obtained in different points, what leads to the consideration of the changes of GNS representation. The properties of the space-time geometry originate then from the same source as the properties of the quantum observables.

4.6 Quantum behaviour vs quantum dynamics

Now we can finally turn to the problem of general description of the behaviour of systems in the quantum field theory. It is widely recognized that the algebraic approach to QFT, while having important virtues in the description of the particular observables and vacuum structure of the free Lorentz-invariant quantum field, as well as being mathematically well-founded, is (in its present form) incapable to provide the effective description of the observed behaviour of quantum fields. In order to describe the behaviour of the quantum phenomena, the Lagrangean path-integral methods are used. These methods (after renormalisation) lead to verifiable predictions, however their relation with the canonical quantization is not well understood beyond the realms of the ordinary nonrelativistic quantum mechanics, and there is still no well-paced mathematical foundations for functional integration [15]. It seems that both these approaches catch some important features of QFT, but none of them is complete and capable to express all meaningful contents of the theory. In this subsection we would like to analyze the problem of description of the behaviour of QFT systems, in order to obtain some new insight on the relation between Hamiltonian and Lagrangean approaches to QFT and – generally – on the essential structure of QFT.

4.6.1 The general quantum theory

First, we will sum up the theory developed in the previous sections. In order to establish the rigorous algebraic formulation of the quantum histories thermodynamics proposition, we have moved from the description of quantum histories statistics in terms of decoherence functional $d = tr_V(\otimes_{t\in\gamma} \alpha^H_t(\rho_{A_t}))$ defined over the histories Hilbert space $V = \otimes_{t\in\gamma} \mathcal{H}_t$ to the algebraic description in terms of expectation value of an observable evaluated along the trajectory of sheaf of germs of algebraic states by the Stieltjes integration of the Radon-Nikodym co cycle

$$\langle A \rangle_{\Omega|\gamma} := \int_{\gamma} D\omega' \frac{D\omega}{D\omega'} (\alpha^H_s(A)).$$

So, the histories Hilbert space became replaced by the sheaf of germs of algebraic states, while the decoherence functional became replaced by the integration over KMS modular automorphisms of unitary evolving observable. Such description of quantum probability enabled us to introduce the Kullback-Leibler informational entropy as the quantity which encodes the information of observer about the system. This entropy, defined as

$$S_{KL}(A) := - \int_{\gamma} D\omega' (\log \frac{D\omega}{D\omega'} (A)) = - \langle \log \frac{D\omega}{D\omega'} (A) \rangle_{\omega',\gamma},$$

generalizes the Araki [7] relative entropy $tr(\rho_1 \log \frac{\rho_1}{\rho_2})$ and plays in general quantum theory the role completely analogous to the role played by the Shannon information entropy in the foundations of
statistical classical mechanics. We have shown that the classical action functional can be derived from the general quantum theory as the Kullback–Leibler entropy of the unitary evolving algebra of observables. Moreover, the classical principle of minimal action is then a straight consequence of the principle of minimization of Kullback–Leibler entropy of algebraic states, which is generalized version of Jaynes’ principle of maximization of Shannon and von Neumann entropies. This way we have achieved the situation in which the principles of quantum theory are sufficient to recover full classical theory. Moreover, this general quantum theory is independent from the Hilbert space structure: the baseing ingredients of the theory are the sheaf of algebraic states and the algebra of observables \( \mathcal{A} \), together with two automorphisms of \( \mathcal{A} \) (Hamiltonian and Liouvillean), which encode the dynamics of \( \mathcal{A} \), and the kinematics of \( \omega_\gamma \), respectively. We have recovered also both elements of present quantum theory which are vague in its predictive success, but are, on the other hand, beyond the frames of ordinary algebraic (local quantum) formalism: quantum geometric phase and Lagrangean path-integral. The former is defined via the Kullback–Leibler entropy, while the latter is defined as the Gaussian average of the full (Hamiltonian plus Liouvillean) automorphism of the observable over all possible observers compatible with the initial and final algebraic states.

