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ON CHEBYSHEV CENTRES, RETRACTIONS, METRIC PROJECTIONS AND HOMOGENEOUS INVERSES FOR BESOV, LIZORKIN–TRIEBEL, SOBOLEV AND OTHER BANACH SPACES*

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We find the exponents and obtain explicit estimates of the constants related to the Hölder-Lipschitz regularity of the relative Chebyshev centre map, various retractions onto convex subsets, metric projection onto a closed convex subset and naturally defined homogeneous right-inverses for closed linear surjective operators. The estimates are first established in the abstract setting of an abstract uniformly convex and/or uniformly smooth Banach space and, then, are transferred to the setting of the subsets of and the closed operators from a wide class of function and other spaces including, in particular, various anisotropic spaces of Besov, Lizorkin-Triebel and Sobolev types endowed with geometrically friendly norms defined in terms of averaged differences, local polynomial approximations, functional calculus, wavelets and other means and a new class of spaces. New approaches are shown to be providing better estimates in the abstract setting as well. Occasionally, attention is paid to the question of the sharpness of the exponents.

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Key words and phrases: Besov, Lizorkin-Triebel and Sobolev spaces, Chebyshev centre, metric projection, retraction, Hölder regularity, right-inverse operator, uniform convexity, uniform smoothness, independently generated spaces, wavelets, functional calculus.

1. Introduction

This article is a part of a larger program of filling the gap between the theory of function spaces, especially Besov, Lizorkin-Triebel and Sobolev and a wide

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class of spaces defined in the end of Section 2, and quantitative functional analysis dealing with abstract Banach spaces. For example, the first natural question, answered here, is: "What is the right equivalent norm for a function space that should be chosen to investigate the geometric properties of this space?" In particular, it appears that the original norm in a Sobolev space $W_p^s(G)$ chosen by Sergey L. Sobolev,

$$\|f|W_p^s(G)_S\|^2 = \|f|L_p(G)\|^2 + \sum_{i=1}^{n} \|D_i^{s_i} f|L_p(G)\|^2,$$

where $G \subset \mathbb{R}^n$ is open, $s \in \mathbb{N}^n$, $p \in (1, \infty)^n$, and $D_i^{s_i}$ is the Sobolev generalised derivative order s_i in the ith axis, is suitable, while the traditional

$$\|f|L_p(G)\| + \sum_{i=1}^{n} \|D_i^{s_i} f|L_p(G)\|$$

is not on many occasions.

The definitions and basic properties of all the spaces under consideration are in §2, while all the necessary aggregated information about them in the form of estimates for their uniform convexity and smoothness parameters and limiting constants is given in §3. Much more information on the general theory of function spaces can be found in the monographs written by Besov, Il'in and Nikol'skii [9], Burenkov [12], Sobolev [27] and Triebel (starting with [28]). Most of the definitions of norms (spaces) is taken from [1].

Section 4 contains examples of the main results and the explanation how the multitude of the main results corresponding to the various classes of spaces (including abstract Banach spaces) and the different problems can be established.

We devote the whole §5 to establishing the abstract part of our results for every particular problem among the following ones presented:

(*i*) the global Hölder regularity of the relative Chebyshev map on the class of the bounded subsets of a uniformly convex Banach space (§5.1.1);

(*ii*) the global Hölder regularity of the retractions onto a closed convex subset of a uniformly convex Banach space (or an isomorphic (to this space) quasi-Banach space) from metric spaces containing this subset isometrically (§5.1.2);

(*iii*) the global Hölder-Lipschitz regularity of the metric projection onto a closed convex subset of either a uniformly convex, or both uniformly convex and uniformly smooth Banach space and a related retraction onto closed convex subsets of the isomorphic quasi-Banach spaces (§5.2);

(*iv*) the construction of a natural homogeneous right-inverse (non-linear) operator for a closed surjective operator defined on either a uniformly convex, or both uniformly convex and uniformly smooth Banach space, or on a quasi-Banach space isomorphic to any of them and the evaluation of its global Hölder-Lipschitz regularity (§5.3);

(v) the properties of the cross-sections of the balls in uniformly convex and/or uniformly smooth Banach spaces that are more explicit then their preceding counterparts used in the earlier proofs of the local Hölder-Lipschitz regularity of the metric projections onto closed subsets (§5.4).

Eventually, we cite the technical results on the existence of uniformly isomorphic and uniformly complemented copies of finite-dimensional l_p spaces in some classes of our spaces under consideration in §6. They are used to establish the sharpness of the main results for some spaces and problems.

A local numbering of formulas is used throughout the article, i.e., the formulae are numbered independently in every formal logical unit of the text, such as a proof of a corollary, lemma, theorem, or a remark.

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2. Definitions, notations, and agreements

$\mathbb{N}$ is the set of the natural numbers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; for $\alpha, \beta \in \mathbb{N}_0^n$, notation $\alpha \leq \beta$ means the partial order relation generated by the coordinate order relations; $\max(\alpha, \beta) = \min\{\gamma : \gamma \geq \alpha, \gamma \geq \beta\}$; $\mathbb{R}^n$ is n-dimensional Euclidean space with the standard base $\langle e^1, \dots, e^n \rangle$, $x = (x_1, \dots, x_n) = \sum_1^n x_i e^i = (x_i)$; $x_{\min} := \min_i x_i$ and $x_{\max} := \max_i x_i$.

We write p' for the conjugate to $p \in [1, \infty]^n$, i.e. $1/p_i' + 1/p_i = 1 (1 \leq i \leq n)$.

For $A \subset \mathbb{N}_0^n$, $|A|$ designates the number of the elements of A, and

$$\hat{A}_\alpha = \{\beta : \beta \in \mathbb{N}_0^n, \beta \leq \alpha, \alpha \in A\}.$$

In what follows, one can assume $\gamma_a = ((\gamma_a)_1, \dots, (\gamma_a)_n) \in (0, \infty)^n$ and $|\gamma_a| = \sum_1^n (\gamma_a)_i = n$ to be fixed.

For $x, y \in \mathbb{R}^n$ and $t > 0$, we write $[x, y]$ for the segment in $\mathbb{R}^n$ with endpoints x and y; $xy = (x_i y_i)$, $t^y = (t^{y_i})$; $x/y = \left(\frac{x_i}{y_i}\right)$ for $y_i \neq 0$, and $t/\gamma_a = \left(\frac{t}{(\gamma_a)_i}\right)$. Assuming that $|x|_{\gamma_a} = \max_{1 \leq i \leq n} |x_i|^{1/(\gamma_a)_i}$, we have the inequality $|x + y|_{\gamma_a} \leq c_{\gamma_a} \left(|x|_{\gamma_a} + |y|_{\gamma_a}\right)$. For $E \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we denote

$$|E|_{\gamma_a} := \inf_{x \in E} |x|_{\gamma_a}, \quad x \pm aE := \{y : y = x \pm az, z \in E\},$$

$$G_t := \{x : x \in G, |x - \partial G|_{\gamma_a} > t\}.$$

Definition 2.1. *For $m \in \mathbb{N}$, $h > 0$, $\gamma \in \mathbb{R}$, $x \in \mathbb{R}^n$, $\Delta_i^m(h)f(x)$ is the m-th order difference of the function f with step h in the direction of e^i at the point x. Further notations are*

$$\Delta_i^m(h,E)f(x) := \begin{cases} \Delta_i^m(h)f(x) & \text{for } [x, x+mhe^i] \in E, \text{ and} \\ 0 & \text{for } [x, x+mhe^i] \notin E, \end{cases}$$

$$\delta_{i,a}^m(t,x,f,E)_\gamma := \Big(\int_{-1}^{1} |\Delta_i^m(t^{(\gamma_a)_i}u, E) f(x + \gamma t^{(\gamma_a)_i}u)|^a \, du \Big)^{1/a}.$$

Sometimes, in the absence of ambiguity, we shall write $\delta_{i,a}^m(t,x,f)$ instead of $\delta_{i,a}^m(t,x,f,E)_\gamma$.

If ϕ is an integrable function on a measurable set $E \subset \mathbb{R}^n$, then $|E|$ is the Lebesgue measure of E, and $\phi_E := |E|^{-1} \int_E \phi \, d\mu$. Let $Q_0 = [-1,1]^n$. For $v \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, we say that $Q_v(x) = x + v^{\gamma_a} Q_0$ is a parallelepiped of γ_a-radius v with center x; χ_E is the characteristic function of E, and $\Theta : \mathbb{R} \to \{0,1\}$ is the Θ–function.

For $n \in \mathbb{N}$, $r \in (0,\infty]^n$, $p \in (0,\infty)$, $q \in (0,\infty]$, a (countable) index set I, and a quasi-Banach space A, let $l_q(A) := l_q(I,A)$ be the space of all sequences $\alpha = \{\alpha_k\}_{k \in I} \subset A$ with finite quasi-norm $\|\alpha\|_{l_q} = (\sum_{k\in I} |\alpha_k|^q)^{1/q} < \infty$. For $m \in \mathbb{N}$ and either $l = l_r(A)$, or $l \in \{l_q(A), l_{p,q}(A), c_0(A)\}$, by l^m we designate the subspace of l satisfying $\alpha_i = 0$ for either $i_{\max} > m$, or $i > m$, correspondingly.

For $p \in (0,\infty]$, let $L_{*p} = L_{*p}(\mathbb{R}_+)$ be the (quasi)normed space of all functions f measurable on $\mathbb{R}_+$ with finite (quasi)norm

$$\|f|L_{*p}(\mathbb{R}_+)\| := \begin{cases} \Big(\int_{\mathbb{R}_+} |f(t)|^p dt/t \Big)^{1/p}, & \text{if } p < \infty, \\ \|f|L_\infty(\mathbb{R}_+)\|, & \text{if } p = \infty. \end{cases}$$

For $G \subset \mathbb{R}^n$ and $f : G \to \mathbb{R}$, we denote by $\overline{f} : \mathbb{R}^n \to \mathbb{R}$ the function

$$\overline{f}(x) := \begin{cases} f(x), & \text{for } x \in G, \\ 0, & \text{for } x \in \mathbb{R}^n \setminus G. \end{cases}$$

For $p \in (0,\infty]^n$, $L_p(G)$ is the space of all measurable functions $f : G \to \mathbb{R}^n$ with the finite mixed quasi-norm

$$\|f|L_p(G)\| = \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \cdots \Big(\int_{\mathbb{R}} |\overline{f}|^{p_1} \, dx_1 \Big)^{p_2/p_1} \cdots \Big)^{p_n/p_{n-1}} dx_n \Big)^{1/p_n},$$

where, for $p_i = \infty$, $\big(\int_{\mathbb{R}} |g(x_i)|^{p_i} \, dx_i \big)^{1/p_i}$ is understood as $\operatorname{ess\,sup}_{x_i \in \mathbb{R}} |g(x_i)|$.

For $s \in \mathbb{N}^n$, $\sigma \in [1, \infty)$ and an open set $G \subset \mathbb{R}^n$, $W_p^s(G) = W_p^s(G)_\varsigma$ will stand for the Banach space of measurable functions f defined on G, possessing Sobolev generalized derivatives $D_i^{s_i} f$ and finite norm

$$\|f|W_p^s(G)\|^\varsigma := \|f|L_p(G)\|^\varsigma + \|f|w_p^s(G)\|^\varsigma = \|f|L_p(G)\|^\varsigma + \sum_{i=1}^{n} \|D_i^{s_i} f|L_p(G)\|^\varsigma.$$

For an ideal space $Y = Y(\Omega)$ for a measurable space (Ω, μ) and a Banach space X, let $Y(\Omega, X)$ be the space of the Bochner-measuarable functions $f : \Omega \mapsto X$ with finite (quasi)norm

$$\|f|Y(\Omega, X)\| := \| \|f(\cdot)\|_X |Y(\Omega)\|.$$

For example, $L_p(\mathbb{R}^n, l_q)$ with $p, q \in [1, \infty]$, is a Banach space of the function sequences $f = \{f_k(x)\}_{k=0}^{\infty}$ with finite norm $\|\|\{f_k(\cdot)\}_{k\in\mathbb{N}_0}|l_q\||L_p(\mathbb{R}^n)\|$.

Definition 2.2. *For $p, q \in (0, \infty)$ and $n \in \mathbb{N}$, let $lt_{p,q}$ be the Banach space of the sequences $\{t_{i,j}\}_{i\in\mathbb{Z}^n}^{j\in J}$ with $J \in \{\mathbb{N}_0, \mathbb{Z}\}$, endowed with the (quasi)norm*

$$\|\{t_{i,j}\}|lt_{p,q}\| := \left\| \left\{ \sum_{i\in\mathbb{Z}^n} t_{i,j} \chi_{F_i^j} \right\}_{j\in J} \middle| L_p(\mathbb{R}^n, l_q(J)) \right\|,$$

where $\{F_i^j\}_{i\in\mathbb{Z}^n}^j \in J$ is a fixed nested family of decompositions of $\mathbb{R}^n$ into unions of congruent parallelepipeds F_i^j satisfying

$$\cup_{i\in\mathbb{Z}^n} F_i^j = \mathbb{R}^n, \ |F_i^j \cap F_k^j| = 0 \text{ for every } j \in J, \ i \neq k,$$

and $F_{i_0}^{j_0} \cap F_{i_1}^{j_1}$ is either $\emptyset$ or $F_{i_0}^{j_0}$ for every i_0, i_1 and $j_0 > j_1$. We call this system regular if the k-th side $l_{k,j}$ of the parallelepipeds $\{F_i^j\}_{i\in\mathbb{Z}^n}$ of the jth decomposition (level) satisfies

$$c_l b^{-j(\lambda_a)_k} \leq l_{k,j} \leq c_u b^{-j(\lambda_a)_k}$$

for some positive constants $b > 1$ and c_l, c_u.

The space $lt_{p,q}$ is isometric to a complemented subspace of $L_p(\mathbb{R}^n, l_q)$, while its dual $lt_{p,q}^*$ is isomorphic to $lt_{p',q'}$ for $p, q \in (1, \infty)$ (see §3.2.4 of [2]).

For an operator T from X into Y, we denote by $D(T)$, Ker T and Im T its domain, kernel and image, and by $\mathcal{C}(X, Y)$ and $\mathcal{L}(X, Y)$ the spaces of closed and bounded operators, respectively.

For $\gamma_a \in (0, \infty)^n$ and $s \in [0, \infty)$, let $A_s^* := \{\alpha : \alpha \in \mathbb{N}_0^n, (\alpha, \gamma_a) \leq s\}$.

For $z \in \mathbb{R}^n$ and $v > 0$, we set $\tau_z f(x) := f(x - z)$ and $\sigma_v f(x) := f(v^{-\gamma_a} x)$. For a Banach space X and $A \subset \mathbb{N}_0^n$, $|A| < +\infty$, let $\mathcal{P}_A(X)$ be the space of the polynomials of the form $\sum_{\alpha\in A} c_\alpha x^\alpha$ with $\{c_\alpha\}_{\alpha\in A} \subset X$, $\mathcal{P}_A = \mathcal{P}_A(\mathbb{R})$. Though the main results can be extended to the case of X-valued spaces, we consider the scalar-valued case for simplicity. For $a \in [1,\infty]^n$, let $p_A \in \mathcal{L}(L_a(Q_0), \mathcal{P}_A)$ be a certain projector onto $\mathcal{P}_A$.

Definition 2.3. *For $a \in [1,\infty]^n$, let $p_{A,v,z} = \tau_z \circ \sigma_v \circ p_A \circ \sigma_v^{-1} \circ \tau_z^{-1}$.*

For $\varepsilon > 0$, $a \in (0,\infty]^n$, let $\pi_{A,v,z} : L_a(Q_v(z)) \to \mathcal{P}_A$ be an operator of the best L_a-approximation satisfying

$$\|f - \pi_{A,v,z} f | L_a(Q_v(z))\| = \min_{g\in\mathcal{P}_A} \|f - g | L_a(Q_v(z))\|, \; f \in L_a(Q_v(z)).$$

For $f \in L_{a,loc}(G)$, $v > 0$ and $a \in (0,\infty]^n$, we define the $\mathcal{D}$-functionals

$$\mathcal{D}_a(v,x,f,G,A) := \begin{cases} \|f | L_a(Q_v(x))/\mathcal{P}_A\| v^{-(\gamma_a, 1/a)}, & \text{if } Q_v(x) \subset G, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{D}_a(v,x,f,G,p_A) := \begin{cases} \|f - p_{A,v,x} | L_a(Q_v(x))\| v^{-(\gamma_a, 1/a)}, & \text{if } Q_v(x) \subset G, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.1. *Note that*

$$\|f - \pi_{A,v,z} f | L_a(Q_v(z))\| \asymp \|f - p_{A,v,x} | L_a(Q_v(x))\|$$

uniformly by v and x when they all are well-defined (see [1]).

While switching from one functional to another provides only an equivalent norm, the geometric properties under consideration will depend on the parameters only.

Let us define the spaces of Besov and Lizorkin-Triebel type. In these definitions, we use a parameter $\varsigma \in (0,\infty]$, which is essential in the study of the geometric properties of function spaces but not the topological (isomorphic) ones (we have equivalent (quasi)norms for different $\varsigma \in (0,\infty]$). It will normally be omitted for the sake of simplicity. If its presence and value should be emphasized, we say that the space under consideration is endowed with *the ς-product norm, or, just, the ς-norm*, and/or add ς as a subindex. For a seminormed (homogeneous) space $x(G)$ of functions defined on $G \subset \mathbb{R}^n$ and an ideal space $Y(G)$, we assume that its intersection $x(G) \cap Y(G)$ is endowed with the ς-norm too:

$$\|f | x(G) \cap Y(G)\|^\varsigma := \|f | Y(G)\|^\varsigma + \|f | x(G)\|^\varsigma.$$

Moreover, *we shall always assume* the parameter ς to be equal to one of the other parameters or its components of $x(G)$ and $Y(G)$ except for the smoothness.

We start with the spaces defined in terms of the averaged (shifted) axis-directional differences. While the study of these norms and the their equivalence with other norms was one of the primary tasks of, for example, [1], we shall refrain from the usage of the results of this type here in order to cover the sets of parameters not covered by the equivalence results and because the geometrical constants depend on the specific equivalent norm chosen.

