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CONDITIONAL EXPECTATIONS ON ORDERED NORMED SPACES

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The notion of conditional expectations have been studied only for algebras, since its definition is based on the notion of the multiplication of elements. In this article, we work with an order unit space, which may be considered as an JBW-algebra, however the multiplication in such an object is not defined. We consider a natural question of extending the notion of conditional expectations to the order unit space. We introduce the definition of conditional expectation in the spaces with an order unit and show that this definition agrees with that given for JBW-algebras. The theorem on existence of conditional expectations on generalized spin factors is proved.

1. Introduction

An order unit space presents a statistic model [7], as a space of affine functions on the space of states. Lately, the theory of those spaces is intensively developed.

In the classical model a space of states is a simplex. Generally, it is an arbitrary convex set in a local convex space. In the case of commutative algebra elements of the order unit space is interpreted as a space of observed on a measure space.

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In the works [2, 8, 9] the conditional expectations on W*- and JBW-algebras are studied. The order unit space which is being an ordered linear normed space, can be considered as an JBW-algebra by the normed structure. Naturally, the question on extension of the notion of conditional expectations to the order unit spaces with order unit is born. In the paper we give the definition of an conditional expectation for the spaces with order unit. We show that this definition agrees with definition given for JBW-algebras and we prove a theorem on existence of conditional expectations on generalized spin factors.

2. Preliminaries

Let A be a real linear ordered space. We denote by A^+ the set of its positive elements. An element $\mathbf{e} \in A^+$ is said to be an order unit if for any $a \in A$, there exists a number $\lambda \in \mathbb{R}^+$ such that $-\lambda\mathbf{e} \leq a \leq \lambda\mathbf{e}$. If the order is Archimedean, then the mapping defined as

$$a \to \|a\| = \inf\{\lambda > 0 : -\lambda\mathbf{e} \leq a \leq \lambda\mathbf{e}\}$$

is a norm in A. If A is a Banach space with respect to this norm then we say that $(A, \mathbf{e})$ is an order unit space with the order unit $\mathbf{e}$ [1].

Let V be a real linear space with the generating cone V^+ possessing a base K, i.e.

$$V = V^+ - V^+, \quad V^+ = \bigcup_{\lambda \geq 0} \lambda K, \quad \lambda K \cap K = \varnothing, \text{ as } \lambda \neq 1.$$

We suppose that the set $B = \operatorname{conv}(K \cup -K)$ is radially compact, i.e. the set $B \cap L$ is a closed bounded segment for any straight line L passing through the zero element of V. In this case the Minkowsky's functional

$\|\rho\| = \inf\{\lambda \geq 0 : \rho \in \lambda B\}$ turns V into a normed space called a base-norm space. Below, we denote this space by (V, K).

Let $(A, \mathbf{e})$ be an order unit space, (V, K) be a base-norm space. Suppose that these spaces are in separated, ordered and normed duality. This duality we denote by $< \cdot, \cdot >$.

Further, a linear weakly *-continuous mapping $R : A \to A$ with $R^2 = R$ is called projection of the space A. A projection R is called smooth if for $a \in \ker^+ R = A^+ \cap \ker R$, the condition $\rho \in V^+$, $< a, \rho >= 0$ implies $< a, \rho >= 0$. A projection Q is said to be a quasicomplement of R if

$$\ker^+ R = im^+ Q, \quad im^+ R = \ker^+ Q.$$

A projection R is called P-projection if R and its quasicomplement Q are smooth and $\|R\| \leq 1$, $\|Q\| \leq 1$. It is known that the smooth quasicomplement of P-projector R is always unique. Further we denote it by R'. Elements of the set $U = \{u = R\mathbf{e} : \ R \text{ is } P\text{-projector in } A\}$ are called projective units.

If A is an JBW-algebra, projective units are idempotents, and P-projectors have the form $Ra = U_p a$, where p is an idempotent.

3. Main Results

Let $(A, \mathbf{e})$ be an order unit space and let B be an order unit subspace of A, which contains $\mathbf{e}$.

