Communications
on Applied Nonlinear Analysis
Volume 24(2017), Number 2, 28 - 48

Generalized Projections and Equivalent Representations of James Orthogonal Decompositions in Banach Spaces

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Communicated by Ram U. Verma

(Received March 23, 2017; Revised Version Accepted April 1, 2017)

www.internationalpubls.com

Abstract

We present a short observe of all the generalized projection operators in Banach spaces. These operators have been created and studied in our works [1, 2, 6, 10]. We pay a special heed to the new properties of the projection operator $\mathcal{P}_\Omega$ which is afterwards used to construct the equivalent decompositions of arbitrary elements of Banach spaces onto their subspaces and cones. These decompositions are complete analogies of the well known classical orthogonal decompositions in Hilbert spaces (see (4.8) and (5.4) and compare with (4.6) and (5.3), respectively). Generalized projections and orthogonal decompositions in Banach spaces have lately found numerous applications in different areas of the functional analysis.

AMS Subject Classification: Primary 46B07, 46B20, 46B25, 46B38; Secondary 47H99, 47N10, 47S20.

Key words and phrases: Banach spaces, metric projection operators, Young-Fenchel transformation, generalized projection operators, proximity properties, subspaces, cones, nonlinear manifolds, equivalent decompositions.

1 Introduction and Preliminaries

In recent two decades there had been a great progress which made it possible for us to establish and study the fundamental principles and facts of Banach spaces earlier known only in Hilbert spaces. It is in view the generalized projection operators in Banach spaces, decompositions of arbitrary elements of Banach spaces and decompositions of Banach spaces themselves. This progress enabled also to adapt to reflexive Banach spaces the classical research methods of nonlinear functional analysis including methods for approximation and optimization problems, dynamical systems and variational inequalities, fixed point problems and operator theory (see, for example, [2-10, 12, 13, 18]).

Let B be a real reflexive strictly convex and smooth Banach space with dual space B^*. Denote the norms of elements $\phi \in B^*$ and $x \in B$ by $\|\phi\|_*$ and $\|x\|$, respectively, and their dual product by $\langle x, \phi\rangle$ $(= \langle \phi, x\rangle)$. Let θ_B and θ_{B^*} be origins of the spaces B and B^*.

We recall that nonlinear, in general, operator $J : B \to B^*$ is called normalized duality mapping in B, if $\langle Jx, x\rangle = \|Jx\|_*\|x\| = \|x\|^2$ for all $x \in B$. Operator $J^* : B^* \to B$ is called normalized duality mapping in B^*, if $\langle J^*\phi, \phi\rangle = \|J^*\phi\|\|\phi\|_* = \|\phi\|_*^2$ for all $\phi \in B^*$. If $I_B : B \to B$ and $I_{B^*} : B^* \to B^*$ are the identity operators then $J^*J = I_B$ and $JJ^* = I_{B^*}$ (see [1, 2, 11, 14, 17, 18] for the geometric characteristics of Banach spaces).

We also recall that the function

$$\delta_B(\epsilon) = \inf\{1 - 2^{-1}\|x + y\| : \|x\| = 1, \|y\| = 1, \|x - y\| = \epsilon\}$$

is called the modulus of convexity of the space B. The function

$$\rho_B(\tau) = \sup\{2^{-1}(\|x + y\| + \|x - y\|) - 1 : \|x\| = 1, \|y\| = \tau\}$$

is called the modulus of smoothness of the space B. Observe that the space B is uniformly convex if and only if $\delta_B(\epsilon) > 0 \quad \forall \epsilon > 0$, and that it is uniformly smooth if and only if $\lim_{\tau\to 0} h_B(\tau) = 0$, where

$$h_B(\tau) = \frac{\rho_B(\tau)}{\tau}. \tag{1.1}$$

Let us present definitions and some properties of the metric projection operator and sunny nonexpansive retraction in a Banach space [1, 2, 10].

1.1. Metric projection operator

Definition 1.1 *Suppose that Ω is a closed convex subset of B. The operator $P_\Omega : B \to \Omega$ is called the metric projection operator onto Ω if it assigns to each $x \in B$ its nearest point $\bar{x} \in \Omega$, i.e., a solution of the following minimization problem:*

$$P_\Omega x = \bar{x} : \quad \bar{x} = \arg\min_{\xi\in\Omega} \|x - \xi\|^2. \tag{1.2}$$

$\bar{x}$ is then called metric projection or P_Ω-projection of a point x onto Ω.

It is clear that $\|x - \bar{x}\|^2 = \inf_{\xi\in\Omega} \|x - \xi\|^2$. The main properties of the metric projection can be expressed as follows:
(i$_0$) $P_\Omega x = x$ for any $x \in \Omega$. Hence, $P_\Omega^2 = P_\Omega$;
(ii$_0$) The point $\bar{x}$ is the metric projection of $x \in B$ onto $\Omega \subset B$ if and only if the inequality

$$\langle J(x - \bar{x}), \bar{x} - \xi)\rangle \geq 0 \quad \forall \xi \in \Omega \tag{1.3}$$

is satisfied [15, 23]. This is the based variational principle for $\bar{x} = P_\Omega x$ in B;
(iii$_0$) If $\theta_B \in \Omega$, then $\|x - P_\Omega x\| \leq \|x\|$ for all $x \in B$;

It is known that in a Hilbert space H, the normalized duality mapping J is an identity operator I_H, therefore (1.3) has the form

$$(x-\bar{x},\bar{x}-\xi)\geq 0 \quad \forall \xi\in\Omega, \tag{1.4}$$

where (x,y) denotes inner (scalar) product of $x,y\in H$. In addition to (i_0) and (ii_0), in a Hilbert space the inequality
(iv_0) $\|\bar{x}-\xi\|^2\leq\|x-\xi\|^2-\|x-\bar{x}\|^2 \quad \forall x\in H,\ \forall\xi\in\Omega$
holds. In a general Banach space (iv_0) is not true. Obviously that

$$\|\bar{x}-\xi\|\leq\|x-\xi\| \quad \forall x\in H,\ \forall\xi\in\Omega.$$

In reality, in a Hilbert space (and only in a Hilbert space)

$$\|\bar{x}-\bar{y}\|\leq\|x-y\| \quad \forall x,y\in H.$$

For comparison, let us provide

1.2. Sunny nonexpansive retraction

Definition 1.2 *Let Ω be a non-empty closed convex subset of B. A mapping $Q_\Omega : B\to\Omega$ is said to be*
a) a retraction onto Ω if $Q_\Omega^2=Q_\Omega$;
b) a nonexpansive retraction if it also satisfies the inequality

$$\|Q_\Omega x-Q_\Omega y\|\leq\|x-y\| \quad \forall x,y\in B;$$

c) a sunny retraction if for all $x\in B$ and for all $0\leq t<\infty$

$$Q_\Omega(Q_\Omega x+t(x-Q_\Omega x))=Q_\Omega x.$$

Let us provide the basic variational principle for sunny nonexpansive retraction in B. It is well known that Q_Ω is sunny nonexpansive retraction if and only if the following inequality holds:

$$\langle x-Q_\Omega x, J(Q_\Omega x-\xi)\rangle\geq 0 \quad \forall x\in B,\ \forall\xi\in\Omega. \tag{1.5}$$

If to compare two definitions above, one can notice that the definition of metric projection operator describes also the way of its calculation by means of the corresponding minimization problem. Unfortunately, the definition of sunny nonexpansive retraction does not give us any effective method to calculate it. This fact essentially restricts the practical applications retractions. Nevertheless, for some special sets there exist several iterative procedures to generate sunny nonexpansive retractions (see, for example, [8] for fixed point sets of nonexpansive and total nonexpansive maps, respectively).