Let Pr_i be the orthogonal projector on the i-th axis in $\mathbb{R}^n$, and, for any $y \in (I - Pr_i)(G)$,

$$In_i(y) := (I - Pr_i)^{-1}(y) \cap G = \{x \in G : x = y + te_i, t \in \mathbb{R}\}.$$

Definition 2.4. *For an ideal space $Y = Y(G)$, $p \in (0,\infty]^n$, $q \in (0,\infty]$, $r > 0$, $s \geq 0$, $s/\gamma_a < m \in \mathbb{N}_0^n$, $a \in (0,\infty]^n$, $\gamma \geq 0$, and an open set $G \subset \mathbb{R}^n$, by $b^s_{Y,q,a}(G)$ we denote the (quasi) semi-normed space of measurable functions $f \in L_{a_i,loc}(In_i(y), dx_i)$ for a.e. $y \in (I - Pr_i)(G)$ with finite (quasi) semi-norm*

$$\|f|b^s_{Y,q,a}(G)\|^\varsigma := \sum_{i=1}^n \left(\int_0^\infty \left\| \delta^{m_i}_{i,a_i}(t,\cdot,f,G_{rt})_\gamma |Y(G)\right\|^q t^{-qs}\,\frac{dt}{t} \right)^{\varsigma/q} \tag{1}$$

or, equivalently, with finite (quasi) semi-norm

$$\left(\sum_{i=1}^n \left\| \{b^{-sj} \|\delta^{m_i}_{i,a_i}(b^{-j},\cdot,G_{rb^{-j}},f)_\gamma |Y(G)\|\}_{j\in\mathbb{Z}} |l_q \right\|^\varsigma \right)^{1/\varsigma}, \tag{2}$$

$$B^s_{Y,q,a}(G) := b^s_{Y,q,a}(G) \cap Y(G),\ b^s_{p,q,a}(G) := b^s_{L_p,q,a}(G).$$

Definition 2.5. *For an ideal space $Y = Y(G)$, $p \in (0,\infty]^n$, $q \in (0,\infty]$, $r > 0$, $s \geq 0$, $s/\gamma_a < m \in \mathbb{N}_0^n$, $a \in (0,\infty]^n$, $\gamma \in \mathbb{R}$, and an open set $G \subset \mathbb{R}^n$, we denote by $l^s_{p,q,a}(G)$ the (quasi) semi-normed space of measurable functions $f \in L_{a_i,loc}(In_i(y), dx_i)$ for a.e. $y \in (I - Pr_i)(G)$ with finite (quasi) semi-norm*

$$\|f|l^s_{Y,q,a}(G)\|^\varsigma := \sum_{i=1}^n \left\| \left(\int_0^\infty \left(\delta^{m_i}_{i,a_i}(t,\cdot,G_{rt},f)_\gamma\right)^q t^{-qs}\,\frac{dt}{t} \right)^{1/q} \middle| Y(G) \right\|^\varsigma \tag{3}$$

or, equivalently, with finite (quasi) semi-norm

$$\left(\sum_{i=1}^n \left\| \|\{b^{-sj}\delta^{m_i}_{i,a_i}(b^{-j},\cdot,G_{rb^{-j}},f)_\gamma\}_{j\in\mathbb{Z}} |l_q\| |Y(G) \right\|^\varsigma \right)^{1/\varsigma}, \tag{4}$$

$$L^s_{Y,q,a}(G) := l^s_{Y,q,a}(G) \cap Y(G),\ l^s_{p,q,a}(G) := l^s_{L_p,q,a}(G).$$

Next, we define the anisotropic local approximation spaces of Besov and Lizorkin-Triebel type in terms of the $\mathcal{AD}$-functional as follows.

Definition 2.6. *For $p \in (0,\infty]^n, q \in (0,\infty]$, $a \in (0,\infty]^n$, $s \in [0,\infty)$, $D \subset \mathbb{N}_0^n$, $|D| < \infty$ and an ideal space $Y = Y(G)$, we denote by $\widetilde{b}^{s,D}_{Y,q,a}(G)$ and $\widetilde{l}^{s,D}_{Y,q,a}(G)$, correspondingly, the anisotropic (quasi) semi-normed space of functions $f \in L_{a,loc}(G)$ with finite (quasi) semi-norm*

$$\|f|\widetilde{b}^{s,D}_{Y,q,a}(G)\| := \Big(\int_0^\infty \|t^{-s}\mathcal{D}_a(t,\cdot,f,G,D)|Y(G)\|^q \frac{dt}{t}\Big)^{\frac{1}{q}},\ \widetilde{b}^{s,D}_{p,q,u}(G) := \widetilde{b}^{s,D}_{L_p,q,u}(G),$$

and

$$\|f|\widetilde{l}^{s,D}_{Y,q,a}(G)\| := \Big\|\Big(\int_0^\infty (t^{-s}\mathcal{D}_a(t,\cdot,f,G,D))^q \frac{dt}{t}\Big)^{\frac{1}{q}}\Big|Y(G)\Big\|,$$

$$\widetilde{l}^{s,D}_{p,q,u}(G) := \widetilde{l}^{s,D}_{L_p,q,u}(G).$$

Assume also that

$$\widetilde{B}^{s,D}_{Y,q,a}(G) = \widetilde{b}^{s,D}_{Y,q,a}(G) \cap Y(G) \quad \text{and} \quad \widetilde{L}^{s,D}_{Y,q,a}(G) = \widetilde{l}^{s,D}_{Y,q,a}(G) \cap Y(G).$$

Remark 2.2. *It is important to note that for $s > 0$ and $Y = L_p$ with $a_{\max} \le p_{\min}$ we obtain equivalent norms in the non-homogeneous Besov and Lizorkin-Triebel spaces defined above by substituting the integration $\int_0^\infty$ in their seminorms with the integration $\int_0^h$ for any fixed $h > 0$. At the same time, the geometric properties will remain the same for every $h \in (0,\infty]$, depending on the parameters only (as in the previous remark).*

Definition 2.7. *Let $G \subset \mathbb{R}^n$, $p \in [1,\infty]^n$, $q, \varsigma \in [1,\infty]$, $s \in \mathbb{R}$, $b > 1$ and $\mathcal{F} = \{F_k\}_{k\in\mathbb{N}_0} \subset \mathcal{C}(L_p(G))$ be a system of closed operators satisfying*

$$f \in L_p(G) \quad \text{and} \quad F_k f = 0 \quad \text{for} \quad k \in \mathbb{N}_0 \Rightarrow f = 0.$$

We denote by $B^s_{p,q,\mathcal{F}}(G)$ and $L^s_{p,q,\mathcal{F}}(G)$, respectively, the Banach spaces of functions defined on G with finite norms

$$\|f|B^s_{p,q,\mathcal{F}}(G)\|^\varsigma = \|f|L_p(G)\|^\varsigma + \Big(\sum_{k\in\mathbb{N}_0} b^{ksq}\|F_k f|L_p(G)\|^q\Big)^{\varsigma/q},$$

$$\|f|L^s_{p,q,\mathcal{F}}(G)\|^\varsigma = \|f|L_p(G)\|^\varsigma + \Big\|\Big(\sum_{k\in\mathbb{N}_0} b^{ksq}|F_k f(\cdot)|^q\Big)^{1/q}\Big|L_p(G)\Big\|^\varsigma.$$

Remark 2.3. *Under the conditions of the last definition, let also $\Omega \subset \mathbb{C}$ be open with $b^{-k}\Omega \subset \Omega$ for $k \in \mathbb{N}_0$, $g \in H_\infty(\Omega)$ (bounded holomorphic function on Ω), and $A \in \mathcal{C}(L_p(G))$ admits the bounded $H_\infty(\Omega)$ functional calculus*

$$H_\infty(\Omega) \ni h \mapsto h(A) \in \mathcal{L}(L_p(G)) \quad \text{with} \quad \|f(A)|\mathcal{L}(L_p(G))\| \le \|f|H_\infty(\Omega)\|.$$

Assuming that $F_k = g(b^{-k}A)$, we obtain the Besov and Lizorkin-Triebel spaces $B^s_{p,q,\mathcal{F}}(G)$ and $L^s_{p,q,\mathcal{F}}(G)$ defined in terms of the bounded H_∞-calculus.

Definition 2.8. *Let X, Y be (quasi)Banach spaces and $\lambda \geq 1$. Then t Banach-Mazur distance $d_{BM}(X, Y)$ between them is equal to ∞ if they a not isomorphic and is, otherwise, defined by*

$$d_{BM}(X, Y)) := \inf\{\|T\| \cdot \|T^{-1}\| : T : X \overset{onto}{\longrightarrow} Y,\ \mathrm{Ker}(T) = 0\}.$$

The space X is λ-finitely represented in Y if for every finite-dimension subspace $X_1 \subset X$,

$$\inf\{d(X_1, Y_1) : Y_1 \text{ is a subspace of } Y\} = \lambda.$$

If λ is equal to 1, then X is simply said to be finitely represented or almo isometrically finitely represented in Y.

For example, it is well-known that, for a Banach space X, its secon conjugate X^{**} is finitely represented in X, and X itself is finitely represente in c_0.

2.1. Independently generated spaces

Definition 2.9. Independently generated spaces
Let $\mathcal{S}$ be a set of ideal (quasi-Banach) spaces, such that every element $Y \in \mathcal{S}$ is either a sequence space $Y = Y(I)$ with a finite or countable I, or a spac $Y = Y(\Omega)$, where (Ω, μ) is a measure space with a countably additive measur μ without atoms.

Leaf growing process (step) *from some $Y \in \mathcal{S}$ is the substitution of Y with:*

- *(Type A)* $\quad Y(I, \{Y_i\}_{i \in I})$ *for some* $\{Y_i\}_{i \in I} \subset \mathcal{S}$ *if* $Y = Y(I)$,
- *(Type B)* $\quad Y(\Omega, Y_0)$ *for some* $Y_0 \in \mathcal{S}$ *if* $Y = Y(\Omega)$.

Here the quasi-Banach space $Y(I, \{Y_i\}_{i \in I})$ is the linear subset of $\prod_{i \in I} Y_i$ of the elements $\{x_i\}_{i \in I}$ with finite quasi-norm

$$\|\{x_i\}_{i \in I} | Y(I, \{Y_i\}_{i \in I})\| := \|\{\|x_i\|_{Y_i}\}_{i \in I}\|_Y .$$

Note that a type B leaf (i.e. of the form $Y = Y(\Omega)$) can grow only one leaf of its own.

We shall also refer to either $\{Y_i\}_{i \in I}$, or Y_0 as to the leaves growing from Y, which could have been a leaf itself before the tree growing process. Let us designate by $IG(\mathcal{S})$ the class of all spaces obtained from an element of $\mathcal{S}$ in a **finite** *number of tree growing steps consisting of the tree growing processes for some or all of the current leaves.*

Thus, there is a one-to-one correspondence between $IG(\mathcal{S})$ and the trees of finite depth with vertices from $\mathcal{S}$, such that every vertex of the form $Y(I)$ has at most I branches and every vertex of the form $Y(\Omega)$ has at most one branch. The tree corresponding to a space $X \in IG(\mathcal{S})$ is designated by $T(X)$.

The set of all vertices (corresponding to elements of $\mathcal{S}$) of the tree corresponding to some $X \in IG(\mathcal{S})$ will be denoted by $\mathcal{V}(X)$.

We shall always assume that the generating set $\mathcal{S}$ of $IG(\mathcal{S})$ is minimal in the sense that there does not exist a proper subset $\mathcal{Q} \subset \mathcal{S}$, such that $\mathcal{S} \subset IG(\mathcal{Q})$.

If the set $\mathcal{S}$ includes only the spaces described by (different) numbers of parameters from $[1, \infty]$ and $X \in IG(\mathcal{S})$, we assume that $I(X)$ is the set of all the parameters of the spaces at the vertices of the tree T corresponding to X and

$$p_{\min}(X) := \inf I(X) \quad \text{and} \quad p_{\max}(X) := \sup I(X).$$

For the sake of brevity, we also set

$$IG := \{X \in IG(l_p, L_p, lt_{p,q}, lt^*_{p',q'}) : \; [p_{\min}(X), p_{\max}(X)] \subset (1, \infty)\}.$$

We say that two IG-spaces are of the same tree type if their trees are congruent and the spaces at the corresponding vertexes are both either l_p-spaces, or $L_p(\Omega)$-spaces, or $lt_{p,q}$-spaces, or $lt^*_{p,q}$-spaces.

Remark 2.4. *a) We shall deal with the set $\{l_p, L_p, lt_{p,q}, lt^*_{p',q'}\}$, where l_p, L_p, $lt_{p,q}$ and $lt^*_{p',q'}$ designate the classes $\{l_p(I)\}_{p\in[1,\infty]}$, $\{L_p(\Omega)\}_{p\in[1,\infty]}$ for all the Ω with some countable non-atomic measure μ on it, $\{lt_{p,q}(I)\}_{p,q\in[1,\infty]}$ and $\{lt^*_{p',q'}(I)\}_{p,q\in[1,\infty]}$, respectively.*

b) The subclass of l_p-spaces can formally be excluded from the definition of the class IG because l_p is isometric to $lt_{p,p} = lt^*_{p,p}$ but is left there for the sake of technical convenience. The subclass of $lt^*_{p,q}$-spaces is included to make IG closed with respect to passing to dual spaces.

c) The Lebesgue or sequence spaces with mixed norm and the l_p-sums of them are particular elements of $IG(l_p, L_p)$.

d) Two $IG(\mathcal{S})$ spaces form a compatible couple of Banach spaces if their trees are congruent and the spaces standing at the corresponding vertexes form compatible couples themselves.

3. Uniform (p, h_c)-convexity and (q, h_s)-smoothness

In this section we define the notions of (p, h_c)-uniform convexity and (q, h_s)-uniform smoothness of functions and spaces, and cite some estimates of the related constants for all the spaces under consideration established in [2] that will be required in the next sections. Essentially more complete information regarding these homogeneous notions of convexity and smoothness and some more general notions in the settings of both abstract and concrete spaces is provided in [2]. The study of the uniform convexity in a very close homogeneous form was initiated by Chekanov, Nesterov and Vladimirov [13] in 1978. In our terms, their uniform convexity corresponds to the case of (p, h_c)-uniform convexity with the constant function $h_c(t) = C_c$. Some further study was undertaken, for example, by Zălinescu [31] and others. Important estimates for the classical moduli of the uniform convexity and uniform smoothness for particular spaces (different from the classical case of the Lebesgue spaces or l_p presented in [19]) were found by Maleev and Trojanski in [21].

Throughout this section, we assume that F is a convex real-valued function with the convex domain $\mathrm{D}(F)$ in a Banach space X with the norm $\|\cdot\|_X$.

Definition 3.1. *For $1 - \nu = \mu \in (0,1)$, the modulii of uniform μ-convexity and uniform μ-smoothness of a function $F : X \supset \mathrm{D}(F) \to \mathbb{R}$ are functions on $\mathbb{R}_+$ defined, correspondingly, by the relations*

$$\delta_\mu(t, F, X) := (\mu\nu)^{-1} \inf\{\mu F(x) + \nu F(y) - F(\mu x + \nu y) : \\ x, y \in \mathrm{D}(F),\ \|x - y\|_X = t\};$$

$$\rho_\mu(t, F, X) := (\mu\nu)^{-1} \sup\{\mu F(x) + \nu F(y) - F(\mu x + \nu y) : \\ x, y \in \mathrm{D}(F),\ \|x - y\|_X = t\}.$$

Let $p, q \in [1, \infty]$ and h_c, h_s be non-negative functions defined on $(0,1)$.

We say that a function F is (p, h_c)-uniformly convex ((q, h_s)-uniformly smooth) if

$$\inf_{t>0} t^{-p} \delta_\mu(t, F, X) \geq h_c(\mu) > 0 \quad \left(\sup_{t>0}(t^{-q} \rho_\mu(t, F, X) \leq h_s(\mu))\right)$$

for all $t \in \mathbb{R}_+$ and $\mu \in (0,1)$.

Since $\delta_\mu(t, F, X) = \delta_\nu(t, F, X)$ and $\rho_\mu(t, F, X) = \rho_\nu(t, F, X)$, we can and will always assume that $h_c(\mu) = h_c(\nu)$ and $h_s(\mu) = h_s(\nu)$.

For $2 \in [q, p] \subset (1, \infty)$, a Banach space X will be said to be (p, h_c)-uniformly convex (respectively, (q, h_s)-uniformly smooth) if the function $F(x) = \|x\|_X^p$ ($\|x\|_X^q$) is (p, h_c)-uniformly convex ((q, h_s)-uniformly smooth).

For the sake of convenience, we shall assume that

$$h_c(0) = \lim_{\mu \to 0} h_c(\mu) \text{ and } h_s(0) = \lim_{\mu \to 0} h_s(\mu)$$

if the corresponding limit exists.

Remark 3.1. *a) In the case of a Hilbert space H and $F(x) = \|x\|^2$, we have*

$$\delta_\mu(t, \|\cdot\|_H^2, H) = \rho_\mu(t, \|\cdot\|_H^2, H) = t^2.$$

Hence, every Hilbert space is $(2,1)$*-uniformly convex and* $(2,1)$*-uniformly smooth.*

b) A result from [13] implies that a Banach space X *is* (p, h_c) *-uniformly convex with* $h_c(\mu) > 0$ *for* $\mu \in (0,1)$ *if and only if it is* p*-uniformly convex in the classical sense (see [19]). Similarly, a Banach space* X *is* (q, h_s)*-uniformly smooth for a finite* h_s *if and only if it is* q*-uniformly smooth in the classical sense.*

c) As noted in [2], the functions

$$\sup_{y\in A} \|x-y\|_X^p + \phi(x) \text{ for a bounded } A \subset X \text{ and } \limsup_{k\to\infty} \|x-y_k\|_X^p + \phi(x)$$

for a bounded $\{y_k\}_{k=1}^{\infty} \subset X$ *and an arbitrary convex function* $\phi: X \to \mathbb{R}$ *provide examples of* (p, h_c)*-uniformly convex functions if* X *is* (p, h_c)*-uniformly convex, while the functions*

$$\inf_{y\in A} \|x-y\|_X^p + f(x) + c \text{ for a bounded } A \subset X \text{ and } \liminf_{k\to\infty} \|x-y_k\|_X^p + f(x) + c$$

for a bounded $\{y_k\}_{k=1}^{\infty} \subset X$ *and an arbitrary* $f \in X^*$ *and* $c \in \mathbb{R}$ *provide examples of* (q, h_s)*-uniformly smooth functions if* X *is* (q, h_s)*-uniformly smooth.*

d) Considering the case of the vectors $x, y \in X$ *satisfying* $\mu x + \nu y = 0$*, we obtain the estimates* $h_c(\mu) \le \mu^{p-1} + \nu^{p-1} \le 1$ *if* X *is* (p, h_c)*-uniformly convex and* $h_s(\mu) \ge \mu^{q-1} + \nu^{q-1} \ge 1$ *if* X *is* (p, h_c)*-uniformly smooth because the function* $g(t) = \mu^t + \nu^t$ *is decreasing on* $(0, \infty)$ *for* $\mu = 1 - \nu \in (0,1)$*.*

To formulate the main theorem of this section determining the (p, h_c)-uniform convexity and (q, h_s)-uniform smoothness of IG-spaces, we define two functions. For $s, t \in (1, \infty)$ and $\mu \in [0, 1/2]$, let

$$\omega_{c,0}(s,t) = \begin{cases} (s-1)2^{2-t} & \text{for} \quad s \le 2, \\ \xi_s(z_s)2^{s-t} & \text{for} \quad s \ge 2; \end{cases}$$

$$\omega_{s,0}(s,t) = \begin{cases} (\xi_s(z_s))^{\frac{t-1}{s-1}} \, 2^{\frac{s-t}{s-1}} & \text{for} \quad s \le 2, \\ (s-1)^{t-1}2^{2-t} & \text{for} \quad s \ge 2, \end{cases}$$

where

$$\xi_s(z) = \frac{1+z^{s-1}}{(1+z)^{s-1}},$$

and

$$z_s = \begin{cases} \text{the positive root of } (s-2)z^{s-1} + (s-1)z^{s-2} = 1 & \text{for} \quad s \ne 2, \\ 1 & \text{for} \quad s = 2; \end{cases}$$

Theorem 3.1. *Let* $Y \in \{B^s_{p,q,\mathcal{F}}(G), L^s_{p,q,\mathcal{F}}(G), B^s_{p',q',\mathcal{F}}(G)^*, L^s_{p',q',\mathcal{F}}(G)^*\}$ *for* $G \subset \mathbb{R}^n$, $p \in (1,\infty)^n$, $q, \varsigma \in (1,\infty)$, $s \in \mathbb{R}$. *Assume further that*

$$[\min(p_{\min}, q, 2), \max(p_{\max}, q, 2)] \subset [r_s, r_c] \subset (1,\infty),$$

and X *is either a subspace, or a quotient, or almost isometrically finitely represented in* Y. *Then the space* X *is* (r_c, h_c)*-uniformly convex with positive* h_c *and* (r_s, h_s)*-uniformly smooth with finite* h_s. *Moreover,*

$$h_c(0) = \omega_{c,0}(\min(p_{\min}, q), r_c) \quad \text{and} \quad h_s(0) = \omega_{s,0}(\max(p_{\max}, q), r_s).$$