DEFINITION 1. A linear mapping $E : A \to B$ is said to be a conditional expectation with respect to B if

1. $E(\mathbf{e}) = \mathbf{e}$;

2. $a \geq 0 \Rightarrow E(a) \geq 0$;

3. $E(Ra) = R(Ea)$ for all P-projectors R in A such that $R\mathbf{e} \in B$ and $a \in A$.

It is not difficult to show that $\|E\| = 1$.

Indeed, let $a \in A$ and $\|a\| \leq 1$, i.e. $-\mathbf{e} \leq a \leq \mathbf{e}$. Owing to the positivity of E we have $E(a + \mathbf{e}) \geq 0$ and $E(\mathbf{e} - a) \geq 0$. Since E is linear and $E(\mathbf{e}) = \mathbf{e}$, the last inequalities mean that $-\mathbf{e} \leq E(a) \leq \mathbf{e}$. Hence, $\|E\| \leq 1$. But $E(\mathbf{e}) = \mathbf{e}$. Therefore $\|E\| = 1$.

Definition 1 implies that E is an idempotent mapping, i.e. $E(E(a)) = E(a)$, for all $a \in A$, in the other words, $E^2 = E$. Indeed, if $u \in B$ is a projective unit, then $u = R\mathbf{e}$, for a P-projector R, and by conditions 3 and 1 of Definition 1 we have $E(u) = E(R\mathbf{e}) = R(E\mathbf{e}) = R\mathbf{e} = u$. Further, let $a = \sum_{i=1}^{n} \lambda_i u_i$ be a simple element of B. Then, obviously, $u_i \in B$ and $E(a) = \sum_{i=1}^{n} E(\lambda_i u_i) = \sum_{i=1}^{n} \lambda_i u_i = a$. It means that $E(a) = a$. Since $E(a) \in B$ for any $a \in A$, the equality $E(E(a)) = E(a)$ holds.

As it was stated above, an JBW-algebra is an example for an order unit space. The conditional expectations on JBW-algebras are defined as follows [2].

Let A be an JBW-algebra with the unit $\mathbf{1}$ and let A_1 be an JBW-subalgebra of A containing $\mathbf{1}$.

DEFINITION 2. A linear mapping $E : A \to A_1$ is called an conditional expectation if

(i) $E(\mathbf{1}) = \mathbf{1}$;

(ii) $x \geq 0 \Rightarrow E(x) \geq 0$;

(iii) $E(ax) = aE(x)\ \forall x \in A,\ \forall a \in A_1$.

This definition agrees with Definition 1, i.e. an conditional expectation on JBW-algebras satisfies also to Definition 1.

THEOREM 1. *Let $M : A \to A_1$ be a linear mapping having the properties*:

1) $M(\mathbf{1}) = \mathbf{1}$;

2) $x \geq 0 \Rightarrow M(x) \geq 0$;

3) $M(U_p(x)) = U_p M(x)\ \forall x \in A$ *and for any idempotent* $p \in A_1$.

Then M is a conditional expectation.

PROOF. The properties (i) and (ii) of Definition 2 are obvious. We will check the condition (iii).

Let $M(U_p(x)) = U_p M(x)$ for any idempotent $p \in A_1$ and $\forall x \in A$. Then we have $U_{p'}(Mx) = M\left(U_{p'}x\right)$, for all $x \in A$ and $p' = \mathbf{1} - p \in A_1$. It is known [5], that the Pierce decomposition $x = U_p x + 2U_{p,p'}x + U_{p'}x$ takes place for $x \in A$ with respect to the idempotent p. Therefore we have

$$M(x) = M(U_p x) + M(2U_{p,p'}x) + M\left(U_{p'}x\right).$$

On the other hand, the Pierce decomposition for Mx is the following:

$$M(x) = U_p(M(x)) + 2U_{p,p'}(M(x)) + U_{p'}(M(x)).$$

Hence, by the condition 3) of the theorem, we get

$$2U_{p,p'}(M(x)) = M(2U_{p,p'}x).$$

Since $U_{p,p'}x = 2px - 2p(px)$ by Definition (see [5]), the last equality means that