### 4.6.2 Quantum histories thermodynamics

This framework enables us to define quantum histories thermodynamics on the base of quantum histories statistics. The elementary quantities of the former theory are the quantum histories temperature and quantum histories (thermodynamic) entropy, defined respectively as

\[
\beta^{-1} := \frac{1}{\hbar} V_t \quad \text{and} \quad S_{qht} := t_p S_{KL},
\]

where \( V_t \) is a Liouville operator. The laws of quantum histories thermodynamics are then as follows:

- **Zeroth:** For a system in a quantum histories thermodynamics equilibrium the sheaf of germs of algebraic states satisfies the KMS equilibrium condition with temperature \( \beta^{-1} = \frac{1}{\hbar} V_t \).

- **First:** The quantum analogue of the Helmholtz equation reads:

\[
dS' := -dS = dH - V_t dt = dH - \frac{V_t}{(V_t) - \frac{1}{2} (\Delta V_t)^2} dS_{qht} = dH - V_{\text{eff}} dS_{qht},
\]

(68)

where \( V_{\text{eff}} \) should be understood as an effective Liouville operator which will be observed by an observer providing the measurements not in terms of an abstract external time \( dt \), but in the entropic (informational) time \( dS_{qht} \).

- **Second:**

\[
S_{qht}(\omega_t || \omega_0) = t_p S_{KL}(\omega_t || \omega_0) \geq 0,
\]

(69)

what follows straightly from the definition of \( S_{KL} \).

- **Third:**

\[
\lim_{(V_t) \rightarrow 0} S_{qht} = t_p \cdot \log g_0 = \int \left( \frac{(\hbar V_t)}{t} \right)^2 dt.
\]

(70)

The condition \( (V^2) = 0 \) is analogous to the thermodynamical condition of perfect crystal lattice, which leads to Planck’s postulate of zero entropy value for zero temperature. One should note that the zero entropy means obtaining complete information about the system. This way for \( (V^2) = 0 \) we obtain quantum Zeno and anti-Zeno paradoxes: the complete information about quantum system is knowing exactly its state and path. On the other hand, the case \( (V^2) \neq 0 \) prevents these paradoxes: even for zero expectation value of the Liouville operator (and, consequently, zero geometric phase), there is always some amount of uncertainty which prevents observer from reaching the limit \( dS_{qht} \rightarrow 0 \) of his subjective time going to zero, so the effects which arise from the constant measurement of the system cannot appear.
4.6. Quantum behaviour vs quantum dynamics

4.6.3 The canonical transformation

The behaviour of the classical field theory is described in terms of the Lagrangean \( L \) and its action \( S \) by the variational equation
\[
0 = \delta S := \delta \int dt L(q_i, \dot{q}_i, t) =: \delta \int dt \int d^3x L(q_i, \dot{q}_i, t),
\]
which leads to the Euler–Lagrange equations
\[
\frac{\partial L(q_i, \dot{q}_i, t)}{\partial q_i} - \frac{\partial}{\partial q_j} \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} = 0.
\]
In order to move from Lagrangean to Hamiltonian description, one performs a Legendre transformation, which eliminates the Liouvillean part from the Lagrangean, leaving only the Hamiltonian:
\[
H(p_i, q_i) := p_i \dot{q}_i - L = V - L.
\]
The dynamics of the field is then described by the Hamiltonian equations:
\[
\dot{q}_i = -\frac{\partial H(p_i, q_i)}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H(p_i, q_i)}{\partial q_i}.
\]
These equations imply that the canonical phase space variables \((p, q) \in T^*Q\) obey the following Poisson bracket commutation relations:
\[
\{p_i, p_j\} = 0, \quad \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}.
\]
The canonical quantization replaces these relations by
\[
[p_i, p_j] = 0, \quad [q_i, q_j] = 0, \quad [p_i, q_j] = -i\hbar \delta_{ij}.
\]
The *-algebra of observables is obtained then from the quantized field operator valued distributions \(p\) and \(q\). This means that the algebras of quantum observables describe solely the Hamiltonian dynamics of a field, obtained from the general behaviour of the system through the elimination of the Liouvillean from the Lagrangean. This explains why the standard algebraic approach is capable to hold enough information about local quantum physics, but cannot properly describe the global aspects, such as interaction of fields and global topological structure of phase space, which, at the present state of art, can be described only in the Lagrangean approach to QFT. The canonical quantization procedure and its effect (the algebra of observables) refer to purely ‘intrinsic’ dynamical part of the action of the system, neglecting the interaction with observer encoded in the Liouvillean, hence assuming that no events happen. However, the effective success of the classical field theory based on the variational principle and the corresponding effective success of the Lagrangean approach to QFT imply that in order to provide the full description of quantum phenomena (behaviour) which is in agreement with experiment, one has to consider also the Liouvillean and its effects. With the help of the quantum histories thermodynamics we have shown how the Liouvillean part of the action can be incorporated in terms of the algebraic approach to QFT. This is provided through the Tomita–Takesaki automorphism theory, which enables to describe the change of the algebraic state in terms of the Radon–Nikodym cocycle, with the KMS condition as the source of dependence of algebraic state on the Liouville operator. From this perspective we can conclude that the functional-analytic approach in terms of Lagrangean provides the non-local (or non-infinitesimal) one, due to the incorporation of the Liouvillean, which describes the changes between subsequent measurements (events, points of external time) performed by an observer. The ordinary canonical algebraic description in terms of Hamiltonian provides only the local (or infinitesimal) description due to the lack
\[\text{More mathematically well-defined canonical quantization procedure is provided by the Weyl quantization relations, which replace the quantization of non-bounded variables \((p, q)\) by the quantization of their bounded exponentialized versions.}\]
of the Liouvillean component. The action functional contains full information about the joint system of ‘observer+observed’.