2 Young-Fenchel transformation and projection operators

Our nearest aim is to construct projection operators in a Banach space based on the Young-Fenchel transformation of conjugate functions. All of them can be calculated by use of corresponding minimization problems.

2.1. The original concept of generalized projections

Let $f : B \to R$ be a given finite functional, $x \in B$ and $\phi \in B^*$. Recall that Young-Fenchel transformation is defined by the relation

$$g(\phi) = \sup_{x \in B}\{\langle \phi, x\rangle - f(x)\}.$$

The functional g is called conjugate to f. If $f(x)$ is Gâteaux differentiable and $f'(x)$ is its Gâteaux derivative, then for $\phi = f'(x)$ we have

$$g(\phi) = \langle \phi, x\rangle - f(x).$$

Otherwise, if $\phi \neq f'(x)$, then

$$g(\phi) > \langle \phi, x\rangle - f(x).$$

Consequently, the functional $\mathcal{W}^f : B^* \times B \to R$ defined by the Lyapunov functional

$$\mathcal{W}^f(\phi, x) = g(\phi) - \langle \phi, x\rangle + f(x), \tag{2.1}$$

is non-negative for all $x \in B$ and for all $\phi \in B^*$. We also recall that $g(\phi)$ is convex for any $f(x)$. Therefore, $\mathcal{W}^f(\phi, x)$ is also convex with respect to $\phi \in B^*$ (with fixed $x \in B$) because $\langle \phi, x\rangle$ is a linear and differentiable functional. Moreover, if f is a convex lower semicontinuous function, then g is also convex lower semicontinuous one.

Definition 2.1 *The operator $\pi_\Omega^f : B^* \to \Omega$ is said to be the f-generalized projection operator if it puts an arbitrary fixed point $\phi \in B^*$ into the correspondence with the point of minimum for the functional $\mathcal{W}^f(\phi, x)$ according to the minimization problem*

$$\pi_\Omega^f \phi = \tilde{\phi}^f : \quad \tilde{\phi}^f = \arg\min_{x \in \Omega} \mathcal{W}^f(\phi, x).$$

The particular cases are:

a) Let $f(x) = \|x\|^2$ for any $x \in B$ and $\phi = Jy$ for any $y \in B$. Then $W_1 : B \times B \to R$ is defined as follows: $W_1(y, x) = \mathcal{W}^f(Jy, x)$, that is

$$W_1(y, x) = \|y\|^2 - 2\langle Jy, x\rangle + \|x\|^2. \tag{2.2}$$

b) Let $f(x) = \|x\|^2$ for any $x \in B$ and $\phi \in B^*$. Then

$$\|\varphi\|_*^2 = \sup_{x \in B}\{2\langle \varphi, x\rangle - \|x\|^2\}$$

is Young-Fenchel transformation and $g(\varphi) = 4^{-1}\|\varphi\|_*^2$ is the conjugate function to $f(x)$. We have: $W_2(\phi, x) = \mathcal{W}^f(\phi, x)$, that is,

$$W_2(\phi, x) = \|\phi\|_*^2 - 2\langle \phi, x\rangle + \|x\|^2.$$

c) Let $f(x)$ be a power, lower semicontinuous, strictly convex and Gáteaux differentiable function for any $x \in dom f$ and $\phi = f'(x)$. Introduce the functional $W^f : B \times B \to R$ defined by the formula

$$W^f(y,x) = \mathcal{W}^f(f'(y), x),$$

that is,

$$W^f(y,x) = f(x) - f(y) - \langle f'(y), x - y\rangle. \tag{2.3}$$

The functional $W(y,x)$ is well known as *the Bregman distance* (see, for example, [7]).

Let us present definitions and main properties of the generalized projection operators in a Banach space.

2.2. The first generalized projection operator

Consider the Lyapunov functional $W_1(x,\xi) : B \times B \to R :$

$$W_1(x,\xi) = \|x\|^2 - 2\langle Jx, \xi\rangle + \|\xi\|^2 \quad \forall x, \xi \in B. \tag{2.4}$$

Definition 2.2 *Suppose that Ω be a closed convex subset of B. Operator $\Pi_\Omega : B \to \Omega$ is called generalized projection operator onto Ω if it assigns to each $x \in B$ a minimum point $\hat{x} \in \Omega$ of the functional $W_1(x,\xi)$, i.e., a solution of the following minimization problem:*

$$\Pi_\Omega x = \hat{x} : \quad \hat{x} = \arg\min_{\xi\in\Omega} W_1(x,\xi).$$

$\hat{x}$ is then called generalized Π_Ω-projection of a point x onto Ω.

It is clear that $W_1(x,\hat{x}) = \inf_{\xi\in\Omega} W_1(x,\xi)$. The main properties of the projection operator Π_Ω can be expressed by (i_1)-(iii_1):

(i_1) $\Pi_\Omega x = x$ for any $x \in \Omega$. Hence, $\Pi_\Omega^2 = \Pi_\Omega$;

(ii_1) The point $\hat{x}$ is the generalized Π_Ω-projection of $x \in B$ onto $\Omega \subset B$ if and only if the inequality

$$\langle Jx - J\hat{x}, \hat{x} - \xi\rangle \geq 0 \quad \forall \xi \in \Omega, \tag{2.5}$$

is satisfied. This is the based variational principle for $\Pi_\Omega x$ in B;

(iii_1) $W_1(\hat{x},\xi) \leq W_1(x,\xi) - W_1(x,\hat{x}) \quad \forall x \in B,\ \forall \xi \in \Omega.$

It follows from the last inequality that the generalized projection operator Π_Ω is conditionally nonexpansive with respect to functional W_1, i.e.,

$$W_1(\hat{x},\xi) \leq W_1(x,\xi) \quad \forall x \in B,\ \forall \xi \in \Omega.$$

This is very important property of the generalized projection operator Π_Ω. Let us present one example of the Π_Ω-projection.

Proposition 2.3 *[4] Let $M_\alpha \subseteq B$ be one-dimensional subspace spanned upon the element e_α with the unit norm. Then the first generalized projection of arbitrary element $x \in B$ on M_α is $\langle Jx, e_\alpha\rangle e_\alpha$, that is,*

$$\Pi_{M_\alpha} x = \langle Jx, e_\alpha\rangle e_\alpha, \tag{2.6}$$

where $\langle Jx, e_\alpha\rangle$ is the generalized Fourier coefficient.

This proposition is used to deduce the Fourier-decomposition of arbitrary elements in Banach spaces (see (3.5)-(3.8) in [5]). In a Hilbert space, (2.6) gives the well known result: $P_{M_\alpha}x = \langle x, e_\alpha \rangle e_\alpha$. It is the fundamental relation in order to construct the orthonormal bases and Fourier-decompositions on these bases in a Hilbert space. As regards to $P_{M_\alpha}x$ and $Q_{M_\alpha}x$, we can only state that there are no representations similar (2.6) in Banach spaces. In conclusion note the following obvious fact: $\Pi_\Omega x = P_\Omega x$ for all $x \in H$ and for all convex closed sets $\Omega \subset H$.