$$2p(M(x)) - 2p(p(M(x))) = M(2px - 2p(px)).$$

Similarly by the condition 3) of the theorem we get

$$2p(p(M(x))) - pM(x) = M(2p(px) - px).$$

Summing these equalities, we obtain

$$pM(x) = M(px) \ \forall x \in A, \ p \in A_1.$$

Since M is weakly continuous and a linear span of idempotents is weakly dense in A_1 we conclude that

$$aM(x) = M(ax), \quad \forall x \in A, \ \forall a \in A_1. \qquad \square$$

Further we shall consider of the following examples

EXAMPLES 1. **1)** Let $(A, \mathbf{e})$ be an order unit space and let ρ be a state on A. For $a \in A$, we put $E(a) = \rho(a)$. Then E is the conditional expectation with respect to the subspace $B = \{\lambda\mathbf{e} : \lambda \in \mathbb{R}\}$.

2) Let Q be a P-projector in A. Put $E(a) = Qa + Q'a$, $\forall a \in A$. Then E is the conditional expectation with respect to the subspace

$$B = \{a \in A : a = Qa + Q'a\} = imQ + imQ'.$$

In fact, for the map E the conditions 1 and 2 of Definition 1 are obvious, since Q is a P-projector. Let us to check the property 3.

Let $R\mathbf{e} \in B$, i.e. $R\mathbf{e} \in imQ + imQ'$. It means that R and Q are compatible, i.e. $RQ = QR$ and $RQ' = Q'R$ (see [1], 5.26). Hence, $RE(a) = E(Ra)$.

DEFINITION 3. A state τ on A is said to be *a trace* if

$$\tau(a) = \tau(Ra) + \tau(R'a)$$

for every $a \in A$ and P-projector R.

Let ρ be a state on A. If $\rho(Ea) = \rho(A)$, then we say that E preserves ρ. It is easy to see, that the conditional expectation E from Examples 1) and 2) preserves the states ρ and τ, respectively.

The following question of interest: when does a conditional expectation for a given subspace exist? The answer still remains open. We can will answer in the special case, when an order unit space is a generalized spin factor.

Let X be a reflexive Banach space and let the unit ball X_1 of X be a smooth, strongly convex set, i.e. the own faces of the unit ball X_1^* of the dual space X^* are only the sets of the form $\{\sigma\}$, where σ is an extreme point of X_1^*, and for each $\sigma \in \partial e X_1^*$ there exists an unique element $x \in \partial e X_1^*$ with $\sigma(x) = 1$.

Consider the spaces $A = \mathbb{R} + X$ and $V = \mathbb{R} + X^*$. The order and norm on A are defined as:

$$a = \alpha + x \geq 0 \;\; \Leftrightarrow \;\; \alpha \geq \|x\|, \;\; \|a\| = |\alpha| + \|x\|$$

$$(\rho = \gamma + \xi \geq 0 \;\; \Leftrightarrow \;\; \beta \geq \|\xi\|, \;\; \|\rho\| = \max(|\beta|, \|\xi\|))$$

for $a \in A$, $\rho \in V$, respectively.

Clearly, A is an order unit space and V is a base-norm space. Moreover these spaces are in separated, ordered and normed duality with respect to the form:

$$\langle \alpha, \rho \rangle = \langle \alpha + x, \beta + \xi \rangle = \alpha\beta + \xi(x), \tag{1}$$

where ξ is a bounded linear functional on X.

The order unit spaces of this construction are called generalized spin factors [4].

The trace τ on a generalized spin factor is unique and defined as follows: $\tau(\alpha + x) = \alpha$.

Since the unit ball of X is a smooth, strongly convex set, elements of the form of $u = \frac{1}{2} + \frac{1}{2}x_0$, where $x_0 \in X$, $\|x_0\| = 1$, are projective units, and P-projector R corresponding to u has the form $Ra = \langle a, \hat{u} \rangle\, u$, where $\hat{u}$ is a continuous linear functional on A having the properties: $\langle u, \hat{u} \rangle = 1$, $\|\hat{u}\| = 1$.