Theorem 3.2. *Let* $G \subset \mathbb{R}^n$, $p, a \in (1,\infty)^n$, $q, \varsigma \in (1,\infty)$, $s \in (0,\infty)^n$ *and*

$$[\min(p_{\min}, q, a_{\min}, 2), \max(p_{\max}, q, a_{\max}, 2)] \subset [r_s, r_c] \subset (1,\infty).$$

Assume also that

$$Y \in \Big\{B^s_{p,q,a}(G),\ \tilde{B}^{s,A}_{p,q,a}(G),\ L^s_{p,q,a}(G),\ \tilde{L}^{s,A}_{p,q,a}(G),\ b^s_{p,q,a}(G),\ \tilde{b}^{s,A}_{p,q,a}(G),\ l^s_{p,q,a}(G),\ \tilde{l}^{s,A}_{p,q,a}(G),\ B^s_{p',q',a'}(G)^*,\ \tilde{B}^{s,A}_{p',q',a'}(G)^*,\ L^s_{p',q',a'}(G)^*,\ \tilde{L}^{s,A}_{p',q',a'}(G)^*,\ b^s_{p',q',a'}(G)^*,\ \tilde{b}^{s,A}_{p',q',a'}(G)^*,\ l^s_{p',q',a'}(G)^*,\ \tilde{l}^{s,A}_{p',q',a'}(G)^*\Big\},$$

and X *is either a subspace, or a quotient, or almost isometrically finitely represented in* Y. *Then the space* X *is* (r_c, h_c)*-uniformly convex with positive* h_c *and* (r_s, h_s)*-uniformly smooth with finite* h_s. *Moreover, one has*

$$h_c(0) = \omega_{c,0}(\min(p_{\min}, q, a_{\min}), r_c) \quad \text{and} \quad h_s(0) = \omega_{s,0}(\max(p_{\max}, q, a_{\max}), r_s).$$

Contrariwise, if the space Y *is* (β_c, h_c)*-uniformly convex with positive* h_c *and* (β_s, h_s)*-uniformly smooth with finite* h_s*, then*

$$[\min(p_{\min}, q, 2), \max(p_{\max}, q, 2)] \subset [r_s, r_c].$$

Theorem 3.3. *Let* $Y \in \{W^s_p(G),\ W^s_{p'}(G)^*\}$ *for* $G \subset \mathbb{R}^n$, $p \in (1,\infty)^n$, $\varsigma \in (1,\infty)$, $s \in \mathbb{N}^n$ *and*

$$[\min(p_{\min}, 2), \max(p_{\max}, 2)] \subset [r_s, r_c] \subset (1,\infty).$$

Assume also that X *is either a subspace, or a quotient, or almost isometrically finitely represented in* Y. *Then the space* X *is* (r_c, h_c)*-uniformly convex with positive* h_c *and* (r_s, h_s)*-uniformly smooth with finite* h_s. *Moreover, one has* $h_c(0) = \omega_{c,0}(p_{\min}, r_c)$ *and* $h_s(0) = \omega_{s,0}(p_{\max}, r_s)$. *In addition, the condition*

$$[\min(p_{\min}, 2), \max(p_{\max}, 2)] \subset [r_s, r_c]$$

is sharp.

Theorem 3.4. *Let X be either a subspace, or a quotient, or almost isometricall finitely-represented in $Y \in IG$ with $[p_{\min}(Y), p_{\max}(Y)] \subset (1, \infty)$ and*

$$[\min(p_{\min}(Y), 2), \max(p_{\max}(Y), 2)] \subset [r_s, r_c] \subset (1, \infty).$$

Then the space X is (r_c, h_c)-uniformly convex with positive h_c and (r_s, h_s)-uniformly smooth with finite h_s. Moreover, one has $h_c(0) = \omega_{c,0}(p_{\min}(Y), r_c)$ and $h_s(0) = \omega_{s,0}(p_{\max}(Y), r_s)$. In addition, the condition

$$[\min(p_{\min}(Y), 2), \max(p_{\max}(Y), 2)] \subset [r_s, r_c]$$

is sharp.

Remark 3.2. *Note that Besov and Lizorki-Triebel spaces $B^s_{p,q}(\mathbb{R}^n)_w$ and $L^s_{p,q}(\mathbb{R}^n)_w$ (and their duals) for $s_* \in \mathbb{R}$, $q \in (1, \infty)$ and $p \in (1, \infty)^n$ endowed with the wavelet norms are isometric to the spaces $l_q(\mathbb{N}, l_p)$ and $lt_{p,q}$ (and their duals) correspondingly. Thus, Theorem 3.4 covers, in particular, the case of Besov and Lizorkin-Triebel spaces with wavelet norms and their duals with*

$$\begin{aligned} p_{\min}(B^s_{p,q}(\mathbb{R}^n)_w) &= p_{\min}(L^s_{p,q}(\mathbb{R}^n)_w) = p_{\min}(B^s_{p',q'}(\mathbb{R}^n)^*_w) \\ &= p_{\min}(L^s_{p',q'}(\mathbb{R}^n)^*_w) = \min(p_{\min}, q) \end{aligned}$$

and

$$\begin{aligned} p_{\max}(B^s_{p,q}(\mathbb{R}^n)_w) &= p_{\max}(L^s_{p,q}(\mathbb{R}^n)_w) = p_{\max}(B^s_{p',q'}(\mathbb{R}^n)^*_w) \\ &= p_{\max}(L^s_{p',q'}(\mathbb{R}^n)^*_w) = \max(p_{\max}, q). \end{aligned}$$

4. Main results and examples

The whole set of the main results can be described in the following way. One takes either one of Theorems 5.1, 5.2, 5.4, 5.5, 5.10, 5.12, or one of Corollaries 5.2, 5.4, 5.5 written in terms of the extreme parameters of the (p, h_c)-uniform convexity and (q, h_s)-uniform smoothness, and applies it to one of the spaces under consideration by using one of the theorems from Section 3 providing these parameters of convexity and smoothness for this particular space. In the case of Theorem 5.5, one needs one of the theorems of §6 on the existence of uniformly isomorphic and uniformly complemented copies of $l_p(I_n)$-spaces.

For the sake of simplicity the spectrum of function spaces for which the presented results hold was reduced. At the same time, some of the original equivalent norms for Besov and Lizorkin-Triebel spaces are hidden inside wider classes. The following remark deals with these issues.

Remark 4.1. *a) The results of this paper, excluding some related to sharpness, hold for the classes of function spaces of Besov and Lizorkin-Triebel* *type (and their duals, subspaces, factor-spaces and finitely represented spaces) with variable smoothness (including the weighted spaces) which are defined by substituting the power t^s in the definitions of the corresponding spaces in §2 with a more general function $\omega(t,x)$ (see [8]).*

b) The classical Besov and Lizorkin-Triebel spaces defined in terms of the differences (used instead of the a-averaged differences in the definition of $B^s_{p,q,a}(G)$ and $L^s_{p,q,a}(G)$ in §2) have the same geometric properties (dealt with in this paper) as $B^s_{p,q}(\mathbb{R}^n)_w$ and $L^s_{p,q}(\mathbb{R}^n)_w$.

c) The spaces $B^s_{p,q}(G)$ and $F^s_{p,q}(G)$ (see [28]) defined in terms of the Littlewood-Paley decomposition are particular cases of $B^s_{p,q,\mathcal{F}}(G)$ and $L^s_{p,q,\mathcal{F}}(G)$ and their subspaces.

Let us consider some examples of the main results.

The following main result is a combination of Theorems 3.2 and 5.1.

Theorem 4.1. *Let $G \subset \mathbb{R}^n$, $p, a \in (1,\infty)^n$, $q, \varsigma \in (1,\infty)$, $s \in (0,\infty)^n$ and*

$$[\min(p_{\min}, q, a_{\min}, 2), \max(p_{\max}, q, a_{\max}, 2)] \subset [r_s, r_c] \subset (1,\infty).$$

Assume also that

$$\begin{aligned} Y \in \{&B^s_{p,q,a}(G),\ \tilde{B}^{s,A}_{p,q,a}(G),\ L^s_{p,q,a}(G),\ \tilde{L}^{s,A}_{p,q,a}(G), b^s_{p,q,a}(G),\ \tilde{b}^{s,A}_{p,q,a}(G),\\ &l^s_{p,q,a}(G),\ \tilde{l}^{s,A}_{p,q,a}(G),\ B^s_{p',q',a'}(G)^*,\ \tilde{B}^{s,A}_{p',q',a'}(G)^*,\ L^s_{p',q',a'}(G)^*,\\ &\tilde{L}^{s,A}_{p',q',a'}(G)^*, b^s_{p',q',a'}(G)^*,\ \tilde{b}^{s,A}_{p',q',a'}(G)^*,\ l^s_{p',q',a'}(G)^*,\ \tilde{l}^{s,A}_{p',q',a'}(G)^*\}, \end{aligned}$$

and X is either a subspace, or a quotient, or almost isometrically finitely represented in Y. Assume, in addition, that $D \subset X$ is its convex locally weakly compact subset, and A and B are bounded subsets of X. Then, for every $\sigma > 0$, the relative Chebyshev centre map $C_D : H(X) \to D$ is well-defined and satisfies

$$\begin{aligned} \|C_D(A) - C_D(B)\|_X \le {r_c}^{1/r_c} &(\omega_{c,0}(\min(p_{\min}, q, a_{\min}), r_c))^{-1/r_c}\\ &\times (\min(r(A,D), r(B,D))(1+\sigma))^{1/r_c'} \left(d_{H(X)}(A,B)\right)^{1/r_c} \end{aligned}$$

for $d_{H(X)}(A,B) \le \sigma \min(r(A,D), r(B,D))$;

$$\begin{aligned} \|C_D(A) - C_D(B)\|_X \le &{r_c}^{1/r_c} (\omega_{c,0}(\min(p_{\min}, q, a_{\min}), r_c))^{-1/r_c}\\ &\times (1 + 1/\sigma)^{1/r_c'} d_{H(X)}(A,B) \end{aligned}$$

for $d_{H(X)}(A,B) \ge \sigma \min(r(A,D), r(B,D))$;

$$\begin{aligned} \|C_D | H^{1/r_c}(H(F), D)\| \le {r_c}^{1/r_c} &(\omega_{c,0}(\min(p_{\min}, q, a_{\min}), r_c))^{-1/r_c}\\ &\times (r(F,D) + d(F))^{1/r_c'} \end{aligned}$$

for every bounded $F \subset X$.

Moreover, the Hölder smoothness of C_D *(for all such* $D \subset X$*) cannot be better then* $(\max(p_{\max}, q, 2))^{-1}$ *if* $X = Y$ *isometrically.*

The next theorem is the combination of Theorem 3.2 and Theorem 5.12.

Theorem 4.2. *Let* $G \subset \mathbb{R}^n$, $p, a \in (1,\infty)^n$, $q, \varsigma \in (1,\infty)$, $s \in (0,\infty)^n$ *and*

$$[\min(p_{\min}, q, a_{\min}, 2), \max(p_{\max}, q, a_{\max}, 2)] \subset [r_s, r_c] \subset (1,\infty).$$

Assume also that

$$\begin{aligned} Y_0 \in \{ & B^s_{p,q,a}(G),\ \tilde{B}^{s,A}_{p,q,a}(G),\ L^s_{p,q,a}(G),\ \tilde{L}^{s,A}_{p,q,a}(G), b^s_{p,q,a}(G),\ \tilde{b}^{s,A}_{p,q,a}(G), \\ & l^s_{p,q,a}(G),\ \tilde{l}^{s,A}_{p,q,a}(G),\ B^s_{p',q',a'}(G)^*,\ \tilde{B}^{s,A}_{p',q',a'}(G)^*,\ L^s_{p',q',a'}(G)^*, \\ & \tilde{L}^{s,A}_{p',q',a'}(G)^*,\ b^s_{p',q',a'}(G)^*,\ \tilde{b}^{s,A}_{p',q',a'}(G)^*,\ l^s_{p',q',a'}(G)^*,\ \tilde{l}^{s,A}_{p',q',a'}(G)^* \}, \end{aligned}$$

and X_0 *is either a subspace, or a quotient, or almost isometrically finitely represented in* Y_0. *Let, in addition,* X *and* Y *be quasi-Banach spaces, and* $d_{BM}(X, X_0) < d$. *Assume that* A *is a closed linear surjective operator from* X *onto* Y, *and that* $F \subset Y$ *is bounded and*

$$c_c = \omega_{c,0}(\min(p_{\min}, q, a_{\min}), r_c) \text{ and } c_s = \omega_{s,0}(\max(p_{\max}, q, a_{\max}), r_s).$$

Then there exists a homogeneous right-inverse operator $B:\ Y \to X$ *satisfying*

$$A \circ B = I,\ B\lambda x = \lambda B x,\ \sup_{y \in B_Y} \|By\|_X \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\|;$$

$$\begin{aligned} \|By - Bx\|_X \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \Bigg(& \|y-x\|_Y + \Big(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\Big)^{1/r_c} \\ & \times \Big(\|x\|_Y^{r_s} + c_s c_c^{-r_s/r_c}\|y-x\|_Y^{r_s}\Big)^{1/r_s - 1/r_c} \|y-x\|_Y^{r_s/r_c} \Bigg) \end{aligned}$$

for every $x, y \in Y$, *and*

$$\begin{aligned} \|B|H^{r_s/r_c}(F,X)\| \le & d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \Bigg(d(F)^{1-r_s/r_c} + \Big(\frac{r_c c_s}{r_s c_c^{1+r_s/r_c}}\Big)^{1/r_c} \\ & \times \Big(r(F,\{0\})^{r_s} + c_s c_c^{-r_s/r_c} d(F)^{r_s}\Big)^{1/r_s - 1/r_c} \Bigg), \end{aligned}$$

where $\tilde{X} = X/\mathrm{Ker}\ A$ *and* $\tilde{A}:\ \tilde{X} \to Y$ *is defined by the canonical factorisation* $A = \tilde{A} \circ Q_{\mathrm{Ker}\ A}$.

Remark 4.2. *In the case of Besov and Lizorkin-Triebel spaces and their duals, the isomorphic (quasi) Banach spaces are often just the same spaces endowed with equivalent (quasi) norms. Many examples of equivalent norms (of the same types as we use here) for function spaces under consideration are provided in [1].*

The following example of a main theorem is a combination of Theorems 3.3 and 5.4 for the main part, and Theorems 5.5 and 6.2 for the sharpness part.

Theorem 4.3. *Let* $Y_0 \in \{W^s_p(G),\ W^s_{p'}(G)^*\}$ *for* $G \subset \mathbb{R}^n$, $p \in (1,\infty)^n$, $\varsigma \in (1,\infty)$, $s \in \mathbb{N}^n$ *and*

$$[\min(p_{\min}, 2), \max(p_{\max}, 2)] \subset [r_s, r_c] \subset (1, \infty).$$

Assume also that X_0 *is either a subspace, or a quotient, or almost isometrically finitely represented in* Y. *Let also* A *be a closed convex subset of a quasi-Banach space* X *that is isomorphic to* X_0 *with* $d_{BM}(X, X_0) < d$ *and* $\sigma > 0$. *For* $a > 0$, *assume further that a metric space* Y *contains an isometric copy* $\tilde{A}$ *of* A *(endowed with the metric inherited from* X*), and* A_a *is the* a*-neighborhood of this copy in* Y. *Then, there exists a retraction* ϕ *of* A_a *onto* $\tilde{A}$ *that does not depend on* σ *and satisfies*

$$d_Y(\phi(x), \phi(y)) \le 8 d r_c^{1/r_c} (\omega_{c,0}(p_{\min}, r_c))^{-1/r_c} ((1+\sigma)r)^{1/r_c'} (d_Y(x,y))^{1/r_c}$$
$$\text{for } d_Y(x,y) \le \sigma \min(a, r(A,A)/8),\ x, y \in A_a;$$

$$d_Y(\phi(x), \phi(y)) \le 8 d r_c^{1/r_c} (\omega_{c,0}(p_{\min}, r_c))^{-1/r_c} (1 + 1/\sigma)^{1/r_c'} d_Y(x,y)$$
$$\text{for } d_Y(x,y) \ge \sigma \min(a, r(A,A)/8),\ x, y \in A_a;$$

$$\|\phi | H^{1/r_c}(Y, \tilde{A})\| \le d(8r_c)^{1/r_c} (\omega_{c,0}(p_{\min}, r_c))^{-1/r_c} (r(A,A) + d(A))^{1/r_c'}.$$

if A *is bounded. If, in addition,* $X = X_0$ *isometrically, then one should take* $d = 1$ *in these estimates.*

If A *is a ball in* X, *while* $X_0 = Y_0$ *isometrically, then the Hölder regularity of any retraction* ϕ *of* A_a *is not greater than* $(\max(p_{\max}, t))^{-1}$.

5. The setting of an abstract Banach space

Definition 5.1. *For a metric space* X, $x, y \in X$, $B \subset X$ *and a bounded* $A \subset X$, *let* $d_X(x,y)$ *be the distance between* x *and* y,

$$d_X(y, B) = \inf_{z \in B} d_X(y, z), \text{ and } r(A, B) = r_X(A, B) = \inf_{x \in B} \sup_{y \in A} d_X(x, y)$$

be the Chebyshev radius of A relative to B. Assume also that

$$r_X(A,x) = r_X(A,\{x\}) \text{ and } r_X(A) = r_X(A,X).$$

The diameter of the set A is

$$d(A) = \sup_{x,y\in A} d_X(x,y).$$

Note that $r_X(\{x\},B) = d_X(x,B)$.

The next definition provides an important example of a metric space.

Definition 5.2. *Let X be a metric space and $B \subset X$. The metric space $H(B)$ is the set of all bounded subsets of B endowed with the Hausdorff metric*

$$d_H(F,G) = \max(\sup_{x\in F} d_X(x,G), \sup_{y\in G} d_X(y,F) \text{ for } F\cup G \subset B.$$

The (closed) ε-neighborhood F_ε of a subset $F \subset M$ in a metric space M is $\{x \in M : \ d_M(x,F) \le r\}$.

Note that, if A and B are subsets of a normed space X and $r > 0$, then

$$d_X(A_r) = d_X(A) + 2r \quad \text{and} \quad r_X(A_r,B) = r_X(A,B) + r.$$

Definition 5.3. *Assume that X and Y are metric spaces, and $\alpha \in (0,1]$. For $f: X \to Y$, its (first order) modulus of continuity on a subset $A \subset X$ is defined, for $t > 0$, by*

$$\omega(t,f) = \omega(t,f,X) := \sup\{d_Y(f(x),f(y) : \ x,y \in A, \ d_X(x,y) < t\}.$$

The mapping f is uniformly continuous on A if

$$\omega(t_0,f,A) < \infty \text{ for some } t_0 > 0, \quad \text{and} \quad \lim_{t\to 0} \omega(t,f,A) = 0.$$

By $H^\alpha(X,Y)$ we designate the family of all mappings $f : X \to Y$ satisfying:

$$\|f|H^\alpha(X,Y)\| := \sup\{d_Y(f(x),f(y))/d_X(x,y)^\alpha : x,y \in X \text{ and } x \ne y\}$$
$$= \sup_{t>0} \frac{\omega(t,f,X)}{t^\alpha} < \infty.$$

Note that $H^\alpha(X,Y)$ is a seminormed space if Y is a (complete) linear metric space, and that $f \in H^\alpha(X,Y)$ is a Hölder (Lipschitz for $\alpha = 1$) mapping.