2.3. The second generalized projection operator

Consider the Lyapunov functional $W_2(x, \phi) : B \times B^* \to R$ defined as:

$$W_2(\phi, x) = \|\phi\|_*^2 - 2\langle \phi, x \rangle + \|x\|^2 \quad \forall x \in B, \ \forall \phi \in B^*. \tag{2.7}$$

We present generalized projection operator $\pi_\Omega : B^* \to \Omega :$

Definition 2.4 *Suppose that Ω be a closed convex subset of B. The operator $\pi_\Omega : B^* \to \Omega$ is called generalized projection operator if it associates to an arbitrary point $\phi \in B^*$ the minimum point of the functional $W(\phi, \xi)$, i.e., a solution to the minimization problem*

$$\pi_\Omega \phi = \tilde{\phi} : \quad \tilde{\phi} = \arg\min_{x \in \Omega} W_2(\phi, x).$$

$\tilde{\phi} \in \Omega$ is then called generalized π_Ω-projection of a point ϕ onto Ω.

The main properties of the generalized projection operator π_Ω can be expressed as follows:
(i_2) if $\pi_\Omega \phi = x$, then $\pi_\Omega J \pi_\Omega \phi = x$.
(ii_2) The point $\tilde{\phi}$ is generalized π_Ω-projection of ϕ onto Ω if and only if the inequality

$$\langle \phi - J\tilde{\phi}, \tilde{\phi} - \xi \rangle \geq 0 \quad \forall \xi \in \Omega$$

holds. This is the basic variational principle for $\pi_\Omega x$ in the dual couple B, B^*;
(iii_2) $W_2(\xi, J\tilde{\phi}) \leq W_2(\xi, \phi) - W_2(\tilde{\phi}, \phi) \quad \forall \phi \in B^*, \ \forall \xi \in \Omega$.

It follows from (iii_2) that the generalized projection operator π_Ω is conditionally nonexpansive with respect to functional W_2, i.e.,

$$W_2(\xi, J\tilde{\phi}) \leq W_2(\xi, \phi) \ \ \forall \phi \in B^*, \ \forall \xi \in \Omega.$$

We have shown in [1, 2] that

$$\Pi_\Omega = \pi_\Omega J \quad \text{or} \quad \pi_\Omega = \Pi_\Omega J^*. \tag{2.8}$$

In a Hilbert space, $B = B^*$, $W_2(x, \phi) = \|x - \phi\|^2$ and $\tilde{\phi}$ coincides with $\hat{x}$ and $\bar{x}$. Very important problem arises for Banach spaces: to establish a connection between the metric projection $\bar{x}$ and generalized projections $\hat{x}$ and $\tilde{x}$. Some results in this direction can be found in [9].

2.4. The third generalized projection operator

Let $\Omega \subset B$. A set $\Omega^* \subset B^*$ is called J-dual if $\Omega^* = J\Omega$. Since J is nonlinear mapping in a Banach space, we observe that J-dual set is not convex, in general, even if Ω is convex. However, there are examples of suitable Ω, such that Ω^* is convex. For instance, 1) Ω is a closed ball in B centered at the original point θ_B, 2) $\Omega = J^*G$, where G is arbitrary convex set in B^*.

Consider again the functional (2.7). Like Definition 2.2, introduce the generalized projection of any element $x \in B$ onto set $\Omega \subset B$. However, we now suppose that Ω is not necessary convex while J-dual set Ω^* is certainly convex. Our goal is to construct some generalized projection onto Ω by the minimization problem for (2.7) on Ω^*, as it was first done in [10] (cf. [20]).

Definition 2.5 *Suppose that Ω is an arbitrary closed subset of B and its J-dual set Ω^* is convex. For all $x \in B$, the generalized projection operator $\mathcal{P}_\Omega : B \to \Omega$ is defined as follows:*

$$\mathcal{P}_\Omega x = \breve{x} : \quad \breve{x} = J^*\tilde{x}, \tag{2.9}$$

where $\tilde{x} \in \Omega^*$ is a unique solution of the minimization problem

$$\tilde{x} = \arg\min_{\varphi \in \Omega^*} W_2(x, \varphi). \tag{2.10}$$

$\breve{x}$ is then called generalized $\mathcal{P}_\Omega$-projection of a point x onto Ω.

Due to Definition 2.4, $\mathcal{P}_\Omega = J^*\pi_{\Omega^*}$ or $\pi_{\Omega^*} = J\mathcal{P}_\Omega$. By (2.8),

$$\mathcal{P}_\Omega = J^*\Pi_{\Omega^*}J \quad \text{or} \quad \Pi_{\Omega^*} = J\mathcal{P}_\Omega J^*.$$

Studying in [10] the properties of the operator $\mathcal{P}_\Omega$, we made sure that it is really a generalized projection operator. Below we give three main properties of $\mathcal{P}_\Omega$. Note that the proof of (ii)$_3$ additionally needs the continuity property of J^* (the last takes place if B^* is strongly smooth, i.e., its norm is Fréchet differentiable). So,

(i$_3$) $\mathcal{P}_\Omega x = x$ for any $x \in \Omega$. Hence, $\mathcal{P}_\Omega^2 = \mathcal{P}_\Omega$;

(ii$_3$) The point $\breve{x} \in \Omega$ is the generalized $\mathcal{P}_\Omega$-projection of x onto $\Omega \subseteq B$ if and only if the inequality

$$\langle x - \breve{x}, J\breve{x} - J\xi \rangle \geq 0 \quad \forall x \in B,\ \forall \xi \in \Omega \tag{2.11}$$

is satisfied. This is the basic variational principle for $\mathcal{P}_\Omega x$ in B;

(iii$_3$) $W_2(\breve{x}, J\xi) \leq W_2(x, J\xi) - W_2(x, J\breve{x}) \quad \forall x \in B,\ \forall \xi \in \Omega$.

In a Hilbert space, $\breve{x}$ coincides with $\tilde{\phi}$, $\hat{x}$ and $\bar{x}$.

The property (iii$_3$) implies the conditional nonexpansivity of $\mathcal{P}_\Omega$ with respect to functional W_2 in the form:

$$W_2(\mathcal{P}_\Omega x, J\xi) \leq W_2(x, J\xi) \quad \forall x \in B,\ \forall \xi \in \Omega. \tag{2.12}$$

Remark 2.6 *(2.9) is equivalent to the minimization problem for the Lyapunov functional $W_3(x, \xi)$ with respect to ξ :*

$$W_3(x, \xi) = \|x\|^2 - 2\langle x, J\xi \rangle + \|J\xi\|_*{}^2 \quad \forall x \in B, \quad \forall \xi \in \Omega.$$

For comparison, we list the basic variational principles above:

1) for the metric projection operator P_Ω

$$\langle J(x - P_\Omega x), P_\Omega x - \xi \rangle \geq 0 \quad \forall x \in B, \quad \forall \xi \in \Omega.$$

2) for the sunny nonexansive retraction Q_Ω

$$\langle x - Q_\Omega x, J(Q_\Omega x - \xi) \rangle \geq 0 \quad \forall x \in B, \quad \forall \xi \in \Omega.$$

3) for the first generalized projection operator Π_Ω

$$\langle Jx - J\Pi_\Omega x, \Pi_\Omega x - \xi \rangle \geq 0 \quad \forall x \in B, \quad \forall \xi \in \Omega.$$

4) for the second generalized projection operator π_Ω

$$\langle \phi - J\pi_\Omega \phi, \pi_\Omega \phi - \xi \rangle \geq 0 \quad \forall \phi \in B^*, \quad \forall \xi \in \Omega.$$