This perspective explains (at least to some extent) the old foundational problem of quantum theory: while the classical systems are invariant under canonical transformations, their quantized counterparts are invariant only under unitary transformations, and there are many canonical transformations which lead to unitary inequivalent representations, so two systems which are canonically isomorphic in the classical theory may be essentially different from the perspective of ordinary quantum theory. The origin of this difference is the following: the ordinary quantum description arises from the quantization of a result of the one fixed canonical transformation, which eliminates the Liouvillean, so it does not contain the kinematical information, lost in the Legendre transformation. Hence, it is well posed only for the description of these properties of the systems which are encoded solely by the Hamiltonian part of the action. In other words, the canonical single-time quantization is well posed only to describe the Hamiltonian systems with no external measurements provided by an observer (environment). The general procedure of quantization is provided only by quantum histories, because it does not forgets the information about the behaviour encoded in the Liouvillean.

4.6.4 The probability and the inference

In the Euclidean approach to QFT the perturbation expansion is generated from the Feynman–Schwinger functional integral, with the interaction Lagrangian \( L_I(\phi) \) expressed in terms of free fields \( \phi \), and the Gaussian measure \( d\mu(\phi) \) over the classical field distributions of the free field theory (the element \( e^{-\int dt L_I} \) is interpreted as deviation from the reference). However, due to observation of de Morgan, Gaussian measures provide almost always the best description of the expectation values in terms of the \( a \) priori (uniform ignorance) probabilities. These measures are based on ignoring the real distribution of errors which appears in experiment, and assuming the contextless situation, with no relation between the prior and posterior knowledge. On the other hand, we have shown in the section 4.4 that the probabilistic knowledge about behaviour of quantum systems is generally contextual, relational and Bayesian, and is given by the Kullback–Leibler divergence of the Radon–Nikodym cocycle derivatives of the algebraic state. We have shown also that the path-integral approach to QFT can be recovered from our under certain simplifying assumptions. So, there appears the question, what is an essential relation between Lagrangian QFT and quantum histories thermodynamics. In order to understand more deeply the relation between these two approaches, let us note first, that every description referring to the notion of probability is involved in the problem of meaning (interpretation) of mathematical structures used to implement the general ideas of the probability and statistical inference. Both statistical mechanics and quantum theory share these problems. Krylov’s [41] essential critique of the notions of probability in statistical and quantum mechanics leads to denial of the possibility of using the frequency or propensity interpretations of probability in these theories [56]. This implies the question: what kind of interpretation of probability is proper for these theories? Through the analysis of quantum histories theory we have shown that the Bayesian interpretation of quantum probabilities naturally follows from the structure of the theory, and reflects the relational and contextual essence of the logic of inference of every particular (quantum relativistic) observer. From the perspective of information theory, the choice of the model which takes under consideration the context of sign leads usually to high reduction of the informational entropy. This means that the context-dependent description provides a better information about the system under consideration. Context describes all this, what is introduced by an observer and is not present in the pure text of the system. In the thermodynamical systems, where environment (observer) acts as a thermostat (stabilizer of the context), the Helmholtz equation is most suitable. In quantum histories setting the text is provided by the inner dynamics of the system \((H(s), A)\), while the context is provided by the observer-dependent kinematics \((V_t, \omega_V, (-, -)\omega_V)\). The complete meaning is given