Remark 5.1. *If X is a convex subset of a normed space endowed with the inherited metric, Y is a metric space, and $f : X \to Y$ with a finite $\omega(t_0, f, X)$ for some $t_0 > 0$, then f is Lipschitz for large distances, i.e., for every $d > 0$,*

$$\omega(t, f) \leq 2\omega(d, f)t/d \quad \text{for } t \geq d.$$

Hence, we will be interested in the smoothness parameters c_α and α for (relatively) small distances:

$$\omega(t, f) \leq c_\alpha t^\alpha \text{ for } t < d.$$

Corollary 5.1. *a) Let X, Y, Z be metric spaces and $\phi \in H^\alpha(X, Y)$, $\psi \in H^\beta(Y, Z)$. Then one has*

$$\|\psi \circ \phi | H^{\alpha\beta}(X, Z)\| \leq \|\phi | H^\alpha(X, Y)\|^\beta \|\psi | H^\beta(Y, Z)\|.$$

b) If X is a bounded metric space with the diameter $d = \sup_{x,y\in A} d_X(x, y)$ and $\emptyset \neq [\beta, \alpha] \subset (0, 1]$, then the norm of the embedding $H^\alpha(X, Y) \subset H^\beta(X, Y)$ is equal to $d^{\alpha-\beta}$.
c) If X and Y are normed spaces, a bounded $F \cup G \subset X$, and $A \in \mathcal{L}(X, Y)$, then

$$d_{H(Y)}(A(F), A(G)) \leq \|A|\mathcal{L}(X, Y)\| d_{H(X)}(F, G)$$

and

$$r(A(F), A(G)) \leq \|A|\mathcal{L}(X, Y)\| r(F, G).$$

Retractions, metric projections and homogeneous inverses of linear operators between Banach spaces are important examples of H^α-mappings considered in what follows.

5.1. Retractions

Definition 5.4. *For a metric space Y and $X \subset Y$, a mapping $f : Y \to X$ is a retraction of Y onto X if $f(x) = x$ for every $x \in X$. The subset X is said to be a retract of Y.*

It is also said to be a uniform or (α, C)-Hölder retract for some $\alpha \in (0, 1]$ and $C > 0$ if the mapping f is uniformly continuous, or $\|f|H^\alpha(Y, X)\| \leq C$, correspondingly. A $(1, C)$-Hölder retract is traditionally called C-Lipschitz.

We shall also say that X is a local (α, ϕ)-Hölder retract of Y for a non-negative non-decreasing function ϕ defined on $(0, \infty)$ if, for every $r > 0$ the subset X is a $(\alpha, \phi(r))$-Hölder retract of its r-neighborhood A_r (with the inherited metric).

A metric space X whose every isometric copy in an arbitrary metric space Y is a retract is called absolute.

According to Part *b*) of the next lemma, $l_\infty(\Gamma)$ is an absolute 1-Lipschitz retract. It is Lemma 1.1 from [6].

Lemma 5.1 ([6]). *a) Every metric space is isometric to a subset of* $l_\infty(X)$.
b) Let Y *be a metric space,* $Z \subset Y$, *and* ω *be a nondecreasing subadditive function defined on* $(0, \infty)$ *with* $\lim_{t\to 0}\omega(t) = 0$. *Assume also that* $f : Z \to l_\infty(\Gamma)$ *satisfies* $\omega(\cdot, f, Z) \leq \omega$. *Then there exists a uniformly continuous extension* $F : Y \to l_\infty(\Gamma)$ *of* f *with* $\omega(\cdot, F, Y) \leq \omega$.

5.1.1. Chebyshev mapping

The Chebyshev cetres and Chebyshev radii are important notions of approximation theory. In its terms the Chebyshev centre is the best uniform approximation of a set by a point, while the Chebyshev radius is the error of this approximation.

Definition 5.5. *Let* X *be a Banach space,* $B \subset X$ *and* $A \subset X$ *is bounded. A Chebyshev centre of* A *relative to* B, *if it exists, is a point* $x \in B$ *satisfying* $r(A, x) = r(A, B)$. *A Chebyshev centre of* A *is a Chebyshev centre of* A *relative to* X. *Let* $B(X)$ *be the class of all bounded subsets of* X.

If the Chebyshev centre relative to B *exists and is unique for every element of a subclass* Σ *of the class of all bounded subsets of* X, *then the Chebyshev centre mapping* $C_B : \Sigma \to B$ *is the mapping that assigns, to every element of* Σ, *its Chenyshev centre relative to* B.

Let us note that a Chebyshev centre of A relative to X is, in fact, a Chebyshev centre of A relative to any $B \supset A_{r_X(A)}$. It was shown by Garkavi [16] that a Banach space X is either a Hilbert space, or of dimension one or two, if and only if a Chebyshev centre of every bounded subset of X belongs to its closed convex envelope.

Remark 5.2. *a) Let us note that a Chebyshev centre of* A *always exists and the set of all Chebyshev centres is compact if* X *is reflexive and* B *is locally weakly compact.*
b) Garkavi [15] has shown that the uniqueness of the Chebyshev centre is equivalent to the uniform convexity of X *in every direction. Every uniformly convex space is uniformly convex in every direction.*
c) Let X *be a normed space and a bounded* $A \cup B \subset X$. *Note that, for every* $\varepsilon > 0$, *a Chebyshev centre of* A *relative to* B *is also a Chebyshev centre of* A_ε *relative to* B *and vice-versa, and that* $r(A_\varepsilon, B) = r(A, B) + \varepsilon$. *Hence, we also have*

$$|r(A, B) - r(C, B)| \leq d_H(A, C).$$

To prove the first main theorem of this subsection we need the following lemma.

Lemma 5.2. *For $p \in (1, \infty)$, let f be a (p, h_c)-uniformly convex function defined on a convex subset $D \subset X$ of a Banach space X, and $x, y \in D$. Assume also that*

$$f(x) = \min_{z \in [x,y]} f(z).$$

Then we have the estimate

$$\limsup_{t \to 0+0} h_c(t) \|y - x\|_X^p \le f(y) - f(x).$$

Proof of Lemma 5.2. Let us consider the convex and monotone on $[0, 1]$ function

$$\phi(t) = f\left(ty + (1 - t)x\right).$$

The (p, h_c)-uniform convexity of f gives us

$$\frac{\phi(1) - \phi(t)}{1 - t} - \frac{\phi(t) - \phi(0)}{t} \ge h_c(t)\|y - x\|_X^p. \tag{1}$$

Noting that ϕ has non-negative right derivative at the origin due to its convexity, we finish the proof by passing to the upper limit with $t \to 0 + 0$:

$$\phi(1) - \phi(0) \ge \phi(1) - \phi(0) - \lim_{t \to 0+0} \frac{\phi(t) - \phi(0)}{t} \ge \limsup_{t \to 0+0} h_c(t)\|y - x\|_X^p.$$

□

The following two assertions are a quantitative counterpart of an important result due to Amir from [4]: a Banach space is uniformly convex if and only if $C_X(A)$ is a singleton for every bounded $A \subset X$, and the Chebyshev map C_X is uniformly continuous on bounded subsets of X (endowed with Hausdorff metric). The next theorem is also a counterpart of Theorem 1.27 from [6] where the classical modulus of convexity was employed.

Theorem 5.1. *For $p \in [2, \infty)$, let X be a (p, h_c)-uniformly convex Banach space, $D \subset X$ be its convex locally weakly compact subset, and A and B be bounded subsets of X. Then, for every $\sigma > 0$, the relative Chebyshev centre map $C_D : H(X) \to D$ is well-defined and satisfies*

$$\begin{aligned}\|C_D(A) - C_D(B)\|_X \le & p^{1/p}\big(\limsup_{t \to 0+0} h_c(t)\big)^{-1/p} \\ & \times(\min\left(r(A, D), r(B, D)\right)(1 + \sigma))^{1/p'}\big(d_{H(X)}(A, B)\big)^{1/p}\end{aligned}$$

for $d_{H(X)}(A, B) \le \sigma \min\left(r(A, D), r(B, D)\right)$;

$$\|C_D(A) - C_D(B)\|_X \le p^{1/p}\left(\limsup_{t \to 0+0} h_c(t)\right)^{-1/p}(1 + 1/\sigma)^{1/p'} d_{H(X)}(A, B)$$

for $d_{H(X)}(A,B) \geq \sigma \min\left(r(A,D), r(B,D)\right)$;

$$\|C_D|H^{1/p}(H(F),D)\| \leq p^{1/p}\left(\limsup_{t\to 0+0} h_c(t)\right)^{-1/p}(r(F,D)+d(F))^{1/p'}$$

for every bounded $F \subset X$. *Moreover, one also has*

$$p^{1/p}\left(\limsup_{t\to 0+0} h_c(t)\right)^{-1/p} \leq \min\left(e^{\frac{\limsup_{t\to 0+0} h_c(t)}{e}}, e^{1/e}\sqrt{\limsup_{t\to 0+0} h_c(t)}\right).$$

Remark 5.3. *a) Theorem* 5.1 *perfectly illustrates the statement of Remark* 5.1*: the first inequality corresponds to relatively small distances, while the second — to relatively large ones.*
b) If B has a non-empty relative interior then the case of A and B being relative balls in B of equal radius providing 1*-Lipschitz estimates does not reflect the convexity of X at all according to the next theorem.*
c) The metric projection map $P_D : X \to D \subset X$ onto a closed convex subset D is the restriction of C_D onto the singletons $\{\{x\}\}_{x\in X}$:

$$P_D x = C_D\left(\{x\}\right).$$

According to Theorems 5.2 *and* 5.8*, this restriction has higher Hölder regularity q/p than the relative Chebyshev centre map itself $(1/p)$ if the Banach space X is both (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth.*

Proof of theorem 5.1. D. P. Milman has shown that every uniformly convex space is reflexive. Hence, according to Parts *a*) and *b*) of Remark 5.2, the map C_D is well-defined on the bounded subsets of X. Assume that $r(A,D) \geq r(B,D)$. Taking an arbitrary $\varepsilon > d_H(A,B)$, we conclude with the aid of Part *c*) of Remark 5.2 that

$$(1)\quad r(A,D) = r(A,C_D(A)) \leq r(A,C_D(B)) \leq r(B_\varepsilon, C_D(B)) = r(B,D)+\varepsilon.$$

Since X is (p,h_c)-uniformly convex, the same is true for the function $f(x) = r(A,x)$ defined on the convex set D. Therefore, according to (1), Lagrange's theorem and Lemma 5.2 with $x = C_D(A)$ and $y = C_D(B)$, we have

$$\begin{aligned}\limsup_{t\to 0+0} h_c(t)\|C_D(B)-C_D(A)\|_X^p &\leq (r(A,C_D(B)))^p - (r(A,C_D(A)))^p\\ &\leq p\,(r(B,D)+\varepsilon)^{p-1}\varepsilon^p. \qquad (2)\end{aligned}$$

This is equivalent to the estimate

$$\begin{aligned}\|C_D(A)-C_D(B)\|_X \leq & p^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p}\Big(\min\left(r(A,D), r(B,D)\right)\\ &+ d_{H(X)}(A,B)\Big)^{1/p'}\Big(d_{H(X)}(A,B)\Big)^{1/p}, \qquad (3)\end{aligned}$$

implying the first three inequalities of the theorem. To establish the last inequality, we also have to take into account that

$$d\,(H(F)) = \sup_{A,B\in H(F)} d_{H(X)}(A,B) = \sup_{x,y\in F} \|x-y\|_X = d(F)$$

and $r(A,D) \le r(F,D)$ for $A \subset F$.

When ε tends to $d_H(A,B)$, one obtains the main estimate of the theorem. The rest follows from the calculus inequalities $(x/c)^{1/x} \le e^{c/e}$ for $x, c > 0$, and $p \ge 2$. □

Corollary 5.2. *For $p \in [2,\infty)$, let X be a quasi-Banach space that is isomorphic to a (p,h_c)-uniformly convex Banach space Y with $d_{BM}(X,Y) < d$, $D \subset X$ be its convex locally weakly compact subset, and let A and B be bounded subsets of X. Then for every $\sigma > 0$ there exists a mapping $\psi_D : H(X) \to D$ satisfying*

$$\psi_D(\{x\}) = x \text{ for every } x \in X;$$

$$\|\psi_D(A) - \psi_D(B)\|_X \le dp^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} \times (\min(r(A,D), r(B,D))\,(1+\sigma))^{1/p'}\big(d_{H(X)}(A,B)\big)^{1/p}$$

for $d_{H(X)}(A,B) \le \sigma \min(r(A,D), r(B,D))$;

$$\|\psi_D(A) - \psi_D(B)\|_X \le dp^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p}(1+1/\sigma)^{1/p'} d_{H(X)}(A,B)$$

for $d_{H(X)}(A,B) \ge \sigma \min(r(A,D), r(B,D))$;

$$\|\psi_D | H^{1/p}(H(F),D)\| \le dp^{1/p}\left(\limsup_{t\to 0+0} h_c(t)\right)^{-1/p} (r(F,D) + d(F))^{1/p'}$$

for every bounded $F \subset X$.

Proof of Corollary 5.2 Since the Banach-Mazur distance between X and Y is less than d, there exists an isomorphism $T : X \to Y$ satisfying

$$(1) \qquad \|T|\mathcal{L}(X,Y)\|.\|T^{-1}|\mathcal{L}(Y,X)\| < d.$$

Observing that TD is a closed convex subset of Y and $H(TD) = T(H(D))$, let us choose $\psi_D = T^{-1} \circ C_{TD} \circ T$. According to Part $c)$ of Corollary 5.1 and the choice of T, one has, for every $A, B \in H(X)$,

$$\|\psi_D A - \psi_D B\|_X \le \|T^{-1}|\mathcal{L}(Y,X)\|.\|C_{TD}TA - C_{TD}TB\|_Y;$$

$$r\,(TA,TD) \le \|T|\mathcal{L}(X,Y)\| r(A,D),$$

$$(2) \qquad d_{H(Y)}(TA,TB) \le \|T|\mathcal{L}(X,Y)\| d_{H(X)}(A,B).$$

The inequality (3) from the proof of Theorem 5.1 holds with TA, TB and TD instead of A, B and D, correspondingly, so we combine it with (1) and (2) to establish the following counterpart of (3) from the proof of Theorem 5.1 for ψ_D:

$$(3) \qquad \|\psi_D(A) - \psi_D(B)\|_X \le dp^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} \left(d_{H(X)}(A,B)\right)^{1/p} \times \left(\min\left(r(A,D), r(B,D)\right) + d_{H(X)}(A,B)\right)^{1/p'}.$$

Proceeding as in the proof of Theorem 5.1, we complete the proof of this corollary. □

The next theorem shows the sharpness of the previous one. It is a quantitative version of the "if"-part of Theorem 5 from [4] with essentially the same proof.

Theorem 5.2. *For $r > 0$, let X be a Banach space, $B \supset rB_X$ be its closed convex subset, and $\alpha \in (0,1]$. If the Chebyshev mapping C_B is well-defined and $C_B \in H^\alpha(H(rB_X), B)$, then the space X is $(1/\alpha, h_c)$-uniformly convex and $\alpha \in (0, 1/2]$.*

Proof of Theorem 5.2. It is an observation that the appropriate part of the proof of Theorem 5 from [4] can be quantified as it is. Thanks to the homogeneity of X, we can assume, without loss of generality, that $r = 1$. Assume also that x, y are arbitrary and satisfy $\|x\|_X = \|y\|_X = 1$. Let A and D be, correspondingly, the convex envelopes of the subsets of B_X

$$\left\{x, -y, \frac{x+y}{\|x+y\|_X}, -\frac{x+y}{\|x+y\|_X}\right\} \quad \text{and} \quad \left\{x, -y, \frac{x+y}{2}, -\frac{x+y}{2}\right\}.$$

Then A contains the diameter of B_X, while D is a parallelogram, and therefore,

$$(1) \qquad C_B(A) = 0 \;\text{ and }\; C_B(D) = \frac{x-y}{4}.$$

In turn, the convexity of the norm implies

$$d_H(A,D) = \left\|\frac{x+y}{\|x+y\|_X} - \frac{x+y}{2}\right\|_X = 1 - \frac{\|x+y\|_X}{2}.$$

Thus, one has

$$\frac{\|x-y\|_X}{4} \le C_\alpha \left(1 - \frac{\|x+y\|_X}{2}\right)^\alpha.$$

This means (see [19]) that X is $1/\alpha$-uniformly convex in the classical sense:

$$\delta_X(t) \geq (4C_\alpha)^{-1/\alpha} t^{1/\alpha}. \tag{2}$$

And, according to Dvoretzky's theorem, $1/\alpha \geq 2$. The classical p-uniform convexity is equivalent to (p, h_c)-uniform convexity, that can be shown as in [17], [5, Proposition II.1.1] and [14].

5.1.2. Retraction problem

A very important tool is the following theorem that is Theorem 1.7 from [6].

Theorem 5.3 ([24, 6]). *Let A be a closed convex subset of a Banach space X. Then $H(A)$ is an absolute 8-Lipschitz retract.*

Combining this theorem with the both parts of Lemma 5.1 and Part *a*) of Corollary 5.1, one obtains the following corollary.

Corollary 5.3 ([6]). *Let A be a closed convex subset of a Banach space X, and let Y be a subset of a metric space Z. Then every Lipschitz mapping $f: Y \to H(A)$ admits an extension $F: Z \to H(A)$ satisfying*

$$\|F|H^1(Z, H(A))\| \leq 8\|f|H^1(Y, H(A))\|.$$

The next assertion is the first main theorem of this subsection.

Theorem 5.4. *For $p \in [2, \infty)$, let A be a closed convex subset of a quasi-Banach space X that is isomorphic to a (p, h_c)-uniformly convex Banach space Z with $d_{BM}(X, Z) < d$ and $\sigma > 0$. For $a > 0$, assume also that a metric space Y contains an isometric copy $\tilde{A}$ of A (endowed with the metric inherited from X), and A_a is the a-neighborhood of this copy in Y. Then, there exists a retraction ϕ of A_a onto $\tilde{A}$ that does not depend on σ and satisfies*

$$d_Y(\phi(x), \phi(y)) \leq 8dp^{1/p}\Big(\limsup_{t \to 0+0} h_c(t)\Big)^{-1/p} ((1+\sigma)r)^{1/p'} (d_Y(x, y))^{1/p}$$

for $d_Y(x, y) \leq \sigma \min(a, r(A, A)/8)$, $x, y \in A_a$;

$$d_Y(\phi(x), \phi(y)) \leq 8dp^{1/p}\Big(\limsup_{t \to 0+0} h_c(t)\Big)^{-1/p} (1 + 1/\sigma)^{1/p'} d_Y(x, y)$$

for $d_Y(x, y) \geq \sigma \min(a, r(A, A)/8)$, $x, y \in A_a$;

$$\|\phi|H^{1/p}(Y, \tilde{A})\| \leq d(8p)^{1/p} \left(\limsup_{t \to 0+0} h_c(t)\right)^{-1/p} (r(A, A) + d(A))^{1/p'},$$

if A is bounded. Moreover, if X is a (p, h_c)-uniformly convex Banach space itself, then one can take $d = 1$ in these estimates.