5) for the third generalized projection operator $\mathcal{P}_\Omega$

$$\langle x - \mathcal{P}_\Omega x, J\mathcal{P}_\Omega x - J\xi \rangle \geq 0 \quad \forall x \in B, \quad \forall \xi \in \Omega.$$

3 Proximity properties of the $\mathcal{P}_\Omega$-projections

The following estimates established in [2] will be used in the proofs below. If B is a uniformly smooth Banach space, then for all $x, y \in B$, such that $\|x\| \leq R$ and $\|y\| \leq R$,

$$\langle x - y, Jx - Jy \rangle \leq 2LR^2 \rho_B(4R^{-1}\|y - x\|), \tag{3.1}$$

where L is the Figiel's constant [2, 18]. It is known that $1 < L < 1.7$. If B is a uniformly convex Banach space, then for all $x, y \in B$ such that $\|x\| \leq R$ and $\|y\| \leq R$

$$\langle x - y, Jx - Jy \rangle \geq (2L)^{-1} R^2 \delta_B(2R^{-1}\|y - x\|). \tag{3.2}$$

Furthermore, the inequality

$$\|Jx - Jy\|_* \leq 8Rh_B(16LR^{-1}\|x - y\|) \tag{3.3}$$

holds for all $\|x\| \leq R$ and $\|y\| \leq R$ in a uniformly smooth space B.

Let G_1 and G_2 be closed subsets of B. The Hausdorff distance between G_1 and G_2 is defined by the following formula:

$$\mathcal{H}(G_1, G_2) = \max\{ \sup_{z_1 \in G_1} \inf_{z_2 \in G_2} \|z_1 - z_2\|, \ \sup_{z_1 \in G_2} \inf_{z_2 \in G_1} \|z_1 - z_2\| \}.$$

In this section we suppose that B is a uniformly convex and uniformly smooth Banach space.

Theorem 3.1 *Assume that Ω_1 and Ω_2 are closed subsets of B such that the Hausdorff distance $\mathcal{H}(\Omega_1, \Omega_2) \le \sigma$, J-dual sets Ω_1^* and Ω_2^* are convex. Then for all $x \in B$*

$$\|\mathcal{P}_{\Omega_1} x - \mathcal{P}_{\Omega_2} x\| \le 2R_0 \delta_B^{-1}\Big(32LR_0^{-2}RR_1 h_B(16LR^{-1}\sigma)\Big), \tag{3.4}$$

where $\delta_B^{-1}(\eta)$ is the inverse function to $\delta_B(\varepsilon)$,

$$R_0 = \|x\| + 2\, max\{\|P_{\Omega_1}\theta_B\|, \|P_{\Omega_2}\theta_B\|\}, \quad R_1 = R_0 + \|x\|, \quad R = R_0 + \sigma.$$

Proof. Denote $\breve{x}_1 = \mathcal{P}_{\Omega_1} x$ and $\breve{x}_2 = \mathcal{P}_{\Omega_2} x$. Since $\mathcal{H}(\Omega_1, \Omega_2) \le \sigma$, there exists $\xi_1 \in \Omega_1$ such that $\|\breve{x}_2 - \xi_1\| \le \sigma$. By (2.11),

$$\langle x - \breve{x}_1, J\breve{x}_1 - J\xi_1 \rangle \ge 0.$$

We have

$$\langle x - \breve{x}_1, J\breve{x}_2 - J\breve{x}_1)\rangle = \langle x - \breve{x}_1, J\xi_1 - J\breve{x}_1)\rangle + \langle x - \breve{x}_1, J\breve{x}_2 - J\xi_1\rangle$$
$$\le \langle x - \breve{x}_1, J\breve{x}_2 - J\xi_1\rangle.$$

This implies

$$\langle x - \breve{x}_1, J\breve{x}_2 - J\breve{x}_1\rangle \le \|x - \breve{x}_1\| \|J\breve{x}_2 - J\xi_1\|_*. \tag{3.5}$$

Similarly, there exists $\xi_2 \in \Omega_2$ such that $\|\breve{x}_1 - \xi_2\| \le \sigma$, and it results that

$$\langle x - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle \le \|x - \breve{x}_2\| \|J\breve{x}_1 - J\xi_2\|_*. \tag{3.6}$$

Summing up (3.5) and (3.6), one gets

$$\langle \breve{x}_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle \le \|x - \breve{x}_1\| \|J\breve{x}_2 - J\xi_1\|_* + \|x - \breve{x}_2\| \|J\breve{x}_1 - J\xi_2\|_*. \tag{3.7}$$

Let us show that $\|\breve{x}_1\|$ and $\|\breve{x}_2\|$ are bounded by some constants depending on $\|x\|$. Indeed, from the inequality (2.11) with $\xi = P_{\Omega_1}\theta_B$, we have

$$\langle x - \breve{x}_1, J\breve{x}_1 - JP_{\Omega_1}\theta_B\rangle \ge 0.$$

This yields

$$\langle \breve{x}_1, J\breve{x}_1\rangle \le \langle x, J\breve{x}_1\rangle - \langle x, JP_{\Omega_1}\theta_B\rangle - \langle \breve{x}_1, JP_{\Omega_1}\theta_B\rangle.$$

Thus,

$$\|\breve{x}_1\|^2 - (\|x\| + \|P_{\Omega_1}\theta_B\|)\|\breve{x}_1\| - \|x\| \|P_{\Omega_1}\theta_B\| \le 0,$$

and we obtain

$$\|\breve{x}_1\| \le \frac{\|x\| + \|P_{\Omega_1}\theta_B\|}{2} + \sqrt{\Big(\frac{\|x\| + \|P_{\Omega_1}\theta_B\|}{2}\Big)^2 + \|x\| \|P_{\Omega_1}\theta_B\|}.$$

Then

$$\|\breve{x}_1\| \le \|x\| + 2\|P_{\Omega_1}\theta_B\| \le R_0. \tag{3.8}$$

Similarly (3.8) we conclude

$$\|\breve{x}_2\| \le \|x\| + 2\|P_{\Omega_2}\theta_B\| \le R_0. \tag{3.9}$$

The estimates (3.8) and (3.9) imply, respectively,

$$\|x - \breve{x}_1\| \le 2(\|x\| + \|P_{\Omega_1}\theta_B\|) \le R_1$$

and

$$\|x - \breve{x}_2\| \le 2(\|x\| + \|P_{\Omega_2}\theta_B\|) \le R_1.$$

Now we are able to evaluate $\|\xi_1\|$ and $\|\xi_2\|$:

$$\|\xi_1\| \le \|\breve{x}_2\| + \sigma \le \|x\| + 2\|P_{\Omega_2}\theta_B\| + \sigma \le R$$

and

$$\|\xi_2\| \le \|\breve{x}_1\| + \sigma \le \|x\| + 2\|P_{\Omega_1}\theta_B\| + \sigma \le R.$$

By (3.3), we can deduce

$$\|J\breve{x}_1 - J\xi_2\|_* \le 8Rh_B(16LR^{-1}\sigma)$$

and

$$\|J\breve{x}_2 - J\xi_1\|_* \le 8Rh_B(16LR^{-1}\sigma).$$

Now (3.7) yields

$$\langle \breve{x}_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2 \rangle \le 16RR_1h_B(16LR^{-1}\sigma).$$

On the other hand, from (3.2), (3.8) and (3.9) one has

$$\langle \breve{x}_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2 \rangle \ge (2L)^{-1}R_0^2\delta_B(\|\breve{x}_1 - \breve{x}_2\|/2R_0).$$

Combining two last inequalities, we get (3.4). The theorem is proved. ■

Remark 3.2 *The right-hand side of (3.4) has the order $\delta_B^{-1}(h_B(\sigma))$. It is $\sqrt{\sigma}$ in a Hilbert space.*

Corollary 3.3 *Suppose that B is a uniformly convex and uniformly smooth Banach space, Ω_n and Ω are closed subsets of B, J-dual sets Ω_n^* and Ω^* are convex and the Hausdorff distance $\mathcal{H}(\Omega_n, \Omega) \to 0$ as $n \to \infty$. Then for all x_n, $x \in B$ such that $x_n \to x$, it follows: $\mathcal{P}_{\Omega_n}x_n \to \mathcal{P}_\Omega x$.*

The following statement gives us another estimate of $\|\mathcal{P}_{\Omega_1}x - \mathcal{P}_{\Omega_2}x\|$.