\footnote{Information is a notion intended as a measure of change of uncertainty of observer and defined as a difference between informational entropy before measurement and after the measurement.}
by the description of the behaviour provided by \((S_\kappa, V)\), or, generally, sheafs of germs of algebraic states and their integrals over the fixed algebra of observables. Thanks to Bayesian reformulation of the contents of QFT in terms of the Kullback–Leibler divergence, we may speak about the meaningful inference using the probability distributions which are, using the words of Jaynes [36], justified in terms of their demonstrable information content, rather than irrelevant frequency conditions. Note that the use of the Lebesgue measure in the case of classical statistical mechanics and finite dimensional quantum theory (for infinite case one introduces Gaussian measure) is based on the assumption of equal (a priori) probabilities of all possible states. Moreover, in both theories one assumes that the evolution of the process has Hamiltonian (purely dynamic/intrinsic) character, what enables the use of Liouville theorem. These two assumptions taken together may be understood as the assumption of the Markovian character of the process. However, if we deny at least one of them, we generally lose the Markov property of independence between prior and posterior knowledge, hence the logical statistical inference has to be provided not in terms of Lebesgue or Gaussian measures, but Kullback–Leibler contextual and relational Bayesian inference. From this perspective, the conceptual problem of the movement from the algebraic to Lagrangean QFT may be understood in the same way as Redei [56] describes the nature of stochastic measures in classical statistic mechanics: Concentrating on the asymptotic behaviour of the trajectories rather than being interested in the precise description of each individual trajectory means quite a radical change of the point of the investigation: whole of details \(\rightarrow\) details of whole. The description in terms of quantum histories Bayesian Kullback–Leibler inference over germs of algebraic states along the external time paths of subsequent events of measurements of particular observer enables one to fill the gap between algebraic and Lagrangean QFT with the structure which in appropriate limits provides both descriptions: whole of details and details of a whole.

since Gaussian sampling distribution almost always leads to the most successful inferences independently from the actual frequency of errors (what was discovered by Augustus de Morgan [49] and brilliantly used by Marek Kac [38]),.

4.6.5 The interaction and renormalisation

Consider now this what is called a ‘physical dynamics’ of the quantum field, that is, the interaction between free fields. The description of the interaction in QFT is provided by introducing new parts in the (classical) Lagrangean. These elements do not constitute the vacuum state of the quantum field, and are not incorporated into the Fock space representation as its building blocks, but are treated as perturbation of the free fields. In particular, the quantum electrodynamics theory is described in terms of the Fock space generated from the canonical quantization of the action of free fields

\[
S = \int dt L = \int dt \int d^3x (\bar{\Psi} i \gamma^\mu (D_\mu - m) \Psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}),
\]

that is without the interaction term \(L_I = j^\mu A_\mu\).