Remark 5.4. *One can also use the numerical estimate given in Theorem 5.1.*

Proof of Theorem 5.4. Let us note that A is isometric to the subset of singletons $\{\{x\}\}_{x\in A} \subset H(A)$. Hence, according to Corollary 5.3, the corresponding isometry of $\tilde{A}$ into $H(A)$ has an 8-Lipschitz extension $f: A_a \to H(A)$. Particularly, one has, for every $x \in A_a$,

$$(1) \qquad r(f(x), A) \le 8d_Y(x, \tilde{A}) \le 8a \quad \text{and} \quad r(f(x), A) \le r(A, A).$$

Thus, considering the composition $\psi_A \circ f$ with ψ_A provided by Corollary 5.2, we combine (1) with the estimate (3) from the proof of Corollary 5.2, to obtain

$$(2) \qquad \|\psi_A \circ f(x) - \psi_A \circ f(y)\|_X \le 8dp^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p}\Big(d_Y(x,y)\Big)^{1/p} \times \Big(\min(a, r(A,A)/8) + d_Y(x,y)\Big)^{1/p'}.$$

This estimate implies the first two inequalities of the statement of the theorem. The last inequality follows from the third estimate for the $1/p$-Hölder norm of ψ_A for bounded $F = A$ from Corollary 5.2, and the composition rule (Part *a*) of Corollary 5.1).

If we, actually, have $X = Z$, then one takes $d = 1$ because we can use the Chebyshev map C_D and Theorem 5.1 instead of ψ_A and Corollary 5.2. □

To investigate the sharpness of the exponents in the last theorem, we need an isomorphic version of Lemma 1.28 from [6] that is the next lemma. To highlight the changes we outline the proof. Let us define some related objects.

For every $p \in [1, \infty]$, $n \in \mathbb{N}$, $a > 0$ and $F \subset I_n$, let x_F be the vector defined by the multiple of the characteristic function $a\chi_F : I_n \to \{0, a\}$. Note that $\|x_F - x_H\|_p = a|F\triangle H|^{1/p}$.

Remark 5.5. *Although this information is not used in what follows, let us note that $\{x_F\}_{F\subset I_n}$ are the extreme points of the translate of the multiple of the unit ball of $l_\infty(I_n)$: $\frac{1}{2}x_{I_n} + \frac{a}{2}B_{l_\infty(I_n)}$. In the next lemma we shall deal with all of them but the origin.*

Lemma 5.3. *For $p \in (1, \infty)$ and $r > 0$, let $T_n : l_p(I_n) \to l_\infty$ realize an isomorphism between $l_p(I_n)$ and $\operatorname{Im} T_n$, and $\{aT_n x_F\}_{F\subset I_n} \subset B \subset \operatorname{Im} T_n$ be a closed and convex uniform retract of its r-neighborhood B_r in l_∞. Then the modulus of continuity of the corresponding retraction f satisfies*

$$\omega(c_0^p r^{1-p} a^p, f) + n^{-1/p} 2^{1+1/p}\omega(r/2, f) \ge a/c_1 \quad \textit{for every} \quad a > 0,$$

where

$$c_0 = \|T_n | \mathcal{L}(l_p(I_n), l_\infty)\| \quad \textit{and} \quad c_1 = \|T_n^{-1} | \mathcal{L}(l_\infty, l_p(I_n))\|.$$

Proof of Lemma 5.3. Following the pattern of the proof of Lemma 1.28 in [6], where the case of isometric copies of l_p and its unit ball was considered, we adapt it to our isomorphic setting.

For every x_F, one defines y_F coordinatewise by the relation

$$(y_F)_i = r/2 + \min_{H \in I_n} (T_n x_H)_i + c_0^p r^{1-p} a^p |F \triangle H|,$$

implying

$$\|y_F - y_H\|_\infty \le c_0^p r^{1-p} a^p |F \triangle H|. \tag{1}$$

The inequality $t/r < 1 + (t/r)^p$ implies

$$\|T_n x_F - T_n x_H\|_\infty < r + \|T|\mathcal{L}(l_p(I_n), l_\infty)\|^p r^{1-p} a^p |F \triangle H|.$$

Hence, one also has

$$\|y_F - T_n x_F\|_\infty \le r/2. \tag{2}$$

The group G_n of the permutations of I_n acts (by isometries) on $l_p(I_n)$:

$$(S_\sigma x)_i = x_{\sigma^{-1}(i)} \text{ for } i \in I_n.$$

Introducing the vectors $\{z_F\}_{F \subset I_n} \subset l_p(I_n)$ defined by

$$z_F = \frac{1}{n!} \sum_{\sigma \in G_n} S_\sigma T_n^{-1} f(y_{\sigma^{-1}(F)}),$$

one sees that $S_\sigma z_F = z_{\sigma(F)}$ for $\sigma \in G_n$ and thus,

$$z_F = \lambda_F \chi_F + \varepsilon_F \chi_{I_n \setminus F}. \tag{3}$$

Since the estimates in (1) and (2) are uniform by F, we also have

$$\|T_n z_F - T_n z_H\|_\infty \le c_0^p r^{1-p} a^p |F \triangle H| \quad \text{and} \quad \|T_n z_F - T_n x_F\|_\infty \le r/2. \tag{4}$$

Considering $F \subset H \subset I_n$ with $|F| = [n/2]$ and $|H \setminus F| = 1$, one deduces from (3) and (4) that

$$|H|^{1/p} |a - \lambda_H| \le \|z_H - x_H\|_p \le c_1 \|T_n z_H - T_n x_H\|_\infty \le c_1 \omega(r/2, f); \tag{5}$$

$$|I_n \setminus F|^{1/p} |\varepsilon_F| \le \|z_F - x_F\|_p \le c_1 \|T_n z_F - T_n x_F\|_\infty \le c_1 \omega(r/2, f). \tag{6}$$

Finally, $(4-6)$ and the definition of c_1 completes the proof:

$$\begin{aligned} \omega(c_0^p r^{1-p} a^p |F \triangle H|) &\ge \|T_n z_F - T_n z_H\|_\infty \ge c_1^{-1} \|z_F - z_H\|_p \\ &\ge c_1^{-1} |\lambda_H - \varepsilon_F| \ge a/c_1 - 2|H|^{-1/p} \omega(r/2, f) \\ &\ge a/c_1 - 2^{1+1/p} n^{-1/p} \omega(r/2, f). \end{aligned}$$

□

The following important technical lemma shows that the balls are among the best retracts of their own normed spaces in the sense of the existence of 2-Lipschitz retractions (that are also metric projections if X is strictly convex).

Lemma 5.4. *Let X be a normed space and B_X be its unit ball. Then there exists a 2-Lipschitz retraction of X onto B_X. It is $\phi_X : X \to B_X$ defined by*

$$\phi_X(x) = \frac{x}{\max(1, \|x\|_X)}.$$

Proof of Lemma 5.4. Since $\phi_X(x) = x$ for $\|x\|_X \leq 1$, we have to consider only two cases.

Assuming that $\|x\|_X > 1$ and $\|y\|_X > 1$, one has, by the triangle inequality and the identity

$$(1) \qquad \frac{x}{\|x\|_X} - \frac{y}{\|y\|_X} = \frac{x-y}{\|x\|_X} + y\left(\frac{1}{\|x\|_X} - \frac{1}{\|y\|_X}\right),$$

$$\left\|\frac{x}{\|x\|_X} - \frac{y}{\|y\|_X}\right\|_X \leq \frac{\|x-y\|_X}{\|x\|_X} + \frac{|\|x\|_X - \|y\|_X|}{\|x\|_X} \leq 2\|x-y\|_X$$

In the case $\|x\|_X > 1$ and $\|y\|_X \leq 1$, we have, by the same argument,

$$\left\|\frac{x}{\|x\|_X} - y\right\|_X \leq \frac{\|x-y\|_X}{\|x\|_X} + \|y\|_X \frac{\|x\|_X - 1}{\|x\|_X} \leq$$

$$\leq \frac{\|x-y\|_X}{\|x\|_X} + \|y\|_X \frac{\|x\|_X - \|y\|_X}{\|x\|_X} \leq 2\|x-y\|_X.$$

□

Theorem 5.5. *For $C_2, C_3 > 0$ and $p \in [2, \infty)$, let X be a separable Banach space containing a C_3-complemented subspace X_n with $d_{BM}(X_n, l_p) \leq C_2$ for every $n \in \mathbb{N}$. Assume also that a closed convex subset B of X contains the unit ball ($B_X \subset B \subset X$), and that B_{r_0} is the r_0-neighborhood of an isometric copy $\tilde{B}$ of B in l_∞, such that there exists a retraction $\phi \in H^\alpha(B_{r_0}, \tilde{B})$ of B_{r_0} onto $\tilde{B}$ satisfying either $\phi \in H^\alpha(B_{r_0}, \tilde{B})$ if B is bounded, or $\omega(t, \phi, B_r) \leq h(r)t^\alpha$ for $t \leq d = d(r)$ with some $r_0 > 0$, $\alpha \in (0, 1]$, and a (necessarily non-decreasing) positive-valued functions $h(r)$ and $d(r)$. Then the estimate $\alpha \leq 1/p$ holds.*

Proof of Theorem 5.5. For every $n \in \mathbb{N}$, let us fix some positive constants c_0, c_1 with $c_0 c_1 \leq C_2$, isomorphisms $T_n : l_p(I_n) \leftrightarrow X_n$ and projectors P_n satisfying

$$(1) \quad \|T_n | \mathcal{L}(l_p(I_n), X_n)\| \leq c_0, \ \|T_n^{-1} | \mathcal{L}(X_n, l_p(I_n))\| \leq c_1, \ \|P_n | \mathcal{L}(X)\| \leq C_3.$$

Let us also choose the parameters $a > 0$ and $r \in (0, r_0]$, such that

$$a = (c_0 n^{1/p})^{-1} \text{ and } \omega(r/2, \phi) < \left(2^{3+1/p} C_2 C_3\right)^{-1}. \tag{2}$$

With this choice, we see that the conditions of Lemma 5.3 are satisfied for the retraction $f = f_n$ of the r-neighborhood $B_{X_n r}$ of the isometric copy $\tilde{B}_{X_n} = J(B_{X_n})$ of the unit ball $B_X \cap X_n$ of X_n in l_∞ defined by

$$f_n = J \circ \phi_{X_n} \circ P_n \circ J^{-1} \circ \phi, \tag{3}$$

where $J: \ B \leftrightarrow \tilde{B}$ is a given isometry and ϕ_{X_n} is the 2-Lipschitz retraction (metric projection) of X_n onto its unit ball B_{X_n}. Particularly, one deduces from the composition rule that $\omega(t, f) \leq 2C_3 \omega(t, \phi)$, and, thus, either

$$f_n \in H^\alpha(B_{X_n r}, \tilde{B}_{X_n}), \quad \text{or} \quad \omega(t, f_n, B_{X_n r}) \leq 2C_3 h(r_0) t^\alpha \tag{4}$$

for the cases of bounded, or unbounded B respectively. Thus, Lemma 5.3 provides, with the help of (1), (2) and (4), the key estimate

$$\omega(r^{1-p}/n, f_n) \geq (C_2 n^{1/p})^{-1} \text{ for } n > r^{1-p}/d,$$

which cpmpletes the proof of the theorem for the cases of both bounded and unbounded B. □

5.2. Chebyshev sets and metric projection

Metric projection is a very important example of a retraction possessing better smoothness than the retractions considered in the previous subsections. In approximation theory it corresponds to the best approximation of a function by a function from a closed convex or linear subclass.

In this subsection we are going to investigate the smoothness of the metric projections on closed convex subsets of either uniformly convex, or both uniformly convex and uniformly smooth spaces. We further provide retractions onto such subsets from the ambient space that is either uniformly convex, or both uniformly convex and uniformly smooth. These retractions possess either better smoothness, or better constants than their counterparts in the previous subsections.

Definition 5.6. *A subset $D \subset X$ of a Banach space X is a Cheyshev set if the metric projection mapping $P_D: \ X \to D$ is well-defined by the relation*

$$\|x - P_D x\|_X = \min_{y \in D} \|x - y\|_X,$$

that is, for every $x \in X$, there exists a unique $y = P_D x$ minimizing the distance between x and D.

As mentioned in Remark 5.3, the metric projection is the restriction of the relative Chebyshev centre map:

$$P_D x = C_D(\{x\}).$$

Note that, by the Hahn-Banach theorem, $P_D y = P_D x$ if $y = \lambda x + (1-\lambda) P_D x$ for some $\lambda \geq 0$.

Remark 5.6. *a) While every closed convex subset of a reflexive and strictly convex Banach space is a Chebyshev set, there exist examples of such Banach spaces with discontinuous (in norm) metric projections on some Chebyshev sets (see [11, 29]). The necessary and sufficient condition on a Banach space for such a continuity was found by L. P. Vlasov [30] (Theorem* 5.6 *below). This condition was introduced by V. L. Shmul'yan in 1940. S. L. Sobolev [27] designed an approach to establish the duality between Lebesgue spaces by utilizing it. Every uniformly convex space satisfies this condition.*

b) There exists an important characterization of inner product spaces due to Phelps [23]: a Banach space with $\dim X > 2$ *is a Hilbert space if and only if the metric projection on every closed convex Chebyshev subset is* 1*-Lipschitz (nonexpansive).*

c) According to Lemma 5.4*, the metric projections onto the balls of a strictly convex normed space* X *are* 2*-Lipschitz. This means that balls are too good subsets to distinguish the peculiar features of the (local) geometry of* X *from the point of view of the metric projection.*

Theorem 5.6 ([30]). *Every closed and convex subset of a Banach space* X *possesses a single-valued and continuous (in norm) metric projection if and only if every subsequence* $\{x_k\}_{k\in\mathbb{N}} \subset X$ *with* $\|x_n\|_X = 1$ *for every* n*, satisfying the condition* $\lim_{k\to\infty} f(x_k) = 1$ *for some* $f \in X^*$ *with* $\|f\|_{X^*} = 1$*, is convergent in* X*.*

The uniform continuity of the set-valued metric projection was investigated by Berdyshev [7], while the same phenomenon for the (single-valued) metric projection in uniformly continuous and uniformly smooth spaces was studied by Björnestål [10] in the case of metric projections onto subspaces and by Benyamini and Lindenstrauss [6] in the case of metric projections onto the closed convex subsets. In the latter case, the estimates for the local uniform continuity, that is for the modulus $\omega(t, P_D, x + r(x)B_X)$ with $r(x) \leq Cd(x, D)$, were established in terms of the moduli of the uniform continuity and uniform smoothness. In some special case, global estimates of similar nature (that cannot be derived from the local ones) were established by Alber [3]. In this section, we establish global estimates in the general setting of an arbitrary closed convex subset providing the same order of the Hölder regularity with explicit numerically friendly constants.

We shall extensively use the following lemma. Its application to the functions of the form $F(x) = \|x - x_0\|_X^r$ with $r \in \{q, p\}$ substitutes the cosine theorem in our non-Euclidean geometry.

An alternative proof (and a generalisation) of this lemma can be found in [2].

Lemma 5.5. *Let X be a Banach space, $x, y \in X$, and let $F: X \to \mathbb{R}$ be convex.*
a) If F is a (p, h_c)-uniformly convex function on X for some $p \in [1, \infty)$ and $f_x \in \partial F(x)$, then we have

$$F(y) \geq F(x) + f_x(y - x) + \limsup_{\theta \to 0+0} h_c(\theta) \|y - x\|_X^p.$$

b) If a (q, h_s)-uniformly smooth function on X for some $q \in [1, \infty)$ and $f_x \in X^*$ is the Frechet (or Gâteaux) derivative of F at x ($f_x = \nabla F(x)$), then we have

$$F(y) \leq F(x) + f_x(y - x) + \liminf_{\theta \to 0+0} h_s(\theta) \|y - x\|_X^q.$$

Proof of Lemma 5.5. As in the proof of Lemma 5.2, we define $\phi(t) = F(ty + (1 - t)x)$.

To establish Part b), we write the condition of (p, h_c)-uniform convexity of F in the form of the second divided difference:

$$\frac{\phi(1) - \phi(t)}{1 - t} - \frac{\phi(t) - \phi(0)}{t} \geq h_c(t) \|y - x\|_X^p. \tag{1}$$

Since ϕ is continuous and $\phi(t) \geq \phi(0) + t f_x(y - x)$, one takes the upper limit in t in (1) to complete the proof of Part a):

$$\phi(1) - \phi(0) - f_x(y - x) \geq \limsup_{\theta \to 0+0} h_c(\theta) \|y - x\|_X^p.$$

To proceed with b) we also write the condition of (p, h_c)-uniform convexity of F in the form of the second divided difference:

$$\frac{\phi(1) - \phi(t)}{1 - t} - \frac{\phi(t) - \phi(0)}{t} \leq h_s(t) \|y - x\|_X^q. \tag{2}$$

Since ϕ is continuous and $\phi(t) = \phi(0) + t f_x(y - x) + o(t)$, one takes the lower limit in t in (2) to complete the proof of the lemma:

$$\phi(1) - \phi(0) - f_x(y - x) \leq \liminf_{\theta \to 0+0} h_c(\theta) \|y - x\|_X^q.$$

□

Since the metric projection onto a closed convex subset of a Banach space is a particular example of a retraction of the whole space onto this subset, it seems reasonable to expect at least the same regularity as given by Theorem 5.4. The theorems below confirm this expectation.

Theorem 5.7. *For $p \in [2, \infty)$, let X be a (p, h_c)-uniformly convex Banach space, and $D \subset X$ be a closed convex subset. Assume also that a bounded $A \subset X$ and $r, \sigma > 0$. Then we have*

$$\|P_D x - P_D y\|_X \leq p^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} ((1+\sigma)r)^{1/p'} \|x-y\|_X^{1/p}$$

for $x \in D_r$, $\|x - y\| \leq \sigma r$;

$$\|P_D x - P_D y\|_X \leq p^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} (1 + 1/\sigma)^{1/p'} \|x-y\|_X$$

for $\|x-y\| \geq \sigma d(x, D)$. In particular, if $f(y - P_D x) \leq 0$ for $f \in X^*$ satisfying

$$\|f\|_{X^*} = 1 \text{ and } f(x - P_D x) = \|x - P_D x\|_X,$$

then one also has

$$\|P_D x - P_D y\|_X \leq \Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-2/p} \|x-y\|_X.$$

Moreover, $P_D \in H^{1/p}(A, D)$ and

$$\| P_D|\, H^{1/p}(A,D)\| \leq p^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} \big(d(A) + r(A,D)\big)^{1/p'}.$$

Proof of Theorem 5.7. According to the definition of P_D and Part $a)$ of Lemma 5.6 below, we have

$$\text{(1)} \qquad f(P_D y - P_D x) \leq 0 \text{ for every } y \in X.$$

The function $F(y) = \|y - x\|_X^p$ is (p, h_c)-uniformly convex and $\partial F(P_D x) = \{-p\|x - P_D x\|_X^{p-1} f\}$. Hence, Part $a)$ of Lemma 5.5 and (1) provide the key estimate

$$\text{(2)} \qquad \|P_D y - x\|_X^p - \|P_D x - x\|_X^p \geq \limsup_{t\to 0+0} h_c(t) \|P_D y - P_D x\|_X^p.$$

Therefore, we obtain, with the aid of Lagrange's theorem and the triangle inequality, the estimate implying the first two inequalities of the theorem

$$\begin{aligned}\text{(3)} \qquad \|P_D y - P_D x\|_X &\leq \Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} p^{1/p} \|P_D y - x\|_X^{1/p'} \|x-y\|_X^{1/p} \\ &\leq \Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} p^{1/p}\big(d(y,D) + \|y-x\|_X\big)^{1/p'} \|x-y\|_X^{1/p}.\end{aligned}$$

Since, in the last inequality of the theorem, the estimate (1) remains true if we substitute $P_D y$ with y, we have the corresponding counterpart of (2):

$$\|y-x\|_X^p - \|P_D x - x\|_X^p \geq \limsup_{t\to 0+0} h_c(t)\|y - P_D x\|_X^p.$$

The latter formula implies

$$\|y - P_D y\|_X \leq \|y - P_D x\|_X \leq \Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} \|y - x\|_X. \tag{4}$$

Hence, changing the roles of x and y in (2), we obtain

$$\|P_D x - y\|_X^p \geq \|P_D x - y\|_X^p - \|P_D y - y\|_X^p \geq \limsup_{t\to 0+0} h_c(t)\|P_D y - P_D x\|_X^p. \tag{5}$$

The combination of (4) and (5) provides the last but one inequality in the statement of the theorem. To finish the proof of the theorem, we note that the estimate for the $H^{1/p}(A,D)$-norm follows directly from (3). □

The next corollary is the isomorphic version of the previous theorem. While the resulting mapping is not a metric projection, it is still a retraction with additional properties and occasionally better constants than those given by Theorem 5.4.