Theorem 3.4 *Suppose that Ω_1 and Ω_2 are closed subsets of B, J-dual sets Ω_1^* and Ω_2^* are convex, the Hausdorff distance $\mathcal{H}(\Omega_1^*, \Omega_2^*) \le \sigma$. Then for all $x \in B$*

$$\|\mathcal{P}_{\Omega_1}x - \mathcal{P}_{\Omega_2}x\| \le 8R_0h_{B^*}\Big(32L\delta_{B^*}^{-1}(4LR_1LR_0^{-2}\sigma)\Big), \tag{3.10}$$

where $\delta_{B^*}^{-1}(\eta)$ is the inverse function to $\delta_{B^*}(\varepsilon)$ and

$$R_0 = \|x\| + 2\,max\{\|P_{\Omega_1^*}\theta_{B^*}\|_*, \|P_{\Omega_2^*}\theta_{B^*}\|_*\}, \quad R_1 = \|x\| + R_0.$$

Proof. Like (3.8) and (3.9), one has

$$\|\pi_{\Omega_1^*}x\|_* \le \|x\| + 2\|P_{\Omega_1^*}\theta_{B^*}\|_* \le R_0$$

and

$$\|\pi_{\Omega_2^*}x\|_* \le \|x\| + 2\|P_{\Omega_2^*}\theta_{B^*}\|_* \le R_0.$$

In addition,

$$\|x - J^*\pi_{\Omega_1^*}x\| \le \|x\| + \|\pi_{\Omega_1^*}x\|_* \le \|x\| + R_0 = R_1$$

and

$$\|x - J^*\pi_{\Omega_2^*}x\| \le \|x\| + \|\pi_{\Omega_2^*}x\|_* \le \|x\| + R_0 = R_1.$$

Then

$$\|\mathcal{P}_{\Omega_1}x - \mathcal{P}_{\Omega_2}x\| = \|J^*\pi_{\Omega_1^*}x - J^*\pi_{\Omega_2^*}x\|_*$$

$$\le 8R_0 h_{B^*}\Big(16LR_0^{-1}\|\pi_{\Omega_1^*}x - \pi_{\Omega_2^*}x\|_*\Big). \tag{3.11}$$

Let us estimate the norm in the right-hand side of (3.11). Since $\mathcal{H}(\Omega_1^*, \Omega_2^*) \le \sigma$, there exists $\phi_1 \in \Omega_1^*$ such that $\|\pi_{\Omega_2^*}x_2 - \phi_1\| \le \sigma$. Consequently, we obtain

$$\langle x - J^*\pi_{\Omega_1^*}x, \pi_{\Omega_2^*}x - \pi_{\Omega_1^*}x\rangle$$

$$= \langle x - J^*\pi_{\Omega_1^*}x, \pi_{\Omega_2^*}x - \phi_1\rangle + \langle x - J^*\pi_{\Omega_1^*}x, \phi_1 - \pi_{\Omega_1^*}x\rangle$$

$$\le \sigma\|x - J^*\pi_{\Omega_1^*}x\|,$$

because $\langle x - J^*\pi_{\Omega_1^*}x, \phi_1 - \pi_{\Omega_1^*}x\rangle \le 0$. By analogy, there exists $\phi_2 \in \Omega_2^*$ such that $\|\pi_{\Omega_1^*}x_1 - \phi_2\| \le \sigma$. Since $\langle x - J^*\pi_{\Omega_2^*}x, \phi_2 - \pi_{\Omega_2^*}x\rangle \le 0$, we have

$$\langle x - J^*\pi_{\Omega_2^*}x, \pi_{\Omega_1^*}x - \pi_{\Omega_2^*}x\rangle \le \sigma\|x - J^*\pi_{\Omega_2^*}x\|.$$

Two last inequalities together give

$$\langle J^*\pi_{\Omega_1^*}x - J^*\pi_{\Omega_2^*}x, \pi_{\Omega_1^*}x - \pi_{\Omega_2^*}\rangle \le \sigma\Big(\|x - J^*\pi_{\Omega_1^*}x\| + \|x - J^*\pi_{\Omega_2^*}x\|\Big) \le 2R_1\sigma. \tag{3.12}$$

On the other hand,

$$\langle J^*\pi_{\Omega_1^*}x - J^*\pi_{\Omega_2^*}x, \pi_{\Omega_1^*}x - \pi_{\Omega_2^*}\rangle \ge (2L)^{-1}R_0^2\delta_{B^*}\Big(\|\pi_{\Omega_1^*}x - \pi_{\Omega_2^*}x\|_*/2R_0\Big). \tag{3.13}$$

Thus, it follows from (3.12) and (3.13) that

$$\|\pi_{\Omega_1^*}x - \pi_{\Omega_2^*}x\|_* \le 2R_0\delta_{B^*}^{-1}(4LR_1LR_0^{-2}\sigma).$$

Now (3.11) implies (3.10). The result holds. ∎

We establish now the estimate of uniform continuity of the generalized projection operator $\mathcal{P}_\Omega$.

Theorem 3.5 *Suppose that B is a uniformly convex and uniformly smooth Banach space, Ω is a closed subset of B and its J-dual set $\Omega^* \subset B^*$ is convex. Then for all $x_1, x_2 \in B$*

$$\|\mathcal{P}_\Omega x_1 - \mathcal{P}_\Omega x_2\| \leq 8Rh_{B^*}\Big(32Lg_{B^*}^{-1}(4LR^{-1}\|x_1 - x_2\|)\Big), \tag{3.14}$$

where $g_B(\varepsilon) = \dfrac{\delta_B(\varepsilon)}{\varepsilon}$, $g_{B^*}^{-1}(\eta)$ *is the inverse function to* $g_{B^*}(\varepsilon)$ *and*

$$R = \ max\{\|x_1\|, \|x_2\|\} + 2\|P_\Omega \theta_B\|.$$

Proof. For $\breve{x}_1 = \mathcal{P}_\Omega x_1$ and $\breve{x}_2 = \mathcal{P}_\Omega x_2$ the condition (2.11) implies

$$\langle \breve{x}_1 - x_1, J\breve{x}_1 - J\breve{x}_2\rangle \leq 0.$$

Then

$$\langle \breve{x}_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle = \langle x_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle + \langle \breve{x}_1 - x_1, J\breve{x}_1 - J\breve{x}_2\rangle$$
$$\leq \langle x_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle.$$

Since

$$\langle x_2 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle \leq 0,$$

we have

$$\langle x_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle \leq \langle x_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle - \langle x_2 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle$$
$$= \langle x_1 - x_2, J\breve{x}_1 - J\breve{x}_2\rangle.$$