Let $A = \mathbb{R} + X$ be a generalized spin factor and let B be its arbitrary subspace. It is not difficult to show that B has the form $B = R + X_0$, where X_0 is a subspace of X.

Theorem 2. *The trace-preserving conditional expectation on A with respect to B exists if and only if there exists a projection T from X into X_0.*

Proof. *Necessity.* Let there exist the trace-preserving conditional expectation E with respect to $B = \mathbb{R} + X_0$. For $\forall a = \alpha + x \in A$, the element $Ea \in B$ has the form

$$Ea = \alpha + Tx. \tag{2}$$

Here T is a projection from X into X_0.

Indeed, let $Ea = \alpha + f(x) + Tx$ for a functional f on X and a linear mapping $T : X \to X_0$. Take $u \in B$ and let $u = R\mathbf{e}$. Since $Eu = u$, the property 3 of conditional expectation (i.e. the equality $ERa = REa$) implies that $\langle a, \hat{u} \rangle\, u = \langle Ea, \hat{u} \rangle\, u$, i.e. $\langle a, \hat{u} \rangle = \langle Ea, \hat{u} \rangle$.

Since the projective unit $u \in B$ has the form $u = \frac{1}{2} + \frac{1}{2}x_0,\ x_0 \in X_0$, $\|x_0\| = 1$, and it corresponds to the state $\hat{u} = 1 + \xi_0$ in B^*, where $\xi_0 \in X_0^*$, $\|\xi_0\| = 1$, $\langle \xi_0, x_0 \rangle = 1$, we have

$$\langle a, \hat{u} \rangle = \langle \alpha + x, 1 + \xi_0 \rangle = \alpha + \xi_0(x),$$

$$\langle Ea, \hat{u} \rangle = \langle \alpha + f(x), 1 + \xi_0 \rangle = \alpha + f(x) + \xi_0(Tx).$$

Hence we get $f(x) = \xi_0(x) - \xi_0(Tx)$. Then due to the fact that $u \in B$ is arbitrary, we have $f(x) = 0$, for all $x \in X$ and $\xi_0(Tx) = \xi_0(x)$. It means that $Ea = \alpha + Tx$. Since E is idempotent, we get

$$\alpha + Tx = Ea = E^2 a = E(\alpha + Tx) = \alpha + T^2 x,$$

which implies $T^2 x = Tx$. So, T is also idempotent.

Now, we will show that $\|T\| \leq 1$. Let $a = \alpha + x \geq 0$, i.e. $\alpha \geq \|x\|$. Then $E(a) = \alpha + Tx \geq 0$, i.e. $\alpha \geq \|Tx\|$. Hence, $\left\|T\left(\frac{x}{\alpha}\right)\right\| \leq 1$. Since $\left\|\frac{x}{\alpha}\right\| \leq 1$, we have $\|T\| \leq 1$. Therefore T is a projection.

Sufficiency. Let a map $T : X \to X_0$ be a projection. We set $E(\alpha + x) = \alpha + Tx$. We shall check that E is a conditional expectation with respect to B.

Indeed, the condition 1 of Definition 1 is obviously satisfied, since $\mathbf{e} = 1 + 0$ in a generalized spin factor. We will check the condition 2. Let $a = \alpha + x \geq 0$, i.e. $\alpha \geq \|x\|$. Since $\|T\| \leq 1$, we have $\|Tx\| \leq \|x\|$. Therefore $\|Tx\| \leq \alpha$. It means that $E(\alpha + x) = \alpha + Tx \geq 0$.