It is striking, because actually this interaction is the most interesting element of all theory! However, due to the technique of quantization, this part has to be introduced only as a perturbation, which needs renormalisation (infinite process of replacement of the theoretical parameters with the measured ones using, to some extent, mathematically questionable techniques). The perturbation expansion is expressed in terms of time-ordered 2-point \(T\)-function, that is, Feynman propagators between two space-time points (or their Fourier transforms in four-momentum space), which are later integrated over all degrees of freedom, that is, over all space-time. The need of renormalisation is then caused by the point-like character of objects under investigation, which leads to infinite values of the quantities under consideration. One should note that the problem of renormalisation is related more with the external space-time distances between points linked by propagator then with the internal time evolution of the Schrödinger equation. The appearance of 2-point \(T\)-functions (equivalent, via the Osterwalder–Schrader theorem to 2-point
Euclidean Schwinger functions) indicates the fact, that important physical quantities cannot be evaluated in terms of Schrödinger unitary description for a single measurement\textsuperscript{15}. This corresponds with the fact that the \( C^* \)-algebras of observables in algebraic QFT provide the good description for some physical quantities only. An example of an important physical quantity which cannot be described as an element of any given \( C^* \)-algebra of observables is the stress-energy tensor

\[
T_{\mu\nu}(x) := \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{2} g_{\mu\nu}(x)(\partial_\lambda \phi(x) \partial_\lambda \phi(x) + m^2 \phi^2(x))
\]

of the Klein–Gordon field \( \phi(x) \). This is caused by the fact that to such quantity cannot be understood as an operator valued distribution or Gelfand triple dual space functional. (Exactly the same problem appears when one considers the Feynman propagators like \( \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) A_\nu(x_3) \) in perturbation expansion of the Lagrangean path-integral). In order to describe such quantities, one has to introduce the alternative definition of the corresponding classical quantity \textsuperscript{73}:

\[
\langle T_{\mu\nu} \rangle := \lim_{x' \to x} \partial_\mu \partial_\nu' F(x,x') - \frac{1}{2} g_{\mu\nu}(x)(\partial_\lambda \partial_\lambda' + m^2) F(x,x'),
\]

where

\[
F(x,x') := \langle \phi(x)\psi(x') \rangle - \langle \Omega | \phi(x)\phi(x') | \Omega \rangle
\]

is a 2-point function. Hence, the 2-point functions appearing in Lagrangean and algebraic QFT should be understood as a tool of approximation of relations between quantities defined in two different points. This suggests that this is in fact an approximation of the relations between two different algebraic states and their GNS-generated Hilbert spaces. This would explain why one has to handle with various objects which are ill-defined in the framework of one, (even rigged) Hilbert space, which in both cases (Lagrangean path-integral and algebraic QFT) need some kind of regularisation of 2-point functions.

\textbf{4.7 The modular Hamiltonian}

In order to understand the meaning of the modular Hamiltonian we will analyse now the Fulling-Unruh effect \textsuperscript{19}, \textsuperscript{71}. Consider a Minkowski space-time \( \mathcal{M} \) divided into four 'Rindler wedges' with the Cauchy surface \( \Sigma_W \cup \Sigma_W' \) of initial data for the solutions of the Klein–Gordon field equation.

\textsuperscript{15}It is often stated that the internal 2-point propagator lines of Feynman diagram also does not correspond to any measurement. However, it is hard to be sure to what extent such claims are true, if actually any concrete Feynman diagram also does not correspond to any measurement, and only an infinite sequence of such does.
One considers a family of accelerating observers in $\mathcal{M}$ by introducing the orbits of isometries generated by the Lorentz boosts along the Killing field $a(x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t})$, where $a$ is a fixed constant, interpreted as an acceleration of the observer on the orbit. Due to Reeh–Schlieder theorem, the restriction of the vacuum state $\omega_{\mathcal{M}}$ on $\mathcal{M}$ to the state $\omega_{\mathcal{W}}$ on the right wedge $\mathcal{W}$ is generally a mixed state. If one will fix his attention to the states referring to the initial data on the Cauchy surface $\Sigma_{\mathcal{W}} \cup \Sigma_{\mathcal{W}'}$, then, in order to obtain the description in terms of the right wedge observer, he has to trace out the elements coming from $\omega_{\mathcal{M}}$ which correspond to the left wedge. In the description in terms of eigenstates $|n_i\rangle$ and eigenvalues $n_i$ of the particle number operator, this leads to the density matrix 

