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ASYMPTOTIC SPECTRA FOR WEYL GEOMETRIES

BY

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1 Preliminaries on Weyl structures. A Weyl structure $\mathcal{W}$ on a smooth (C^∞) manifold M of dimension $m \geq 2$ consists of a torsion free connection $\tilde{\nabla}$, called the Weyl connection, a conformal class $\mathcal{C} = \{h\}$ of semi-Riemannian metrics (here and in the following we identify metrics in $\mathcal{C}$ which merely differ by a constant positive factor), and a class $\mathcal{T} := \{\hat{\theta}_h \mid h \in \mathcal{C}\}$ of 1-forms satisfying the compatibility conditions

$$\tilde{\nabla} h = 2\hat{\theta} \otimes h := 2\hat{\theta}_h \otimes h \tag{1.1}$$

The compatibility condition described in equation (1.1) is invariant under so-called gauge transformations

$$h \mapsto h^{\#} = \beta h, \quad \hat{\theta} \mapsto \hat{\theta}^{\#} = \hat{\theta} + \tfrac{1}{2} d\, lg\, \beta, \tag{1.2}$$

where $0 < \beta \in C^\infty(M)$. We call $(M, \mathcal{W}) := (M, \tilde{\nabla}, \mathcal{C}, \mathcal{T})$ a Weyl manifold.

It is well known that a Weyl structure $\mathcal{W}$ can be generated from a given pair $\{h, \hat{\theta}\}$ (where h is a semi-Riemannian metric and where $\hat{\theta}$ is a 1-form) in the following way. Let $u, v, w, \ldots$ be vector fields on M, let $\hat{\nabla} := \nabla(h)$ be the Levi-Civita connection of h, and let θ be the vector field defined by

$$h(\theta, v) := \hat{\theta}(v) \tag{1.3}$$

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The construction $\tilde{\nabla}_u v := \hat{\nabla}_u v - \{\hat{\theta}(u)v + \hat{\theta}(v)u - h(u,v)\theta\}$ gives a torsion free connection $\tilde{\nabla}$ which, together with $\{h, \theta\}$, satisfies the compatibility condition given in equation (1.1). We use equation (1.2) to generate the classes $\mathcal{C}$ and $\mathcal{T}$ from the given pair $\{h, \hat{\theta}\}$ and thus get a Weyl structure; see for example [11, p.125] or [4].

For $h \in \mathcal{C}$, Weyl introduced the 2-form $F(v,w) := d\hat{\theta}_h(v,w)$ as a gauge invariant of a given Weyl structure. He called it the length curvature or distance curvature [11, p.124]. We denote by $\tilde{R}$ the curvature tensor of the Weyl connection $\tilde{\nabla}$ where we adopt the sign convention of [8] for the curvature tensors. We have that F and $\tilde{R}$ are related by the equation

$$h(z, \tilde{R}(u,v)z) = F(u,v)h(z,z) \tag{1.4}$$

Weyl defined the directional curvature K by

$$K(u,v)w := \tilde{R}(u,v)w - F(u,v)w \tag{1.5}$$

Relations (1.4) and (1.5) imply the orthogonality relation $h(K(u,v)w, w) = 0$ for any $h \in \mathcal{C}$ and for any vector field w. Moreover, F and K satisfy the following symmetry relations $h(F(u,v)w, z) = h(F(u,v)z, w)$, and skew-symmetry relations $h(K(u,v)w, z) = -h(K(u,v)z, w)$. As a local result the following is known: if the Weyl connection $\tilde{\nabla}$ is metric, then the length curvature vanishes identically. Conversely, if $F = d\hat{\theta}_h = 0$, equation (1.2) implies that the cohomology class $[\hat{\theta}_h(\mathcal{W})] \in H^1(M)$ of the closed form $\hat{\theta}_h(\mathcal{W})$ does not depend on the choice of a metric in $\mathcal{W}$. Higa [7] has shown that the distance curvature F and the cohomology class $[\hat{\theta}_h(\mathcal{W})]$ are the obstructions to having $\tilde{\nabla}$ be a metric connection:

1.6 Proposition (Higa). *Let $\tilde{\nabla}$ be the Weyl connection of a Weyl structure $\mathcal{W}$ on M. Then the following two conditions are equivalent:*

a) *We have $F(\mathcal{W}) = 0$ and $[\hat{\theta}_h(\mathcal{W})] = 0$.*

b) *There is a Riemannian metric h in $\mathcal{C}(\mathcal{W})$ such that $\tilde{\nabla}h = 0$.*

2 Weyl structures and Codazzi structures. In this section, we shall investigate the relations between Weyl structures and Codazzi structures on M. We introduce the following notational conventions. Recall that two torsion free connections ∇ and $\nabla^{\#}$ are projectively equivalent if and only if there exists a 1-form τ such that

$$\nabla^{\#}_u v - \nabla_u v = \tau(u)v + \tau(v)u \tag{2.1}$$

or equivalently if the unparametrized geodesics of ∇ and $\nabla^{\#}$ coincide, see [3, §32] for details. The projective class $\mathcal{P}(\nabla)$ is the set of torsion free connections which are projectively equivalent to ∇.