Thus,

$$\langle \breve{x}_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle \leq \|x_1 - x_2\|\|J\breve{x}_1 - J\breve{x}_2\|_*.$$

By (3.2), the following lower inequality holds:

$$\langle \breve{x}_1 - \breve{x}_2, J\breve{x}_1 - J\breve{x}_2\rangle \geq (2L)^{-1}R^2\delta_{B^*}(\|J\breve{x}_1 - J\breve{x}_2\|_*/2R).$$

Consequently,

$$\delta_{B^*}(\|J\breve{x}_1 - J\breve{x}_2\|_*/2R) \leq 2LR^{-2}\|x_1 - x_2\|\|J\breve{x}_1 - J\breve{x}_2\|_*,$$

that is,

$$\frac{2R\delta_{B^*}(\|J\breve{x}_1 - J\breve{x}_2\|_*/2R)}{\|J\breve{x}_1 - J\breve{x}_2\|_*} \leq 4LR^{-1}\|x_1 - x_2\|.$$

From this one gets

$$\|J\breve{x}_1 - J\breve{x}_2\|_* \leq 2Rg_{B^*}^{-1}(4LR^{-1}\|x_1 - x_2\|).$$

By (3.3), we deduce the final estimate

$$\|\breve{x}_1 - \breve{x}_2\| \leq 8Rh_{B^*}(16LR^{-1}\|J\breve{x}_1 - J\breve{x}_2\|_*)$$
$$\leq 8Rh_{B^*}(32Lg_{B^*}^{-1}(4LR^{-1}\|x_1 - x_2\|).$$

The proof is accomplished. ■

4 Equivalent decompositions for subspaces

Let us give the nontrivial example of a closed set C for which the J-dual set C^* is convex and closed. We proved in [5] that any uniformly convex and uniformly smooth Banach space B admits the following decomposition:

$$B = M \uplus J^* M^{\perp}, \tag{4.1}$$

where M is a closed linear subspace of B and $M^{\perp}$ is its annihilator. The latter is defined as

$$M^{\perp} = \{\phi \in B^* : \langle \phi, x \rangle = 0 \ \ \forall x \in M\}. \tag{4.2}$$

It is well known that $M^{\perp}$ is a closed linear subspace of B^*. Any subspace of strictly convex and smooth Banach space B is strictly convex and smooth. Consequently, $M^{\perp}$ is convex and smooth [26]. Denote $C = J^* M^{\perp}$. Since J-dual set $C^* = JC = M^{\perp}$, we conclude that C^* is convex, smooth and closed. It was shown in [5] that C is a smooth and closed manifold, however, it is nonlinear and nonconvex in general. We now notice the following: under our conditions, for any $x \in B$ and for any $\phi \in B^*$,
1) a subspace M has the unique metric projection $\bar{x}$ and unique generalized projections $\hat{x}$ and $\bar{\phi}$ (the same is true for $M^{\perp}$);
2) a manifold C has a unique generalized projection $\check{x}$.

Reproduce the following result from [4]: any element $x \in B$ admits the unique orthogonal decomposition

$$x = P_M x + J^* \Pi_{M^{\perp}} J x, \tag{4.3}$$

where

$$\langle \Pi_{M^{\perp}} J x, v \rangle = 0 \quad \forall v \in M, \tag{4.4}$$

that is, the second term in the right-hand side of (4.3) is James orthogonal to M. We recall the original definition of James' orthogonality [22]:

Definition 4.1 *An element $x \in B$ is James orthogonal to $y \in B$ if*

$$\|x\| \le \|x + ty\| \quad \forall t \in R. \tag{4.5}$$

We say that $x \in B$ is James orthogonal to a subset $K \subset B$ if x is James-orthogonal to each $y \in K$.

As it has been shown in [5], this definition is equivalent to

Definition 4.2 *An element $x \in B$ is James orthogonal to $y \in B$ if $\langle Jx, y \rangle = 0$. An element $x \in B$ is James orthogonal to a subset $K \subset B$ if $\langle Jx, v \rangle = 0$ for all $v \in K$.*

Namely Definition 4.2 implies (4.4). It is clear that the James' orthogonality is not symmetric in a Banach space because in general case $\langle Jx, y \rangle \neq \langle x, Jy \rangle$, and only in an inner product space it is symmetric [27]. It is obvious that in a Hilbert space, (4.3) and (4.4) can be rewritten as

$$x = P_M x + P_{M^{\perp}} x \tag{4.6}$$

and

$$(P_{M^\perp}x, v) = 0 \quad \forall v \in M \tag{4.7}$$

(see more detailed in [24]). Due to (4.2), $M^\perp$ means here an orthogonal complement to M and $P_{M^\perp}x$ is the metric projection of x onto $M^\perp$. Let us emphasize that in [4] and [5] we could not show that the last term in (4.3) is a projection of x onto some set in B. It has been done here below.

By virtue of Definition 2.5, we can now present the following theorem:

Theorem 4.3 *Suppose that B is a reflexive strictly convex and smooth Banach space with dual space B^*. If M is a closed subspace of B, $M^\perp \subset B^*$ is its annihilator, $C = J^*M^\perp$, then every element x of B has one and only one decomposition*

$$x = P_M x + \mathcal{P}_C x, \tag{4.8}$$

where $\mathcal{P}_C x$ is James orthogonal to M, i.e.,

$$\langle J\mathcal{P}_C x, v\rangle = 0 \quad \forall v \in M.$$

It follows from (4.8) that if P_M is the metric projection operator onto a subspace M, then $I - P_M$ is the generalized projection operator $\mathcal{P}_C$ onto C. It is easy to see that $P_M x = \theta_B$ for all $x \in C$ and $\mathcal{P}_C x = \theta_B$ for all $x \in M$. This implies $\mathcal{P}_C P_M x = \theta_B$ and $P_M \mathcal{P}_C x = \theta_B$ for all $x \in B$. We have also shown in [4] that $\Pi_M x = \theta_B$ if $x \in C$. Consequently, $\Pi_M \mathcal{P}_C x = \theta_B$ for all $x \in B$.

The previous result (2.11) gives

$$\langle x - \mathcal{P}_C x, J\mathcal{P}_C x - J\xi\rangle \geq 0 \quad \forall \xi \in C,$$

that is,

$$\langle P_M x, J\mathcal{P}_C x - J\xi\rangle \geq 0 \quad \forall \xi \in C.$$

Since $J\mathcal{P}_C x \in M^\perp$ and $J\xi \in M^\perp$, in fact, we have

$$\langle P_M x, J\mathcal{P}_C x - J\xi\rangle = 0 \quad \forall \xi \in C.$$

It is also clear that

$$\langle P_M x, J\mathcal{P}_C x\rangle = \langle P_M x, J\xi\rangle = 0 \quad \forall \xi \in C. \blacksquare$$

It is known that if $x \in B$ and $y \in M$, then for any constants $-\infty < \alpha,\ \beta < +\infty$

$$P_M(\alpha x + \beta y) = \alpha P_M x + \beta P_M y \tag{4.9}$$

(it is so called "the conditional linearity property" of P_M. This means that

$$P_M(\alpha x + \beta y) = \alpha P_M x + \beta y. \tag{4.10}$$

In particular, $P_M(\alpha x) = \alpha P_M x$. We can prove the similar equalities for generalized projection operator $\mathcal{P}_C$.