$$\rho = \prod_{i} \sum_{n=0}^{\infty} e^{-\pi n_i w_i/a} |n_i\rangle_{WW} \langle n_i|,$$

obtained from tracing out the $\mathcal{W}'$-component of the state

$$\prod_{i} \sum_{n=0}^{\infty} e^{-n_i \pi w_i/a} |n_i\rangle_{\mathcal{W}} \otimes |n_i\rangle_{\mathcal{W}'} ,$$

where $a$ is an acceleration of the orbits of a family of observers, and $w_i$ is a Fourier transform frequency. If we will define now the Hamiltonian $H_{\mathcal{W}}$ which is the generator of time translations of observer identified with the $a$-accelerating trajectories in the Rindler wedge $\mathcal{W}$, then the quantity $n w_i$ is just the energy of the state $|n_i\rangle_{\mathcal{W}}$ 'as seen by the accelerating observer' [73]. In effect, the state (73) may be considered as a thermal state $e^{-H_{\mathcal{W}} T^{-1}}$, with the 'temperature' $T = \frac{\pi}{a}$ specified completely by the properties of an observer what enables us to call it the 'temperature of an observer'. However, such formulation of the state $\omega_{\mathcal{W}}$ as a thermal state lacks enough rigour. The rigorous proof of the Fulling–Unruh effect can be provided only in the equilibrium KMS setting, what was done by Bisognano and Wichmann in [11] and [12]. The Bisognano-Wichmann theorem provides the concrete link between the modular operators of the von Neumann algebra over the Rindler wedge in a Minkowski space-time and the Lorentz transformations.
which describe the kinematic properties of an observer. The Lorentz boosts which map \( W \) into itself can be described as

\[
\Lambda_W(s) = \begin{pmatrix}
\cosh s & \sinh s & 0 & 0 \\
\sinh s & \cosh s & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The Bisognano–Wichmann theorem states that if there is given a Minkowski vacuum vector, then one can associate to \( W \) the modular operator

\[
\Delta^{i\tau}(W) = U(\Lambda(2\pi \tau)),
\]

and

\[
J(W)A(O)J(W) = A(j(O)) \quad \forall A(O) \in A(O),
\]

where \( j : (s, x, y, z) \mapsto (-s, -x, y, z) \) is a space-time plane reflection. This implies that \( \Theta := JU(R) \) for \( R \) given by spatial rotation around \( x \) by the angle of \( \pi \) is a PCT operator, and that the modular operator is given by

\[
\Delta(W) = e^{-2\pi K},
\]

so the modular automorphisms \( \sigma_\tau = \Delta^{i\tau} \) in the wedge \( W \) act as Lorentzian boosts \( U(\Lambda(s)) \) with the parameter \( s = -2\pi \tau \). If we consider now a family of observers in \( W \) given by the ‘accelerating’ trajectory

\[
\Lambda_W(s) \cdot \begin{pmatrix}
0 \\
1/a \\
0 \\
0
\end{pmatrix},
\]

then the proper time of this family is given by \( s' = as \). Hence, if \( K \) is a generator of time transformations associated with \( s \), then \( K' = K/a \) is a generator of time transformations associated with \( s' \). The simple calculation

\[
U_K(\tau) = \Delta^{i\tau} = e^{-K\tau} = e^{isK/2\pi} = e^{is(K/a)\tau a/2\pi} = e^{isK'\beta^{-1}} = U_{K',\beta}(s)
\]

shows that the observer associated with the accelerating trajectory described by the modular Hamiltonian \( K' \) observes the vacuum as the KMS equilibrium state with the temperature parameter \( T = \frac{\beta}{2\pi} \). Hence, \( K' = H_W \). One should note that the Fulling–Unruh effect and Bisognano–Wichmann theorem describe only the kinematical properties of the observer. It does not refer to any dynamics of the observed process (Klein–Gordon field). This means that, due to Bisognano–Wichmann theorem, the modular Hamiltonian encodes the information about the changes of the kinematic relativistic frame along the time trajectory of the observer. This is reflected in the fact, that the spectrum of \( K' \) generally ranges from \(-\infty\) to \(+\infty\). Hence, the quantum histories thermodynamics modular automorphism encodes the following information:

- The change of the relativistic frame of the observer (given by the orthonormal tetrad), encoded in the modular Hamiltonian operator associated with the given trajectory.
- The change of the quantum frame of the observer (given by the geometric phase), encoded in the Liouville operator associated with the given trajectory.