Corollary 5.4. *For $p \in [2,\infty)$, let X be a quasi-Banach space that is isomorphic to a (p,h_c)-uniformly convex Banach space Y with $d_{BM}(X,Y) < d$, and $D \subset X$ be a closed convex subset. Assume also that $A \subset X$ is bounded and $r, \sigma > 0$. Then there exists a retraction ψ_D of X onto D satisfying:*

$$\psi_D(X \setminus D) \subset \partial D;$$

$$\|\psi_D x - \psi_D y\|_X \leq dp^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} ((1+\sigma)r)^{1/p'} \|x-y\|_X^{1/p}$$

for $x \in D_r$, $\|x-y\| \leq \sigma r$;

$$\|\psi_D x - \psi_D y\|_X \leq dp^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} (1+1/\sigma)^{1/p'} \|x-y\|_X$$

for $\|x-y\| \geq \sigma d(x,D)$. Moreover, $P_D \in H^{1/p}(A,D)$ and

$$\|\, \psi_D |\, H^{1/p}(A,D)\| \leq dp^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} (d(A) + r(A,D))^{1/p'}.$$

Proof of Corollary 5.4. According to the condition on the Banach-Mazur distance between X and Y, there exists an isomorphism $T:\ X \leftrightarrow Y$ satisfying $\|T|\mathcal{L}(X,Y)\|\|T^{-1}|\mathcal{L}(Y,X)\| < d$. We define

$$\psi_D = T^{-1} \circ P_{TD} \circ T. \tag{1}$$

This means, in particular, that one also has the identity $T\partial D = \partial TD$ implying the inclusion $\psi_D(X \setminus D) \subset \partial D$. In the same manner as one proceeds in the proof of Corollary 5.2, we use Parts $a)$ and $c)$ of Corollary 5.1 to deduce the following counterpart of the key estimate (3) from the proof of Theorem 5.7

$$\begin{aligned}
\|\psi_D y - \psi_D x\|_X &\le \|T^{-1}|\mathcal{L}(Y,X)\|\|P_{TD}Ty - P_{TD}Tx\|_Y \\
&\le \|T^{-1}|\mathcal{L}(Y,X)\|\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} p^{1/p} \\
&\qquad \times (d(Ty,TD) + \|Tx - Ty\|_Y)^{1/p'}\,\|Tx - Ty\|_Y^{1/p} \\
&\le d\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} p^{1/p}(d(y,D) + \|y - x\|_X)^{1/p'}\|x - y\|_X^{1/p}.
\end{aligned} \tag{2}$$

It implies all the statements of the corollary in exactly the same manner as the estimate (3) in the proof of Theorem 5.7. □

Note that a Hilbert space is $(2,1)$-uniformly convex, while the metric projection is Lipschitz in this case. This observation suggests that the global regularity of the metric projection could be higher if the space is not only (p,h_c)-uniformly convex but also (q,h_s)-uniformly smooth. It is the subject of the next theorem. Since the approach of [10] and Section 2.2 in [6] based on developing the geometry of the cross-sections of the unit ball (that is of quite of interest by itself) leads to local estimates, we have to design an alternative approach based on our counterpart of the cosine theorem for Banach spaces (Lemma 5.5).

We shall need the following simple but useful lemma.

Lemma 5.6. *Let X be a Banach space, $D \subset X$ be its closed convex subset, $x \in X \setminus D$, and $r > 0$. Then the following statements hold.*

a) A point $y \in D$ is the closest point to x in D if and only if there exists $f \in X^*$ with $\|f\|_{X^*} = 1$ satisfying

$$\max_{z\in D} f(z) = f(y) \text{ and } f(x-y) = \|x-y\|_X = \|x-y\|_X.$$

b) If a point $y \in D$ is the closest point to $x \in X \setminus D_r$ in D, then $y_r = (1 - r/d(x,D))\,y + xr/d(x,D)$ is a closest point to x in the closed convex neighborhood D_r of D, and this point is unique if X is strictly convex.

c) For every bounded subsets $A \cup B \subset X$, we have $\mathrm{co}(A+B) = \mathrm{co}(A) + \mathrm{co}(B)$ *and* $\mathrm{co}(A_r) = (\mathrm{co}(A))_r$.

Proof of Lemma 5.6. Part $c)$ follows immediately from the definition of the convex envelope implying, in particular, the convexity of $D_r = D + rB_X$. The latter set is also closed because $D_r = g^{-1}([0,r])$ for the continuous $g(x) = d(x,D)$. Therefore, the nearest to x point of D_r (or any other closed convex subset) is unique if X is strictly convex in Part c). The well-known Part a) follows from the geometric form of the Hahn-Banach theorem applied to the convex sets D and $x + d(x, D)B_X$. Finally, the main assertion of Part b) follows from a) because

$$\max_{z\in D_r} f(z) = f(y_r) \text{ and } f(x - y_r) = \|x - y_r\|_X.$$

□

Now we are in a position to prove the main theorem of this subsection showing that the (global) regularity of the metric projection is higher if, in addition, the space X is also (q, h_s)-uniformly smooth.

Theorem 5.8. *For $2 \in [q,p] \subset (1,\infty)$, let X be a (p, h_c)-uniformly convex and (q, h_s)-uniformly smooth Banach space and let $D \subset X$ be closed and convex. Assume also that $A \subset X$ is a bounded subset of X, $\sigma, r > 0$ and*

$$c_c = \limsup_{t\to 0+0} h_c(t) \text{ and } c_s = \liminf_{t\to 0+0} h_s(t).$$

Then we have

$$\|P_D y - P_D x\|_X \le \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left((1 + c_s c_c^{-q/p}\sigma^q)^{1/q} r\right)^{1-q/p} \|y - x\|_X^{q/p}$$

for every $x \in D_r$ and $y \in X$ satisfying $\|y - x\|_X \le \sigma r$;

$$\|P_D y - P_D x\|_X \le \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(c_s c_c^{-q/p} + \sigma^{-q}\right)^{1/q-1/p} \|y - x\|_X$$

for every $x, y \in X$ satisfying $\|y - x\|_X \ge \sigma d(x, D)$;

$$\left\| P_D \middle| H^{q/p}(A, D)\right\| \le \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(r(A, D)^q + c_s c_c^{-q/p} d(A)^q\right)^{1/q-1/p}.$$

Moreover, if $p = q = 2$, we also have

$$\| P_D | H^1(X, D)\| \le \frac{c_s^{1/2}}{c_c},$$

that is, P_D is $c_s^{1/2}/c_c$-Lipschitz.

Remark 5.7. *The exponent q/p of the Hölder-Lipschitz regularity of the metric projection is sharp when both p and q are sharp for a wide subclass of the independently generated spaces IG, function spaces defined in terms of the wavelet norms and their duals. The proof will appear elsewhere.*

Proof of Theorem 5.8. First assume that $x \in X$ and $y \in X \setminus D$. Then, with the aid of Part $a)$ of Lemma 5.6 providing $g \in X^*$ satisfying $\|g\|_{X^*} = 1$, $g(y - P_D y) = d(y, D)$, $g(x - P_D y) \leq 0$ and $-p\|P_D y - y\|_X^{p-1} g \in \partial\| \cdot - y\|_X^p$, one obtains from Part $a)$ of Lemma 5.5 ("Cosine theorem", as in the proof of Theorem 5.7, that

$$(1) \qquad c_c\|x - P_D y\|_X^p \leq \|y - x\|_X^p - \|y - P_D y\|_X^p < \|y - x\|_X^p.$$

Now assume that $x, y \in X \setminus D$ and $d(y, D) > d(x, D) = a$. Then, according to Part $b)$ of Lemma 5.6 and (1), we have $x \in \partial D_a$,

(2)

$$\|P_{D_a} y - x\|_X \leq c_c^{-1/p}\|y - x\|_X, \; y' = P_{D_a} y = (1 - a/d(y, D))y + ya/d(y, D),$$

and $d(y', D) = a$. Note also that one has $P_D y' = P_D y$ by Lemma 5.6 a).

Applying once more Lemma 5.5 a) together with Lemma 5.6 a), implying the existence of $f \in X^*$ with $\|f\|_X = 1$,

$$(3) \quad f(P_D y - P_D x) \leq 0, \; f(x - P_D x) = a \text{ and } -p\|x - P_D x\|_X^{p-1} f \in \partial\| \cdot - x\|^p,$$

that

$$(4) \qquad \begin{aligned} c_c\|P_D y - P_D x\|_X^p &\leq \|P_D y - x\|_X^p - a^p \\ &\leq \frac{p}{q}\|P_D y - x\|_X^{p-q}\left(\|P_D y - x\|_X^q - a^q\right), \end{aligned}$$

where the second inequality follows from the Lagrange theorem.

At the same time, changing the roles of x and y' ($P_D y' = P_D y$) in (3), one finds $h \in X^*$ satisfying

$$(5) \quad \|h\|_{X^*} = 1, h(P_D x - P_D y) \leq 0 \text{ and } h(y' - P_D y) = \|y' - P_D y\|_X = a.$$

Therefore, the expression for y' in (2) and (5) imply $p\|x - P_D x\|_X^{q-1} h = \nabla\| \cdot - P_D x\|_X^q$ and

$$(6) \qquad \begin{aligned} h(x - y') &= h\left(x - P_D x - (y' - P_D y)\right) + h(P_D x - P_D y) \\ &\leq a - a + h(P_D y - P_D x) \leq 0. \end{aligned}$$

Hence, we can use Lemma 5.5 b), and then (2) to obtain

$$(7) \qquad \begin{aligned} \|P_D y - x\|_X^q - a^q = \|x - P_D y\|_X^q - \|y' - P_D y\|^q &\leq c_s\|x - y'\|_X^q \\ &\leq c_s c_c^{-q/p}\|y - x\|_X^q. \end{aligned}$$

Next, using (7) twice (to estimate the right-hand side of (4), and $\|P_D y - x\|_X^q$ itself), we establish the key estimate

$$(8) \qquad \begin{aligned} \|P_D y - P_D x\|_X &\leq \left(\frac{p c_s}{q c_c^{1+q/p}}\right)^{1/p} \|y - x\|_X^{q/p} \\ &\times \left(d(x, D)^q + c_s c_c^{-q/p}\|y - x\|_X^q\right)^{1/q - 1/p}, \end{aligned}$$

implying immediately all the inequalities of the theorem except for the last one. To prove it, we just skip the usage of the Lagrange theorem and omit the last inequality in (4) and use (7) only once (along with $p = q = 2$):

$$c_c \|P_D y - P_D x\|_X^p \le \|P_D y - x\|_X^p - a^p \le c_s c_c^{-q/p} \|y - x\|_X^q. \tag{9}$$

□

We obtain the following corollary relying on Theorem 5.8 in the same manner as we deduced Corollary 5.5 from Corollary 5.4. As before, we do not have the metric projection for the isomorphic version but still have a very smooth retraction with some additional properties.

Corollary 5.5. *For $2 \in [q,p] \subset (1,\infty)$, let X be a quasi-Banach space that is isomorphic to a (p, h_c)-uniformly convex and (q, h_s)-uniformly smooth Banach space Y with $d_{BM}(X,Y) < d$, and let $D \subset X$ be closed and convex. Assume also that $A \subset X$ is a bounded subset of X, $\sigma, r > 0$ and*

$$c_c = \limsup_{t\to 0+0} h_c(t) \text{ and } c_s = \liminf_{t\to 0+0} h_s(t).$$

Then there exists a retraction ψ_D of X onto D satisfying

$$\psi_D(X \setminus D) \subset \partial D;$$

$$\|\psi_D y - \psi_D x\|_X \le d \left(\frac{p c_s}{q c_c^{1+q/p}} \right)^{1/p} \left((1 + c_s c_c^{-q/p} \sigma^q)^{1/q} r \right)^{1-q/p} \|y - x\|_X^{q/p}$$

for every $x \in D_r$ and $y \in X$ satisfying $\|y - x\|_X \le \sigma r$;

$$\|\psi_D y - \psi_D x\|_X \le d \left(\frac{p c_s}{q c_c^{1+q/p}} \right)^{1/p} \left(c_s c_c^{-q/p} + \sigma^{-q} \right)^{1/q-1/p} \|y - x\|_X$$

for every $x, y \in X$ satisfying $\|y - x\|_X \ge \sigma d(x, D)$;

$$\left\| \psi_D |\, H^{q/p}(A, D) \right\| \le d \left(\frac{p c_s}{q c_c^{1+q/p}} \right)^{1/p} \left(r(A,D)^q + c_s c_c^{-q/p} d(A)^q \right)^{1/q-1/p}.$$

Moreover, if $p = q = 2$, then we also have

$$\| \psi_D |\, H^1(X, D) \| \le \frac{d c_s^{1/2}}{c_c},$$

that is P_D is $d c_s^{1/2}/c_c$-Lipschitz.

Proof of Corollary 5.5. Since $d_{BM}(X,Y) < d$, there exists an isomorphism $T:\ X \leftrightarrow Y$ with

$$\|T|\mathcal{L}(X,Y)\|.\|T^{-1}|\mathcal{L}(Y,X)\| < d. \tag{1}$$

We define ψ_D by the formula (1) from the proof of Corollary 5.4. Moreover, one also has the identity $T\partial D = \partial TD$ implying the inclusion $\psi_D(X\setminus D) \subset \partial D$. As in the proofs of Theorem 5.7 and Corollary 5.2, all the statements of our corollary except for the last one follow immediately from our key estimate

$$\|\psi_D y - \psi_D x\|_X \le d\left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \|y-x\|_X^{q/p} \times \left((d(x,D)^q + c_s c_c^{-q/p}\|y-x\|_X^q\right)^{1/q-1/p}, \tag{2}$$

that is a consequence of the key estimate (8) from the proof of Theorem 5.8 applied to the metric projection P_{TD} of Y onto its closed and convex subset TD, (1) and the observations

$$\|\psi_D y - \psi_D x\|_X \le \|T^{-1}|\mathcal{L}(Y,X)\|\|P_{TD}Ty - P_{TD}Tx\|_Y$$

and $d(Tx,TD) \le \|T|\mathcal{L}(X,Y)\|d(x,D)$. In the Lipschitz case $p = q = 2$, we just use Part *a*) of Corollary 5.1 and the estimates of the Lipschitz norms of the mappings T, T^{-1} and P_{TD} provided by (1) and Theorem 5.8 correspondingly to finish the proof of the corollary. □

5.3. Homogeneous right-inverse operators

In different branches of mathematics, one needs to find a solution of an equation of the form $Ax = y$ with a closed operator A from a quasi-Banach space X onto a quasi-Banach space Y, where the solution is not unique. It is better when the solution x depends continuously on y. Sometimes it is also preferred for the solution to have a minimal possible norm, or comparable to it. For example, A could be a linear partial differential operator, or an operator corresponding to a boundary problem in PDE.

According to the Banach theorem on inverse operator, every injective and surjective bounded linear operator A from a Banach space X into a Banach space Y admits its (left and right) bounded linear inverse A^{-1}. If the kernel of A is not empty but complemented subspace of X (with a projector $P:\ X \to \text{Ker } A$), then the same theorem provides the (bounded linear) right inverse B of A satisfying

$$A \circ B = I \text{ and } B \circ A = I - P.$$

In fact, the bounded right inverse (linear) operator B with respect to a bounded linear operator A exists if and only if Ker A is complemented in X because $(BA)^2 = BA$ if $AB = I$. Unfortunately, it was established by Lindenstrauss and Tzafriri [20] that the only Banach spaces possessing only complemented subspaces are those that are isomorphic to the Hilbert spaces (see also Theorem 5.14 below). It was shown by Skaletskiy [26] that, if X has a uniform normal structure, then there exists a bounded homogeneous right inverse B that is uniformly continuous on every bounded subset of Y (see also [24, 6]).

This subsection is devoted to the existence of the right homogeneous (but non-additive) inverses for the closed linear surjective operators from a (p, h_c)-uniformly convex (and, possibly, (q, h_s)-uniformly smooth) Banach space X onto a Banach space Y that are also Hölder-regular mappings on bounded subsets of Y. We use the results from the previous subsections.

We start with finding a smooth right inverse of the quotient map with the aid of the properties of the metric projection established in the previous subsection.

Theorem 5.9. *For $p \in [2, \infty)$, let X be a (p, h_c)-uniformly convex Banach space, $Z \subset X$ be its subspace, F be a bounded subset of the quotient space $\tilde{X} = X/Z$, and $Q_Z : X \to \tilde{X}$ be the corresponding quotient mapping. Then the mapping $B_Z : \tilde{X} \to X$ defined by $B_Z : \tilde{x} \mapsto P_{Q_Z^{-1}(\tilde{x})} 0$ satisfies*

$$B_Z(\lambda \tilde{x}) = \lambda B_Z(\tilde{x}) \quad \text{for every } \lambda \in \mathbb{R} \text{ and } \tilde{x} \in \tilde{X},$$

$$Q_Z \circ B_Z = I \text{ on } \tilde{X}, \ \|B_Z \tilde{x}\|_X = \|\tilde{x}\|_{\tilde{X}};$$

$$\|B_Z \tilde{y} - B_Z \tilde{x}\|_X \le \|\tilde{y} - \tilde{x}\|_{\tilde{X}} + p^{1/p} \Big(\limsup_{t \to 0+0} h_c(t) \Big)^{-1/p} (\|\tilde{x}\|_{\tilde{X}} + \|\tilde{y} - \tilde{x}\|_{\tilde{X}}))^{1/p'} \times \|\tilde{y} - \tilde{x}\|_{\tilde{X}}^{1/p}$$

for every $\tilde{x}, \tilde{y} \in \tilde{X}$, and

$$\|B_Z | H^{1/p}(F, X)\| \le (d(F))^{1/p'} + p^{1/p} \Big(\limsup_{t \to 0+0} h_c(t) \Big)^{-1/p} (d(F) + r(F, \{0\}))^{1/p'}.$$

Proof of Theorem 5.9. The first three properties of the operator B_Z follow immediately from the definition of the metric projection and the quotient space $\tilde{X}$. In particular, the homogeneity follows from the uniqueness of the metric projection $P_{Q_Z^{-1}(\tilde{x})} 0$ for every $\tilde{x} \in \tilde{X}$ and the linearity of Q_Z.