Now we will check the condition 3. Let $u = \frac{1}{2} + \frac{1}{2}x_0$, and let it correspond to the state $\hat{u} = 1 + \xi_0$ in B^*, $\xi_0 \in X_0^*$. We have

$$ERa = \langle a, \hat{u}\rangle\, u = \langle \alpha + x, 1 + \xi_0\rangle\, u = (\alpha + \xi_0(x))\, u,$$

$$REa = \langle Ea, \hat{u}\rangle\, u = \langle \alpha + Tx, 1 + \xi_0\rangle\, u = (\alpha + \xi_0(Tx))\, u.$$

Since T is a projection from X into X_0, then T^* is a projection from X_0^* into X^*. It means that $\xi_0(Tx) = \xi_0(x)$ for all $x \in X$, $\xi_0 \in X_0^*$. Therefore we have $REa = ERa$.

Preservation of the trace for E follows from the definition of a trace.

□

Remark. A similar theorem for JBW-algebras is proved in [3].

Theorem 3. *Let $A = \mathbb{R} + X$ be a generalized spin factor, $\rho = 1 + \xi$ be a state on A and let $B = \mathbb{R} + X_0$ be its subspace. Then ρ-preserving conditional expectation with respect to B exists if and only if there exists a projection T from X into X_0 with the condition $T^*\xi = \xi$.*

Proof. *Necessity.* Let $E : A \to B$ be the ρ-preserving conditional expectation with respect to B. By Theorem 2, there exists the projection T from X onto X_0 and it has the form $E(\alpha + x) = \alpha + Tx$. Therefore,

$$\rho(Ea) = \langle E(\alpha + x), \rho\rangle = \langle \alpha + Tx, 1 + \xi\rangle = \alpha + \xi(Tx),$$

$$\rho(a) = \langle \alpha + x, 1 + \xi\rangle = \alpha + \xi(x).$$

Since E preserves the state ρ: $\rho(Ea) = \rho(a)$, we have

$$\xi(Tx) = \xi(x), \text{ i.e. } \langle T^*\xi, x\rangle = \langle \xi, x\rangle \ \forall x \in X.$$

Hence, $T^*\xi = \xi$.

Sufficiency follows from Theorem 2. □

COROLLARY 1. *Let $A = \mathbb{R} + X$ be a generalized spin factor, ρ be a state on A, and let $B = R(A) + R'(A)$ for the P-projector R. Then the conditional expectation with respect to B preserves ρ if and only if $\rho = \hat{u}$, where $u = R\mathbf{e}$.*

It is known [6], that if a Banach space admits a bounded projection on its subspace, then it is a Hilbert space. The similar statement takes place for generalized spin factors:

COROLLARY 2. *Let $A = \mathbb{R} + X$ be a generalized spin factor. A conditional expectation with respect to arbitrary subspace $A_1 \subset A$ exists if and only if X is a Hilbert space.*

REFERENCES

[1] E. M. ALFSEN AND F. W. SHULTZ, Non commutative spectral theory for affine function space on convex sets, Memo. Amer. Math. Soc., 172. Providence R.I.: AMS, 1976, 122.

[2] SH. A. AYUPOV, Martingale convergence and strong laws of large in Jordan algebras, Anal. Univ. Craiova. Ser. Mat.Fiz.-Chim., **9**(1981), 29-34.

[3] SH. A. AYUPOV, M. A. BERDIKULOV AND E. YU. AZIZOV, Conditional expectations on spin factors, Uzbek Math. Journal, **3**(1991), 3-9 (in Russian).

[4] M. A. BERDIKULOV, The Banach ball property for the state space of order unit spaces, Malaysian Journal of Mathematical Sciences, **4**(1)(2010), 77-84.

[5] K. A. Jevlakov, A. M. Slinko, I. P. Shestakov and A. I. Shirshov, Rings that are Nearly Associative, Translated from the Russian by Harry F. Smith, Academic Press, New York, 1982.

[6] L. V. Kantorovich and G. P. Akilov, Functional Analysis, Pergamon Press, 1982.

[7] A. S. Kholevo, Probabilistic and Statistic Aspects of Quantum Theory, Elsevier Science & Technology Books, 1982.

[8] M. Takesaki, Conditional expectations in Von Neumann algebras, J. Funct. Anal., **9**(1972), 306-321.

[9] H. Umegaki, Conditional expectation in an operator algebra, II. Tohoku Math. J., **8**(1956), 86-100.

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