5 Black hole thermodynamics

6 Summary

One may understand the qualitative role of the modular operators in quantum theory by considering the basic structure of the ordinary quantum mechanics. Two essential constituents of this theory are the
space of solutions of Schrödinger equation and the scalar product. The first element, despite writing it most often in the complex form, can be equivalently described in terms of real Hilbert space. This means that the presence of the structure of complex numbers in quantum theory comes from the inner product. One may see the appearing of the complex structure from the inner product on the level of states:

\[(\psi, \varphi) := \int dx \bar{\psi}(x,t)\varphi(x,t),\]

or on the level of observables:

\[(A^* \psi, \varphi) := (\psi, A\varphi).\]

In both cases the inner product introduces not only the complex structure, but also the inversion in time (\(\psi(x,t)\) is not a solution of the Schrödinger equation, but it is a solution of the Schrödinger equation with the inversed time). The Tomita–Takesaki modular operator

\[\mathcal{S}A|\Omega\rangle := J\Delta^{1/2}A|\Omega\rangle := A^*|\Omega\rangle\]

(74)

reflects both these properties. It is introduced by an algebraic state (the algebraic source of the inner product), and specifies not only the complex star structure, but also the time inversion via

\[\langle J\psi|J\varphi \rangle = \langle \varphi|\psi \rangle\]

and

\[JU(s)J = U(-s)\]

satisfied for any unitary group with positive generator such that \(U(s)|\Omega\rangle = |\Omega\rangle \forall s \in \mathbb{R}\) [67]. This suggests that one could think about inverting the direction of the definition (74) into the form

\[J\Delta^{1/2}A|\Omega\rangle =: A^*|\Omega\rangle,\]

that is, defining the complex structure on observables from the objects which encode the essential properties of algebraic state (inner product). This way the modular automorphisms of algebra of observables, leading to the appearance of such effects as quantum geometric phase, would describe the evolution of the complex structure in quantum theory. This would be the concrete realisation of the proposition of Anastopoulos that the complex structure is intrinsically linked both to its [quantum theory] probability structure and the way the notion of sucession is encoded [3]. From the perspective provided in the present paper we can state this as follows:

- the Hamiltonian internal dynamics of algebra of observables is totally deterministic
- the description in terms of uncertainty is introduced by the (germs of) algebraic state(s)
- the modular automorphisms generated by the Liouvillean enable the description of the ‘kinematic evolution’ of the algebraic states (and the induced complex structure) along the trajectory of external time of observation.

The essential link between the algebraic state (inner product), the notion of time sucession and the complex structure in quantum theory reflects the fact, that actually we never get any expectation value in one single measurement. In order to define the scale we always fix (assume) some prepared state of the system, by setting it via \(\langle \text{test}|\text{prepared}\rangle\), and measure the \(\langle \text{output}|\text{question}|\text{prepared}\rangle\) in a next event. Between those two events there is always, by definition, some amount (partial order) of time. The measurement introduces the change of information of observer what is literally reflected in the description of the time of observer in terms of an informational ‘relative’ entropy of algebraic states on a category of \(C^\ast\)-algebras with inclusions as morphisms. However, if we take seriously the possibility of definition of the star structure via the modular operators, it would suffice to consider the category of \(C\)-algebras.
In the context of histories approach, Kullback–Leibler divergence was first introduced by Isham and Linden (however, without any reference to algebraic states and the conclusions drawn in the present paper), while the possibility of Bayesian contents was conjectured and analyzed (from different point of view) by ....

In our opinion this provides the key insight for the quantum gravity in general, and for the black hole thermodynamics in particular: for a given action $S$ of the classical theory, its Liouville part $V$ describes the change of the representation $\omega_V$ over space-time of the algebra of observables generated by fields which internal dynamics is described by the Hamiltonian part of the action.

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