To establish the last estimate of the theorem, let us note the identity

$$a + P_{y+Z}(x - a) = P_{a+y+Z} x, \tag{1}$$

following from the uniqueness of the metric projection of any particular point. For an arbitrary $\varepsilon > 0$ and $\tilde{x}, \tilde{y} \in F$, we choose $x = B_Z\tilde{x}$ and $y \in Q_Z^{-1}(\tilde{y})$ satisfying

$$\|y - x\|_X \leq \|\tilde{y} - \tilde{x}\|_{\tilde{X}} + \varepsilon. \tag{2}$$

The identity (1) and the triangle inequality yield $B_Z\tilde{y} = y - x + P_{Q_Z^{-1}(\tilde{x})}(x - y)$ and

$$\|B_Z\tilde{y} - B_Z\tilde{x}\|_X \leq \|y - x\|_X + \left\|P_{Q_Z^{-1}(\tilde{x})}(x - y) - P_{Q_Z^{-1}(\tilde{x})}0\right\|_X. \tag{3}$$

To estimate the difference of the metric projections in (3) , we use the estimate (3) from the proof of Theorem 5.7

$$\left\|P_{Q_Z^{-1}(\tilde{x})}(x - y) - P_{Q_Z^{-1}(\tilde{x})}0\right\|_X \leq \left(\limsup_{t\to 0+0} h_c(t)\right)^{-1/p} p^{1/p} \times \left(\|\tilde{x}\|_{\tilde{X}} + \|y - x\|_X\right)^{1/p'} \|x - y\|_X^{1/p}. \tag{4}$$

Now one uses $(2-4)$ to obtain

$$\|B_Z\tilde{y} - B_Z\tilde{x}\|_X \leq \|\tilde{x} - \tilde{y}\|_{\tilde{X}} + \varepsilon + \left(\limsup_{t\to 0+0} h_c(t)\right)^{-1/p} p^{1/p} \left(\|\tilde{x} - \tilde{y}\|_{\tilde{X}} + \varepsilon\right)^{1/p} \times \left(\|\tilde{x}\|_{\tilde{X}} + \|\tilde{x} - \tilde{y}\|_{\tilde{X}} + \varepsilon\right)^{1/p'}$$

and hence, the key estimate

$$\|B_Z\tilde{y} - B_Z\tilde{x}\|_X \leq \|\tilde{x} - \tilde{y}\|_{\tilde{X}} + \left(\limsup_{t\to 0+0} h_c(t)\right)^{-1/p} p^{1/p} \times \left(\|\tilde{x}\|_{\tilde{X}} + \|\tilde{x} - \tilde{y}\|_{\tilde{X}}\right)^{1/p'} \|\tilde{x} - \tilde{y}\|_{\tilde{X}}^{1/p}. \tag{5}$$

We accomplish the proof by noticing that

$$\|\tilde{x}\|_{\tilde{X}} \leq r(F, \{0\}) \text{ and } \|\tilde{x} - \tilde{y}\|_{\tilde{X}} \leq (d(F))^{1/p'} \|\tilde{x} - \tilde{y}\|_{\tilde{X}}^{1/p} \text{ for every } \tilde{x}, \tilde{y} \in F.$$

□

The following theorem deals with the case of a closed surjective operator A defined on a quasi-Banach space that is (linearly) isomorphic to a (p, h_c)-uniformly convex Banach space.

Theorem 5.10. *For $p \in [2, \infty)$, let X, Y be quasi-Banach spaces, and let X be isomorphic to a (p, h_c)-uniformly convex Banach space Z with $d_{BM}(X, Z) < d$. Assume also that A is a closed linear surjective operator from X onto Y, and that a bounded $F \subset Y$. Then there exists a homogeneous right-inverse operator $B : Y \to X$ satisfying*

$$A \circ B = I, \; B\lambda x = \lambda Bx, \; \sup_{y \in B_Y} \|By\|_X \leq d\|\tilde{A}^{-1}|\mathcal{L}(Y, \tilde{X})\|;$$

$$\|By - Bx\|_X \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \Big(\|y-x\|_Y + p^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} \times \big(\|x\|_Y + \|y-x\|_Y\big)^{1/p'} \|y-x\|_Y^{1/p} \Big)$$

for every $x, y \in Y$, and

$$\|B|H^{1/p}(F,X)\| \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \Big((d(F))^{1/p'} + p^{1/p} \times \Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} \Big(d(F) + r(F,\{0\})\Big)^{1/p'} \Big),$$

where $\tilde{X} = X/\mathrm{Ker}\ A$ and $\tilde{A} : \tilde{X} \to Y$ is defined by the canonical factorisation $A = \tilde{A} \circ Q_{\mathrm{Ker}\ A}$. Moreover, if X is a (p, h_c)-uniformly convex Banach space itself, one can take $d = 1$ in these estimates.

Remark 5.8. *Taking $A = I : \tilde{X} \to \tilde{X}$ and $X = Z$ (isometrically), one can see that Theorem* 5.9 *is a particular case of Theorem* 5.10.

Proof of Theorem 5.10. Since the Banach-Mazur distance between X and Z is less then $d > 1$, there exists an isomorphism $T : X \leftrightarrow Z$ and, thus, the induced isomorphism $\tilde{T} : \tilde{X} \leftrightarrow Z/T\mathrm{Ker}\ A = \tilde{Z}$ satisfying

(1) $\|T|\mathcal{L}(X,Z)\|\|T^{-1}|\mathcal{L}(Z,X)\| < d$ and $\|T|\mathcal{L}(X,Z)\| \ge \|\tilde{T}|\mathcal{L}(\tilde{X},\tilde{Z})\|$.

Since the operator $\tilde{A} : \tilde{X} \to Y$, defined by the factorisation $A = \tilde{A}Q_{\mathrm{Ker}\ A}$ with the quotient map $Q_{\mathrm{Ker}\ A} : X \to \tilde{X}$, is closed and surjective ($\mathrm{Im}\ \tilde{A} = \mathrm{Im}\ A = Y$), and the factor-space $\tilde{X} = X/\mathrm{Ker}\ A$ is quasi-Banach, there exists a bounded linear inverse $\tilde{A}^{-1} \in \mathcal{L}(Y,\tilde{X})$ .We define a mapping B by the formula

(2) $$B = T^{-1} \circ B_{T\mathrm{Ker}\ A} \circ \tilde{T} \circ \tilde{A}^{-1},$$

where $B_{T\mathrm{Ker}\ A} : \tilde{Z} \to Z$ is the mapping provided by Theorem 5.9.

The homogeneity of B follows from the homogeneity of all the other mappings in (2). The identity $A \circ B = I$ is a consequence of the observation that $A \circ T^{-1}$ is a closed operator from Z onto Y satisfying

(3) $$\widetilde{A \circ T^{-1}} = \tilde{A} \circ \tilde{T}^{-1} \quad \text{and} \quad (\widetilde{A \circ T^{-1}})^{-1} = \tilde{T} \circ \tilde{A}^{-1}$$

because of the identity $Q_{\mathrm{Ker}\ A} \circ T^{-1} = \tilde{T}^{-1} \circ Q_{T\mathrm{Ker}\ A}$. The upper estimate for $\sup_{y\in B_Y} \|By\|_X$ is implied by (1), (2), and the isometric property of $B_{T\mathrm{Ker}\ A} : \tilde{Z} \to Z$ provided by Theorem 5.10.

Eventually, the estimate for the Hölder norm of B on bounded subsets is deduced from the corresponding inequality for $B_{T\mathrm{Ker}\ A}$ given in Theorem 5.9 by means of multiple applications of Parts $a)$ and $b)$ of Corollary 5.1 in exactly the same manner as it is done in the proof of Corollary 5.2. Namely, (1) and (2) imply

$$\|By - Bx\|_X \leq \|T^{-1}|\mathcal{L}(Z,X)\| \left\| B_{T\mathrm{Ker}\ A}\tilde{T}\tilde{A}^{-1}y - B_{T\mathrm{Ker}\ A}\tilde{T}\tilde{A}^{-1}x \right\|_Z$$

and

$$\left\|\tilde{T}\tilde{A}^{-1}z\right\|_{\tilde{Z}} \leq \|T|\mathcal{L}(X,Z)\|.\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\|\|z\|_X \text{ for } z \in X. \tag{4}$$

Combining these Lipschitz estimates ((4)) with Corollary 5.1, Parts *a*) and *b*), and the estimate for the Hölder norm of $B_{T\mathrm{Ker}\ A}$ provided by Theorem 5.9, we obtain the first proof of the Hölder norm estimate in the statement of the theorem.

Alternatively, we can deduce from (1), (2), (4) and the first inequality in Theorem 5.9 for $B_{T\mathrm{Ker}\ A}$ the key estimate

$$\|By - Bx\|_X \leq d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \Big(\|y-x\|_Y + p^{1/p}\Big(\limsup_{t\to 0+0} h_c(t)\Big)^{-1/p} \times \left(\|x\|_Y + \|y-x\|_Y\right)^{1/p'} \|y-x\|_Y^{1/p}\Big),$$

and then, using the estimates

$$\|y-x\|_Y \leq (d(F))^{1/p'}\|y-x\|_Y \text{ and } \|x\|_Y \leq r(F,\{0\}) \quad \text{for } x,y \in F, \tag{5}$$

to finish the proof. Of course, if T is an isometry, we can take

$$d = \|T|\mathcal{L}(X,Z)\|\|T^{-1}|\mathcal{L}(Z,X)\| = \|\tilde{T}|\mathcal{L}(\tilde{X},\tilde{Z})\|\|T^{-1}|\mathcal{L}(Z,X)\| = 1.$$

□

With Theorem 5.9 in mind, it is natural to expect the existence of more regular homogeneous inverses for bounded linear operators from both (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth Banach spaces. In the same manner as we have established Theorems 5.9 and 5.10 relying on Theorem 5.7, we obtain the next two theorems on the basis of Theorem 5.8.

Let us start with a counterpart of Theorem 5.9.

Theorem 5.11. *For $p \in [2,\infty)$, let X be a (p,h_c)-uniformly convex Banach space, $Z \subset X$ be its subspace, F be a bounded subset of the quotient space $\tilde{X} = X/Z$, and $Q_Z : X \to \tilde{X}$ be the corresponding quotient mapping. Then the mapping $B_Z : \tilde{X} \to X$ defined by $B_Z : \tilde{x} \mapsto P_{Q_Z^{-1}(\tilde{x})}0$ satisfies*

$$B_Z(\lambda\tilde{x}) = \lambda B_Z(\tilde{x}) \quad \textit{for every } \lambda \in \mathbb{R} \textit{ and } \tilde{x} \in \tilde{X},$$

$$Q_Z \circ B_Z = I \quad \textit{on } \tilde{X}, \ \|B_Z\tilde{x}\|_X = \|\tilde{x}\|_{\tilde{X}};$$

$$\|B_Z\tilde{y} - B_Z\tilde{x}\|_X \le \|\tilde{y} - \tilde{x}\|_{\tilde{X}} + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(\|\tilde{x}\|_{\tilde{X}}^q + c_s c_c^{-q/p} \|\tilde{y} - \tilde{x}\|_{\tilde{X}}^q\right)^{1/q-1/p} \|\tilde{y} - \tilde{x}\|_{\tilde{X}}^{q/p}$$

for every $\tilde{x}, \tilde{y} \in \tilde{X}$, *and*

$$\|B_Z|H^{q/p}(F, X)\| \le (d(F))^{1-q/p} + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \times \left(r(F, \{0\})^q + c_s c_c^{-q/p} d(F)^q\right)^{1/q-1/p}.$$

If, in addition, $p = q = 2$*, then we also have*

$$\|B_Z|H^{q/p}(Y, X)\| \le 1 + \frac{c_s^{1/2}}{c_c}.$$

Remark 5.9. *The exponent* q/p *of the Hölder-Lipschitz regularity of* B_Z *is sharp when both* p *and* q *are sharp for a wide subclass of independently generated spaces* IG*, function spaces defined in terms of the wavelet norms and their duals. The proof will appear elsewhere.*

Proof of Theorem 5.11. We follow the steps of the proof of Theorem 5.9 stressing on the differences only. Choosing x and y as in that proof, we use Theorem 5.8 to obtain our counterpart of the inequality (4) from the proof of Theorem 5.9

$$(1) \quad \|P_{Q_Z^{-1}(\tilde{x})}(x - y) - P_{Q_Z^{-1}(\tilde{x})}0\|_X \le \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(\|\tilde{x}\|_{\tilde{X}}^q + c_s c_c^{-q/p} \|y - x\|_X^q\right)^{1/q-1/p} \|x - y\|_X^{q/p},$$

implying in exactly the same manner our key estimate

$$(2) \quad \|B_Z\tilde{y} - B_Z\tilde{x}\|_X \le \|\tilde{y} - \tilde{x}\|_{\tilde{X}} + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \times \left(\|\tilde{x}\|_{\tilde{X}}^q + c_s c_c^{-q/p} \|\tilde{y} - \tilde{x}\|_{\tilde{X}}^q\right)^{1/q-1/p} \|\tilde{y} - \tilde{x}\|_{\tilde{X}}^{q/p}.$$

We accomplish the proof in the general case in the same way as the proof of Theorem 5.9.

In the case $p = q = 2$, the estimate (1) and the key estimate become, according to Theorem 5.8,

$$\left\|P_{Q_Z^{-1}(\tilde{x})}(x - y) - P_{Q_Z^{-1}(\tilde{x})}0\right\|_X \le \frac{c_s^{1/2}}{c_c} \|y - x\|_X,$$

and

$$\|B_Z\tilde{y} - B_Z\tilde{x}\|_X \le \left(1 + \frac{c_s^{1/2}}{c_c}\right) \|\tilde{y} - \tilde{x}\|_{\tilde{X}}, \tag{3}$$

respectively. The proof is complete. □

Theorem 5.12. *For $2 \in [q,p] \subset (1,\infty)$, let X and Y be quasi-Banach spaces, and let X be isomorphic to a (p,h_c)-uniformly convex and (q,h_s)-uniformly smooth Banach space Z with $d_{BM}(X,Z) < d$. Assume that A is a closed linear surjective operator from X onto Y, and that $F \subset Y$ is bounded and*

$$c_c = \limsup_{t\to 0+0} h_c(t) \text{ and } c_s = \liminf_{t\to 0+0} h_s(t).$$

Then there exists a homogeneous right-inverse operator $B: Y \to X$ satisfying

$$A \circ B = I,\ B\lambda x = \lambda Bx,\ \sup_{y\in B_Y} \|By\|_X \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\|;$$

$$\|By - Bx\|_X \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \Bigg(\|y-x\|_Y + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(\|x\|_Y^q + c_s c_c^{-q/p}\|y-x\|_Y^q\right)^{1/q-1/p} \|y-x\|_Y^{q/p} \Bigg)$$

for every $x, y \in Y$, and

$$\|B|H^{q/p}(F,X)\| \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X}) \Bigg(d(F)^{1-q/p} + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \left(r(F,\{0\})^q + c_s c_c^{-q/p} d(F)^q\right)^{1/q-1/p} \Bigg),$$

where $\tilde{X} = X/\mathrm{Ker}\, A$ and $\tilde{A}: \tilde{X} \to Y$ is defined by the canonical factorization $A = \tilde{A} \circ Q_{\mathrm{Ker}\, A}$.

If, in addition, $p = q = 2$, then we also have

$$\|B|H^{q/p}(Y,X)\| \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \left(1 + \frac{c_s^{1/2}}{c_c}\right).$$

Moreover, if X is a (p,h_c)-uniformly convex and a (q,h_s)-uniformly smooth Banach space itself, one may take $d = 1$ in these estimates.

Proof of Theorem 5.12. Using the notation from the proof of Theorem 5.10, we use the same B defined by

$$B = T^{-1} \circ B_{T\mathrm{Ker}\ A} \circ \tilde{T} \circ \tilde{A}^{-1}, \tag{1}$$

where B_Z is provided by Theorem 5.9. The estimate for $\sup_{y \in B_Y} \|By\|_X$, the identity $A \circ B = I$ and the homogeneity of B have been already established in that proof. As in the proof of Theorem 5.10, we combine the relations (1), (2) and (4) from that proof and the first inequality of Theorem 5.11 with $B_{T\mathrm{Ker}\ A}$, $\tilde{T}\tilde{A}^{-1}x$ and $\tilde{T}\tilde{A}^{-1}y$ instead of B_Z, $\tilde{x}$ and $\tilde{y}$ respectively to establish the key estimate

$$\|By - Bx\|_X \le d\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \left(\|y-x\|_Y + \left(\frac{pc_s}{qc_c^{1+q/p}}\right)^{1/p} \times \left(\|x\|_Y^q + c_s c_c^{-q/p}\|y-x\|_Y^q\right)^{1/q-1/p} \|y-x\|_Y^{q/p} \right). \tag{2}$$

Now we finish the proof of the general case of p and q exactly as it is done in the proof of Theorem 5.10 with the aid of the estimates (5) from that proof. In the case $p = q = 2$, we combine relations (1), (2) and (4) from the proof of Theorem 5.10 with (3) from the proof of Theorem 5.11, where $B_{T\mathrm{Ker}\ A}$, $\tilde{T}\tilde{A}^{-1}x$ and $\tilde{T}\tilde{A}^{-1}y$ are taken instead of B_Z, $\tilde{x}$ and $\tilde{y}$ respectively, to finish the proof of the theorem. □

As a practical remark, let us note that, if there exists any bounded (nonlinear) right-inverse operator B_0 for A in Theorems 5.10 and 5.12, then

$$\|\tilde{A}^{-1}|\mathcal{L}(Y,\tilde{X})\| \le \sup_{x \in X\setminus\{0\}} \frac{\|B_0 x\|_{\tilde{X}}}{\|x\|_Y}.$$

The next theorem exposes the limitations of the linear and Lipschitz settings for the right-inverse operators.

Theorem 5.13. *Let X be a Banach space. Then the following properties are equivalent.*

a) The space X is isomorphic to a Hilbert space.

b) For every bounded linear operator A from X onto a Banach space Y, there exists its right inverse $B \in \mathcal{L}(Y,X)$.

c) For every closed linear operator A from $D(A) \subset X$ onto a Banach space Y, there exists its right inverse $B \in \mathcal{L}(Y,X)$.

d) The space X is reflexive, and for every bounded linear operator A from X onto a Banach space Y, there exists its right inverse B, that is Lipschitz on Y and $B0 = 0$.

e) The space X is reflexive, and for every closed linear operator A from $D(A) \subset X$ onto a Banach space Y, there exists its right inverse B, that is Lipschitz on Y and $B0 = 0$.

Proof of Theorem 5.13. It is enough to show the implications

$$a) \Rightarrow c) \Rightarrow b) \Rightarrow a) \text{ and } a) \Rightarrow e) \Rightarrow d) \Rightarrow a).$$

The implications $c) \Rightarrow b)$ and $e) \Rightarrow d)$ are trivial.

If X is isomorphic to a Hilbert space, then there is a bounded linear projector P onto Ker A. Let $Q: X \to \tilde{X} = X/\text{Ker } A$ is the quotient map. From the inverse mapping theorem, there exists bounded linear $\tilde{A}^{-1}: Y \to \tilde{X}$. Now we define the right inverse B by

$$B: y \mapsto (I-P)x, \quad \text{where } x \in Q^{-1}\left(\tilde{A}^{-1}y\right).$$

It is correctly defined (does not depend on a particular $x \in Q^{-1}\left(\tilde{A}^{-1}y\right)$) and $\|B\| \le \|I-P\|\|\tilde{A}^{-1}\|$.

If $b)$ holds, we take an arbitrary subspace $Z \subset X$ and choose $A = Q: X \to \tilde{X}$ and $Y = \tilde{X}$. Now the composition $P = B \circ A$ is a bounded projector with Ker $P = \ker A = Z$ meaning that Z is complemented in X. From the celebrated result of Lindenstrauss and Tzafriri (see [20]), X is isomorphic to a Hilbert space because Z was arbitrary.

After noticing that every space isomorphic to a reflexive (particularly, Hilbert) space is reflexive, and a bounded linear operator is Lipschitz, we prove $a) \Rightarrow e)$ exactly as $a) \Rightarrow c)$.