Lemma 4.4 *Let any $x \in B$, $y \in M$ and let $-\infty < \alpha,\ \beta < +\infty$. Then*

$$\mathcal{P}_C(\alpha x + \beta y) = \alpha \mathcal{P}_C x + \beta \mathcal{P}_C y, \tag{4.11}$$

that is,

$$\mathcal{P}_C(\alpha x + \beta y) = \alpha \mathcal{P}_C x. \tag{4.12}$$

In particular, $\mathcal{P}_C(\alpha x) = \alpha \mathcal{P}_C x$.

Proof. Indeed, since $\mathcal{P}_C z = z - P_M z$ for all $z \in B$, one gets

$$\begin{aligned} \mathcal{P}_C(\alpha x + \beta y) &= \alpha x + \beta y - P_M(\alpha x + \beta y) \\ &= \alpha(x - P_M x) + \beta(y - P_M y) \\ &= \alpha \mathcal{P}_C x + \beta \mathcal{P}_C y, \end{aligned}$$

i.e., (4.11) holds. Obviously, (4.12) is also true because of $\mathcal{P}_C y = 0$. Setting in (4.12) $y = \theta_B \in M$, we obtain $\mathcal{P}_C(\alpha x) = \alpha \mathcal{P}_C x$. ■

Observe that if in (4.12) $x \in C$ and $y \in M$, then $\mathcal{P}_C(\alpha x + \beta y) = \alpha x$. Now we consider arbitrary $x, y \in B$.

Lemma 4.5 *Let any $x,\ y \in B$ and let α and β be arbitrary constants. Then*

$$\mathcal{P}_C(\alpha x + \beta y) = \mathcal{P}_C(\alpha \mathcal{P}_C x + \beta \mathcal{P}_C y) \tag{4.13}$$

Proof. It is not difficult to see that

$$\alpha x + \beta y = \alpha(P_M x + \mathcal{P}_C x) + \beta(P_M y + \mathcal{P}_C y)$$

$$= (\alpha P_M x + \beta P_M y) + (\alpha \mathcal{P}_C x + \beta \mathcal{P}_C y).$$

Since $\alpha P_M x + \beta P_M y \in M$, we have $\mathcal{P}_C(\alpha P_M x + \beta P_M y) = 0$. Consequently, the result (4.13) is true due to Lemma 4.4. ■

In particular, if $y \in M$ then we obtain (4.12) because $\mathcal{P}_C y = 0$ and $\mathcal{P}_C(\alpha \mathcal{P}_C x) = \alpha \mathcal{P}_C \mathcal{P}_C x = \alpha \mathcal{P}_C x$. If $y \in C$ then

$$\mathcal{P}_C(\alpha x + \beta y) = \mathcal{P}_C(\alpha \mathcal{P}_C x + \beta y).$$

Remark 4.6 *If in Lemma 4.5 the set $C = J^* M^\perp$ is linear, then $\alpha \mathcal{P}_C x + \beta \mathcal{P}_C y \in C$ and we have in (4.13)*

$$\mathcal{P}_C(\alpha x + \beta y) = \alpha \mathcal{P}_C x + \beta \mathcal{P}_C y.$$

We conjecture that such situation may take place only in Hilbert space with $C = M^\perp$, where $M^\perp$ is the orthogonal complement to M.

In order to prove the next theorem, we will beforehand provide the following lemma.

Lemma 4.7 *If x is an arbitrary point of B, ϕ is an arbitrary point of $M^{\perp}$ and $\tilde{x}$ is $\pi_{M^{\perp}}$-projection of x onto $M^{\perp}$, then*

$$W_2(x,\phi) = W_2(x,\tilde{x}) + W_2(J^*\tilde{x},\phi). \tag{4.14}$$

Proof. We use the property (ii$_2$) in the form

$$\langle x - J^*\tilde{x}, \tilde{x} - \phi\rangle \geq 0 \quad \forall \phi \in M^{\perp}.$$

Since $\tilde{x} \in M^{\perp}$ and $\phi \in M^{\perp}$, we can prove, as it was done in [3] (Lemma 2.2), that

$$\langle x - J^*\tilde{x}, \tilde{x}\rangle = 0 \tag{4.15}$$

and

$$\langle x - J^*\tilde{x}, \phi\rangle \leq 0 \quad \forall \phi \in M^{\perp}.$$

By [4] (p.89), we have in reality

$$\langle x - J^*\tilde{x}, \phi\rangle = 0 \quad \forall \phi \in M^{\perp}. \tag{4.16}$$

From the equality (4.16), one gets

$$\langle x, \phi\rangle = \langle J^*\tilde{x}, \phi\rangle \quad \forall \phi \in M^{\perp}.$$

In view of the equality (4.15),

$$\begin{aligned} \langle x, \phi\rangle &= \langle J^*\tilde{x}, \phi\rangle + \langle x - J^*\tilde{x}, \tilde{x}\rangle \\ &= \langle J^*\tilde{x}, \phi\rangle - \|\tilde{x}\|^2 + \langle x, \tilde{x}\rangle. \end{aligned}$$

This implies

$$\begin{aligned} &\|x\|^2 - 2\langle x, \phi\rangle + \|\phi\|_*^2 \\ &= (\|x\|^2 - 2\langle x, \tilde{x}\rangle + \|\tilde{x}\|_*^2) + (\|J^*\tilde{x}\|^2 - 2\langle J^*\tilde{x}, \phi\rangle + \|\phi\|_*^2), \end{aligned}$$

which is equivalent to (4.14). ∎

Theorem 4.8 *Let M be a subspace of B, $M^{\perp} \subset B^*$ be an annihilator, $G^* \subseteq M^{\perp}$ be a closed convex set, $G = J^*G^*$, $C = J^*M^{\perp}$. Then for any $x \in B$*

$$\mathcal{P}_G x = \mathcal{P}_G \mathcal{P}_C x. \tag{4.17}$$

Proof. By (2.9) and (4.14), we have

$$\begin{aligned} \mathcal{P}_G x &= J^* \arg\min_{\phi \in G^*} W_2(x,\phi) \\ &= J^* \arg\min_{\phi \in G^*} (W_2(x, \pi_{M^{\perp}} x) + W_2(J^*\pi_{M^{\perp}} x, \phi)). \end{aligned}$$

Since $W_2(x, \pi_{M^{\perp}} x)$ does not depend on ϕ, one gets

$$\mathcal{P}_G x = J^* \arg\min_{\phi \in G^*} W_2(J^*\pi_{M^{\perp}} x, \phi) = J^*\pi_{G^*} J^*\pi_{M^{\perp}} x.$$

By the conditions, we know that $\mathcal{P}_G = J^*\pi_{G^*}$ and $\mathcal{P}_C = J^*\pi_{M^{\perp}}$. Therefore (4.17) holds. ∎

Remark 4.9 *Under the conditions of Theorem 4.8, for all $x \in B$*

$$\mathcal{P}_G \mathcal{P}_C x = \mathcal{P}_C \mathcal{P}_G x.$$

Lemma 4.10 *An element $x \in B$ is James orthogonal to M if and only if $x \in J^* M^\perp$. If $x_n \in J^* M^\perp$ and $x_n \to x$ as $n \to \infty$, then x is James orthogonal to M.*

Proof. If $x \in B$ is James orthogonal to M, then $\langle Jx, v \rangle = 0$ for any $v \in M$. This implies that $Jx \in M^\perp \subset B^*$ and $x \in J^* M^\perp = C$. Conversely, if $x \in C$ then $Jx \in M^\perp$ and $\langle Jx, v \rangle = 0$ follows. The second assertion appears from the fact that C is a closed set [5]. ■

5 Equivalent decompositions for cones

We use now the concepts of dual and polar cones.