Assume that $d)$ holds. As before, we take an arbitrary subspace $Z \subset X$ and choose $A = Q: X \to \tilde{X}$ and $Y = \tilde{X}$. Then the function $f(x) = x + B \circ Q(-x)$ is a Lipschitz retract of X onto Z. Hence, by Theorem 5.13, there exists a bounded linear projector P of X onto Z since $X = X^{**}$ holds by the reflexivity of Z. Since all subspaces of A are complemented, the application of the same theorem due to Lindenstrauss and Tzafriri finishes the proof. □

5.4. Geometry of the cross-sections of the unit ball

As was mentioned in Part $b)$ of Remark 5.3 and Part $c)$ of Remark 5.6 (Lemma 5.4), both the (relative) Chebyshev centre map restricted to the balls of equal radius and the metric projection onto a ball do not distinguish the pertaining features of the geometry of a Banach space from the point of view of the (p, h_c)-uniform convexity and (q, h_s)-uniform smoothness. It makes it important to observe that the geometry of the cross-sections of the balls determined completely by these properties of the ambient Banach space. The cross-sections of the balls in uniformly convex and uniformly smooth spaces were extensively studied by Björnestål [10] and Benyamini and Lindenstrauss (see Section 2.2 in [6]) in order to establish the local regularity of the metric projections. This subsection contains counterparts of their results on cross-sections from the point of view of (p, h_c)-uniform convexity and (q, h_s)-uniform smoothness.

Definition 5.7. *Let X be a smooth Banach space (i.e. every $x \in X$ possesses unique norming $f \in X^*$ with $f(x) = \|x\|_X$), B_X and S_X be its unit ball and unit sphere respectively, $x \in S_X$, $f \in S_{X^*}$, $f(x) = 1$ and $\tau \in (0, 1]$.*

The cross-section $B_{\tau,x}$ is defined by

$$B_{\tau,x} = B_X \cap f^{-1}(1-\tau) = B_X \cap x_\tau + \mathrm{Ker}\, f,$$

where $x_\tau = (1-\tau)x$ and $f(x) = 1$. Its diameter $d(B_{\tau,x})$ and the internal and external radii $r_I(B_{\tau,x})$ and $r_E(B_{\tau,x})$ are defined as

$$d(B_{\tau,x}) = \sup_{x,y \in B_{\tau,x}} \|x-y\|_X, \; r_I(B_{\tau,x}) = \inf_{y \in S_X \cap B_{\tau,x}} \|y - x_\tau\|_X$$

and

$$r_E(B_{\tau,x}) = \sup_{y \in S_X \cap B_{\tau,x}} \|y - x_\tau\|_X$$

There is a simple estimate of the internal radius of the cross-section in terms of its diameter:

$$2r_I(B_{\tau,x}) \le d(B) \le 2r_E(B_{\tau,x})$$

The opposite estimate is established in the next lemma.

Lemma 5.7. *Let X be a Banach space, $x \in S_X$, $2 \in [q,p] \subset (1,\infty)$ and $\tau \in (0,1)$. Then,*

a) if X is (p, h_c)-uniformly convex, we have

$$\limsup_{t\to 0+0} h_c(t) r_E(B_{\tau,x})^p \le 1 - (1-\tau)^p$$

and

$$\max\left(2^{-2}h_c(1/2), 2^{-p}\limsup_{t\to 0+0} h_c(t)\right)(d(B_{\tau,x}))^p \le 1 - (1-\tau)^p;$$

b) if X is (q, h_s)-uniformly smooth, we have

$$\liminf_{t\to 0+0} h_s(t)\,(r_I(B_{\tau,x}))^q \ge 1 - (1-\tau)^q;$$

c) if X is both (p, h_c)-uniformly convex and (q, h_s)-uniformly smooth, we have

$$r_E(B_{\tau,x}) \le \left(\frac{p \liminf_{t\to 0+0} h_s(t)}{q \limsup_{t\to 0+0} h_c(t)}\right)^{1/p} (r_I(B_{\tau,x}))^{q/p}$$

and

$$d(B_{\tau,x}) \le \left(\frac{p \liminf_{t\to 0+0} h_s(t)}{q \max\left(2^{-2}h_c(1/2), 2^{-p}\limsup_{t\to 0+0} h_c(t)\right)} \right)^{1/p} (r_I(B_{\tau,x}))^{q/p}.$$

Remark 5.10. *In many spaces under consideration, including L_p and l_p, one has*

$$2^{p-2}h_c(1/2) = 1 > \limsup_{t\to 0+0} h_c(t).$$

(See Theorem 3.4.*)*

Proof of Lemma 5.7. For every $\varepsilon > 0$, one can choose $y, z \in B_{\tau,x}$ with $\|y-z\|_X \ge d(B_{\tau,x}) - \varepsilon$. Remembering the definition of the (p, h_c)-uniform convexity, we obtain

$$\begin{aligned}(1)\qquad h_c(1/2)\,(d(B_{\tau,x})-\varepsilon)^p &\le h_c(1/2)\|y-z\|_X^p \\ &\le \frac{\|y\|_X^p + \|z\|_X^p}{2} - \left\|\frac{y+z}{2}\right\|_X^p \le 1-(1-\tau)^p,\end{aligned}$$

where we have also used $P_{B_{\tau,x}}(0) = x_\tau = (1-\tau)x$ and $\frac{y+z}{2} \in B_{\tau,x}$. At the same time, Part *a*) of Lemma 5.5 with $F(y) = \|y\|^p$, and, thus, $f_x = p\|x\|_X^{p-1}f$, yields the estimate

$$(2)\qquad \limsup_{t\to 0+0} h_c(t)\|v - x_\tau\|_X^p \le \|v\|_X^p - \|x_\tau\|_X^p \le 1-(1-\tau)^p$$

for every $v \in B_{\tau,x}$, implying the first inequality of Part *a*) after taking supremum over $v \in B_{\tau,x}$.

Combining the first inequality of *a*) and $d(B) \le 2r_E(B_{\tau,x})$ with (1), we complete the proof of *a*).

To establish *b*), it is enough to notice that, for every $y \in S_X \cap B_{\tau,X}$, Lemma 5.5, b) with the same F implies

$$1-(1-\tau)^q = \|y\|_X^q - \|x_\tau\|_X^q \le \liminf_{t\to 0+0} h_s(t)\|y-x_\tau\|_X^q.$$

To prove *c*) we denote

$$\begin{aligned}(3)\qquad c_d &= \max\left(2^{-2}h_c(1/2), 2^{-p}\limsup_{t\to 0+0} h_c(t)\right), \quad c_c = \limsup_{t\to 0+0} h_c(t), \\ c_s &= \liminf_{t\to 0+0} h_s(t).\end{aligned}$$

Let us note that *a*) and *b*) read as

$$c_c\,(r_E(B_{\tau,x}))^p \le 1-(1-c_s\,(r_I(B_{\tau,x}))^q)^{p/q} \le \frac{p}{q}c_s\,(r_I(B_{\tau,x}))^q$$

and

$$(4) \qquad c_d\,(d(B_{\tau,x}))^p \le 1-(1-c_s\,(r_I(B_{\tau,x}))^q)^{p/q} \le \frac{p}{q}c_s\,(r_I(B_{\tau,x}))^q,$$

where we have used the inequality $(1-x)^\alpha \ge 1-\alpha x$ for $\alpha \ge 1$. This finishes the proof of the lemma. □

We need the next elementary calculus lemma.

Lemma 5.8. *For $p>1$, let $h_p(x) = ((1+x)^p-1)^{1/p}$ on $[0,\infty)$, and $a>0$. Then the function h_p is strictly increasing and concave and satisfies*

$$\max(x,(px)^{1/p}) < h_p(x) < 1+x,\ h_p(x) < p^{1/p}(1+a)^{1/p'}x^{1/p} \quad \text{for } x\in(0,a],$$

and $h_p'(x) > 1$.

Moreover, its asymptotic line at ∞ is $x+1$.

The last lemma relates the internal radius $r_I(B_{\tau,x})$ and $\|x-y\|_X$ under the notation of Lemma 5.7. In fact, it demonstrates to which degree we can approximate the metric projection onto a closed convex subset P_D with the degenerated metric projection onto one point $P_{P_D x}$. To establish it we will follow the idea of Björnestål [10] (see Section 2.2 in [6]) to use the properties of the cross-sections of balls but in a different fashion providing explicit estimates for the constants.

Lemma 5.9. *For $p\in[2,\infty)$, let X be a (p,h_c)-uniformly convex Banach space, and let $D\subset X$, x, f, y, v and τ be as in Lemma 5.7. Assume also that $\|y-x\|_X \le \varepsilon$, and that $\eta>0$ satisfies $h_0(\eta)<1$ for $h_0(t)=2g(t)-t$, where $g(x) = \left(\limsup_{t\to 0+0} h_c(t)\right)^{-1/p} h_p(x)$, and $h_p(x)$ is defined in Lemma 5.8.*

Then we have

$$\tau \le r_I\,(B_{\tau,x}) \le \|v-x_\tau\|_X \le 2\left(\frac{2g(\eta)}{1-h_0(\eta)}+\frac{1}{1-\eta}\right)\varepsilon$$

for every $\varepsilon\in[0,\eta]$. Moreover, if D is a subspace of X, we also have

$$\tau \le r_I\,(B_{\tau,x}) \le \|v-x_\tau\|_X \le 2\left(\frac{g(\eta)}{1-\eta+g(\eta)}+\frac{1}{1-\eta}\right)\varepsilon$$

for $\varepsilon\in[0,\eta]$ with any $\eta<1$ chosen to satisfy $g(\eta)-\eta<1$.

Proof of Lemma 5.9. We start with the case of arbitrary closed and convex $D\subset X$, and then consider the simplification when D is a subspace.

Let us note that g, h_0 and $g(t)-t$ are positive strictly increasing functions on $(0,\infty)$. Moreover, according to Part $d)$ of Remark 3.1, $g(t)\ge h_p(t)$, meaning that $\eta < 1$ if $h_0(t) = 2f(t) - t$ and $h_0(\eta) < 1$. Hence, $f(y) \geq f(x) - \varepsilon > 0$, and there exists

$$(1) \qquad w = [v, P_D] \cap f^{-1} \text{ and } v = \lambda w + (1-\lambda)y \text{ for some } \lambda \leq 1.$$

Note that $w = \beta y + (1-\beta)P_D y$ for some $\beta \in [0, 1)$.

The estimate $\tau \leq r_I(B_{\tau,x})$ follows from the triangle inequality $1 \leq \|x_\tau\|_X + r_I(B_{\tau,x})$.

To estimate $\|v - x_\tau\|_X$, we borrow its (implicit) representation from the proof of Lemma 2.6 in [6]

$$(2) \qquad v - (1-\tau)x = \tau w + (1-\tau)(y-x) + (\tau - \lambda)(y - w),$$

following from (1), and employ the triangle inequality. We already have the estimates

$$(3) \qquad \|y - x\|_X \leq \varepsilon, \; \|y - w\|_X \leq \|y - P_D y\|_X \leq \|y - p_D x\|_X \leq 1 + \varepsilon.$$

In addition, the inequality (2) from the proof of Theorem 5.7 and the monotonicity of $g(x)$ yield

$$(4) \qquad \|P_D y\|_X \leq g(\varepsilon),$$

which implies

$$\begin{aligned} 0 = f(w) = \beta f(y) + (1-\beta) f(P_D y) &\geq \beta(f(x) - \varepsilon) - (1-\beta)\|P_D y\|_X \\ &\geq \beta(1-\varepsilon) - (1-\beta) g(\varepsilon), \end{aligned}$$

hence

$$\beta \leq \frac{g(\varepsilon)}{1 - \varepsilon + g(\varepsilon)}.$$

The last estimate, along with the triangle inequality, (3) and (4), permits us to estimate $\|w\|_X$:

$$(5) \; \|w\|_X \leq \beta\|y\|_X + (1-\beta)\|p_D y\|_X \leq \beta(1 + \varepsilon - g(\varepsilon)) + g(\varepsilon) \leq \frac{2g(\varepsilon)}{1 - \varepsilon + g(\varepsilon)}.$$

To estimate $|\tau - \lambda|$ (and $1 - \lambda$), we note that

$$1 - \tau = f(v) = (1-\lambda) f(y) \text{ and thus, } \lambda - \tau = (1-\lambda) f(y - x),$$

implying with the aid of the triangle inequality that

$$(6) \quad 1 - \lambda = \frac{1-\tau}{f(y)} \leq \frac{1-\tau}{1-\varepsilon} \text{ and therefore, } |\lambda - \tau| \leq (1-\lambda)\varepsilon \leq \frac{1-\tau}{1-\varepsilon}\varepsilon.$$

Now combining (3), (5) and (6), we derive from (2) the key estimate

$$\begin{aligned}(7)\qquad \tau &\le r_I\,(B_{\tau,x}) \le \|v - x_\tau\|_X \\ &\le \tau\|w\|_X + (1-\tau)\|y-x\|_X + |\tau-\lambda|\|y-w\|_X \\ &\le \tau\frac{2g(\varepsilon)}{1-\varepsilon+g(\varepsilon)} + (1-\tau)\varepsilon + (1-\tau)\varepsilon\frac{1+\varepsilon}{1-\varepsilon} \\ &= \tau\left(\frac{2g(\varepsilon)}{1-\varepsilon+g(\varepsilon)} - \frac{2\varepsilon}{1-\varepsilon}\right) + \frac{2\varepsilon}{1-\varepsilon}.\end{aligned}$$

Whence, one has

$$(8)\qquad 2\varepsilon \ge \tau\left(1+\varepsilon - \frac{2(1-\varepsilon)g(\varepsilon)}{1-\varepsilon+g(\varepsilon)}\right) \ge \tau(1-h_0(\varepsilon)),$$

where, for the last inequality, we have used Lemma 5.8. Plugging the outcome of (8) back into (7), we obtain

$$\begin{aligned}(9)\qquad r_I\,(B_{\tau,x}) &\le \tau\left(\frac{2g(\varepsilon)}{1-\varepsilon+g(\varepsilon)} - \frac{2\varepsilon}{1-\varepsilon}\right) + \frac{2\varepsilon}{1-\varepsilon} \le 2g(\varepsilon)\tau + \frac{2\varepsilon}{1-\varepsilon} \\ &\le 2\varepsilon\left(\frac{2g(\varepsilon)}{1-h_0(\varepsilon)} + \frac{1}{1-\varepsilon}\right) \le 2\left(\frac{2g(\eta)}{1-h_0(\eta)} + \frac{1}{1-\eta}\right)\varepsilon,\end{aligned}$$

where we have used Lemma 5.8 for the second inequality and the strict monotonicity of g and h_0 for the last inequality.

If D is a subspace, $w = P_D y$ and we do not need (5), while the estimates $(7-9)$ become, respectively,

$$\tau \le r_I\,(B_{\tau,x}) \le \|v - x_\tau\|_X \le \tau\|P_D y\|_X + (1-\tau)\|y-x\|_X + |\tau-\lambda|\|y - P_D y\|_X$$

$$(10)\qquad \le \tau g(\varepsilon) + (1-\tau)\varepsilon + (1-\tau)\varepsilon\frac{1+\varepsilon}{1-\varepsilon} = \tau\left(g(\varepsilon) - \frac{2\varepsilon}{1-\varepsilon}\right) + \frac{2\varepsilon}{1-\varepsilon},$$

$$(11)\qquad 2\varepsilon \ge \tau\,(1+\varepsilon-(1-\varepsilon)g(\varepsilon)) \ge \tau(1-h_0(\varepsilon)),$$

and

$$r_I\,(B_{\tau,x}) \le \tau\left(g(\varepsilon) - \frac{2\varepsilon}{1-\varepsilon}\right) + \frac{2\varepsilon}{1-\varepsilon} \le g(\varepsilon)\tau + \frac{2\varepsilon}{1-\varepsilon}$$

$$(12)\qquad \le 2\varepsilon\left(\frac{g(\varepsilon)}{1-\varepsilon+g(\varepsilon)} + \frac{1}{1-\varepsilon}\right) \le 2\left(\frac{g(\eta)}{1-\eta+g(\eta)} + \frac{1}{1-\eta}\right)\varepsilon.$$

The proof of Lemma 5.9 is complete. □

6. Complemented isomorphic copies of finite-dimensional l_p-spaces

Definition 6.1. *For $\lambda \geq 1$, we say that a quasi-Banach space X is λ-isomorphic to a quasi-Banach space Y if there exists an isomorphism $T : X \leftrightarrow Y$ with*

$$\|T|\mathcal{L}(X,Y)\| \|T^{-1}|\mathcal{L}(Y,X)\| \leq \lambda.$$

For $\beta \geq 1$, we say that a subspace Z of X is β-complemented in X if there exists a projector P onto Z with $\|P|\mathcal{L}(X)\| \leq \beta$.

Theorem 6.1. *Let $G \subset \mathbb{R}^n$, $p, a \in (1,\infty)^n$, $q, \varsigma \in (1,\infty)$, $s \in (0,\infty)^n$ and*

$$r \in \{p_{\min}, p_{\max}, q, 2\}.$$

Assume also that

$$Y \in \{B^s_{p,q,a}(G),\ \tilde{B}^{s,A}_{p,q,a}(G),\ \tilde{b}^{s,A}_{p,q,a}(G),\ L^s_{p,q,a}(G),\ \tilde{L}^{s,A}_{p,q,a}(G),\ b^s_{p,q,a}(G),\ l^s_{p,q,a}(G),\ \tilde{l}^{s,A}_{p,q,a}(G),\ B^s_{p',q',a'}(G)^*,\ \tilde{B}^{s,A}_{p',q',a'}(G)^*,\ L^s_{p',q',a'}(G)^*,\ \tilde{L}^{s,A}_{p',q',a'}(G)^*,\ b^s_{p',q',a'}(G)^*,\ \tilde{b}^{s,A}_{p',q',a'}(G)^*,\ l^s_{p',q',a'}(G)^*,\ \tilde{l}^{s,A}_{p',q',a'}(G)^*\}.$$

Then there are constants $C_0, C_1 > 0$, such that Y contains an C_0-isomorphic and C_1-complemented copy of $l_r(I_n)$ for every $n \in \mathbb{N}$.

Theorem 6.2. *Let $Y \in \{W^s_p(G),\ W^s_{p'}(G)^*\}$ for $G \subset \mathbb{R}^n$, $p \in (1,\infty)^n$, $\varsigma \in (1,\infty)$, $s \in \mathbb{N}^n$ and*

$$r \in \{p_{\min}, p_{\max}, 2\}.$$

Then there are constants $C_0, C_1 > 0$, such that Y contains an C_0-isomorphic and C_1-complemented copy of $l_r(I_n)$ for every $n \in \mathbb{N}$.

Theorem 6.3. *Let $Y \in \{B^s_{p,q}(\mathbb{R}^n)_w,\ L^s_{p,q}(\mathbb{R}^n)_w,\ B^s_{p',q'}(\mathbb{R}^n)^*_w,\ L^s_{p',q'}(\mathbb{R}^n)^*_w\}$ for $p \in (1,\infty)^n$, $\varsigma \in (1,\infty)$, $s \in \mathbb{N}^n$ and*

$$r \in \{p_{\min}, p_{\max}, q, 2\}.$$

Then there are constants $C_0, C_1 > 0$, such that Y contains an C_0-isomorphic and C_1-complemented copy of $l_r(I_n)$ for every $n \in \mathbb{N}$.

Remark 6.1. *Similar result holds for the spaces from the class IG (with a simpler proof) but it requires additional definition classifying the set of indexes, and therefore is omitted here.*

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