Definition 5.1 *A set $K \subset B$ is said to be a cone if it contains λx together with an element $x \in K$, $\lambda > 0$.*

Any cone $K \subset B$ induces dual cone $K^* \subset B^*$ and polar cone $K^0 \subset B^*$.

Definition 5.2 *The dual cone is given by the formula*

$$K^* = \{\phi \in B^* : \langle x, \phi \rangle \geq 0 \quad \forall x \in K\}.$$

The polar cone is given by the formula

$$K^0 = \{\phi \in B^* : \langle x, \phi \rangle \leq 0 \quad \forall x \in K\}.$$

In general, $K^* = -K^0$. If K is a subspace $M \subset B$ then $K^* = K^0 = M^\perp$, i.e.,

$$M^\perp = \{\phi \in B^* : \langle x, \phi \rangle \geq 0 \quad \forall x \in M\}.$$

In [4] we have proved the following theorem (cf. [28]):

Theorem 5.3 *If $K \subset B$ is a closed convex cone with the vertex at θ_B, $K^0 \subset B^*$ is the polar cone, x is an arbitrary element of the Banach space B, then*

$$x = P_K x + J^* \Pi_{K^0} J x, \tag{5.1}$$

where

$$\langle \Pi_{K^0} J x, P_K x \rangle = 0. \tag{5.2}$$

In a Hilbert space, (5.1) and (5.2) respectively accept the forms [25]:

$$x = P_K x + P_{K^0} x \tag{5.3}$$

and

$$(P_{K^0} x, P_K x) = 0.$$

Next we present the equivalent decomposition theorem for cones.

Theorem 5.4 *If $K \subset B$ is a non-empty closed convex cone with the vertex at θ_B, $K^0 \subset B^*$ is the polar cone, $S = J^*K^0$, x is an arbitrary element of the space B, $P_K x$ is a metric projection of x onto K, $\mathcal{P}_S x$ is a $\mathcal{P}_S$- projection of x onto S, then*

$$x = P_K x + \mathcal{P}_S x, \tag{5.4}$$

where

$$\langle J\mathcal{P}_S x, P_K x\rangle = 0. \tag{5.5}$$

Proof. First of all, we show that an element $\bar{x} \in K$ is the metric projection of $x \in B$ onto K if and only if (1) $\langle J(x-\bar{x}), \bar{x}\rangle = 0$ and (2) $\langle J(x-\bar{x}), v\rangle \leq 0 \quad \forall v \in K$.

(i) We assert that the inclusion $\bar{x} \in K$ and equality $\bar{x} = P_K x$ follow from (1) and (2). Indeed, subtracting the equality (1) from (2), one gets

$$\langle J(x-\bar{x}), \bar{x} - v\rangle \geq 0 \quad \forall v \in K, \tag{5.6}$$

therefore, the claim holds thanks to (1.3).

(ii) Let $\bar{x} = P_K x$. Then (1.3) holds, i.e.,

$$\langle J(x - P_K x), P_K x - \xi\rangle \geq 0 \quad \forall \xi \in K. \tag{5.7}$$

For $\xi = \theta_B \in K$ we have

$$\langle J(x - P_K x), P_K x\rangle \geq 0. \tag{5.8}$$

We now set $\xi = 2P_K x$ in (5.7). Since $2P_K x = P_K 2x$, therefore $\xi \in K$ and

$$\langle J(x - P_K x), P_K x - 2P_K x\rangle \geq 0.$$

This implies

$$\langle J(x - \mathcal{P}_K x), \mathcal{P}_K\rangle \leq 0. \tag{5.9}$$

The inequalities (5.8) and (5.9) together give (1). It is not difficult to see that (2) arises from (5.6) and (1). Indeed, for all $v \in K$

$$\langle J(x-\bar{x}), v\rangle = \langle J(x-\bar{x}), v\rangle - \langle J(x-\bar{x}), \bar{x}\rangle = \langle J(x-\bar{x}), v - \bar{x}\rangle \leq 0.$$

The inequality (2) shows that $J(x - P_K x) \in K^0$ or $x - P_K x \in S$. Denote $w = x - P_K x$. Then

$$x = P_K x + w,$$

where, due to (1), an element $w \in S$ is James orthogonal to $P_K x$, i.e.,

$$\langle Jw, P_K x\rangle = 0, \tag{5.10}$$

or

$$\langle Jw, x - w\rangle = 0.$$

Choose $\xi \in S$ and get

$$\langle Jw - J\xi, x - w\rangle + \langle J\xi, x - w\rangle = 0.$$

It is clear that $\langle J\xi, x - w\rangle \le 0$. Therefore,

$$\langle Jw - J\xi, x - w\rangle \ge 0 \quad \forall \xi \in S.$$

In view of (2.11), this implies $w = \mathcal{P}_S x$ and then $x = P_K x + \mathcal{P}_S x$ with (5.10) which is rewritten as $\langle J\mathcal{P}_S x, P_K x\rangle = 0$. The theorem is true. ■

In addition, from (5.10) it follows $\|w\|^2 = \langle Jw, x\rangle$. Hence, $\|\mathcal{P}_S x\|^2 = \langle \pi_{K^0} x, x\rangle$. Denoting $S = J^* K^0$ we have $\mathcal{P}_S = J^* \Pi_{K^0} J$.

Next we prove that $\mathcal{P}_S(\alpha x) = \alpha \mathcal{P}_S x$. However, previously let us state

Lemma 5.5 *Let $K \subset B$ be a non-empty closed convex cone with the vertex at θ_B. If $\alpha > 0$ then $P_K(\alpha x) = \alpha P_K x$ for all $x \in B$.*

Proof. The factors (1) and (2) in Theorem 5.4 assert that

$$\langle J(\alpha x - \alpha \bar{x}), \alpha \bar{x}\rangle = \alpha^2 \langle J(x - \bar{x}), \bar{x}\rangle = 0$$

and

$$\langle J(\alpha x - \alpha \bar{x}), v\rangle = \alpha \langle J(x - \bar{x}), v\rangle \le 0 \quad \forall v \in K.$$

Since $\alpha \bar{x} \in K$ (see Definition 5.1), this means that $\alpha P_K x = P_K(\alpha x)$. The claim is valid. ■

Lemma 5.6 *Let $K \subset B$ be a non-empty closed convex cone with the vertex at θ_B. If $\alpha > 0$ then $\mathcal{P}_S(\alpha x) = \alpha \mathcal{P}_S x$,*

Proof. Due to Lemma 5.5, from the decomposition (5.4) we have

$$\mathcal{P}_S(\alpha x) = \alpha x - P_K(\alpha x) = \alpha x - \alpha P_K x = \alpha(x - P_K x) = \alpha \mathcal{P}_S x. \blacksquare$$

In the conclusion we present the statement which is very important for applications.

Theorem 5.7 *Let K be a non-empty closed convex cone in the Banach space B with the vertex in θ_B, K^0 be polar cone and $S = J^* K^0$. An element $\breve{x} \in S$ is the generalized $\mathcal{P}_S$-projection of a point x onto S if and only if*
1) $\langle x - \breve{x}, J\breve{x}\rangle = 0$,
2) $\langle x - \breve{x}, Jv\rangle \le 0 \quad \forall v \in S$.

The proof is the same as in Theorem 5.4.

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