There is an interesting relation between a projective class $\mathcal{P}$ of torsion free connections and a conformal class $\mathcal{C}$ of semi-Riemann metrics, defined via Codazzi equations, see [9, §3; 1, §6; 10, §1.6]. We call $\mathcal{P}$ and $\mathcal{C}$ Codazzi compatible if there exists a pair $\{\nabla, h\}$ with $\nabla \in \mathcal{P}$ and with $h \in \mathcal{C}$ such that both satisfy Codazzi equations

$$(\nabla_u h)(v, w) = (\nabla_v h)(u, w); \tag{2.2}$$

we shall call $\{\nabla, h\}$ a Codazzi pair. This relation is preserved under a so called Codazzi transformation

$$h \mapsto h^{\#} = \beta h, \quad \nabla \mapsto \nabla^{\#}, \tag{2.3}$$

where $\nabla^{\#}$ satisfies equation (2.1) for $\tau = d\log(\beta)$, with $\beta \in C^{\infty}(M)$ and $\beta > 0$. A Codazzi structure $\{\mathcal{P}, \mathcal{C}\}$ consists of Codazzi compatible classes $\mathcal{P}$ and $\mathcal{C}$.

2.4 Remark. The class $\mathcal{P}$ need not to be Ricci-symmetric, in this case we don't have a pairing in general, but only an injective mapping $\mathcal{C} \longrightarrow \mathcal{P}$.

For a given Weyl structure $\mathcal{W} = \{\tilde{\nabla}, \mathcal{C}, \mathcal{T}\}$, we consider a fixed pair $\{h, \hat{\theta}\} \in \mathcal{C} \times \mathcal{T}$ satisfying equation (1.1). We use equation (1.3) to define the symmetric (1.2)-tensor field $C = C(h, \hat{\theta})$ and the associated totally symmetric cubic form $\hat{C} = \hat{C}(h, \hat{\theta})$ by

$$C(v, w) := \hat{\theta}(v)w + \hat{\theta}(w)v + h(v, w)\theta$$
$$\hat{C}(u, v, w) := h(C(u, v), w) = \hat{\theta}(u)h(v, w) + \hat{\theta}(v)h(w, u) + \hat{\theta}(w)h(u, v).$$

The connections $\nabla^* := \hat{\nabla} + C$ and $\nabla := \hat{\nabla} - C$ are torsion free and depend on the given pair $\{h, \hat{\theta}\}$.

2.5 Proposition. *The class $\mathcal{P}^* := \{\nabla^* = \hat{\nabla} + C \mid \{h, \hat{\theta}\} \in \mathcal{C} \times \mathcal{T}$ satisfying equation (1.1)$\}$ is a projective class generated by the gauge transformations described in equation (1.2) of the Weyl structure. Furthermore, we have $\mathcal{P}^* = \mathcal{P}(\tilde{\nabla})$.*

The following two propositions deal with the relations between Weyl structures and Codazzi structures on a manifold M. For that purpose, we sketch how to construct a Weyl structure from a given Codazzi structure $\{\mathcal{P}, \mathcal{C}\}$. Consider a fixed pair $\{\nabla, h\} \in \mathcal{P} \times \mathcal{C}$ and the Levi-Civita connection $\hat{\nabla}$ of h. Define the symmetric (1.2)-tensor $c := \nabla - \hat{\nabla}$ and a 1-form $\hat{\theta}$ by its trace $(m+2)\hat{\theta} := \mathrm{Tr}(c)$. According to section 1, the pair $\{h, \hat{\theta}\}$ generates a Weyl structure, and then a Codazzi transformation of $\{\mathcal{P}, \mathcal{C}\}$ induces a gauge transformation in $\mathcal{W}$.

2.6 Proposition. *A Weyl structure $\mathcal{W} = \{\tilde{\nabla}, \mathcal{C}, \mathcal{T}\}$ on M induces a Codazzi structure $\{\mathcal{P}^*, \mathcal{C}\}$, where $\mathcal{P}^* = \mathcal{P}(\tilde{\nabla})$ is as above. We call it the canonical Codazzi structure of $\mathcal{W}$. A gauge transformation as described in equation (1.2) induces a Codazzi transformation as defined in equation (2.3).*

2.7 Proposition. *Let $\{\mathcal{P}, \mathcal{C}\}$ and $\{\bar{\mathcal{P}}, \mathcal{C}\}$ be two Codazzi structures. Then we have*

a) *There is a unique, symmetric (1.2)-tensor field γ such that for any two Codazzi pairs $\{\nabla, h\}$ and $\{\bar{\nabla}, h\}$ we have $\gamma(v, w) = \nabla_v w - \bar{\nabla}_v w$.*

b) *The Codazzi structures $\{\mathcal{P}, \mathcal{C}\}$ and $\{\bar{\mathcal{P}}, \mathcal{C}\}$ define the same Weyl structure if and only if γ is polar. This means that $\mathrm{Tr}\,\{u \to \gamma(u, \cdot)\} = 0$.*

3 Second order differential operators on Weyl manifolds

3.1 Definition. *Let ∇ be an affine connection on M.*

a) *Let $f \in C^\infty(M)$ and let u and v be tangent vector fields. The* **Hessian** *$H_\nabla f$ is the (0.2) tensor field given by $(\mathrm{Hess}_\nabla f)(u, v) := u(v(f)) - df(\nabla_u v)$. We have that the second order differential operator Hess_∇ is symmetric, i.e. $(\mathrm{Hess}_\nabla f)(u, v) = (\mathrm{Hess}_\nabla f)(v, u)$ if and only if ∇ is torsion free.*

b) *We say that a second order partial differential operator D on $C^\infty(M)$ is of* **Laplace type** *if its leading symbol is positive definite and thus defines a Riemannian metric h on M. This means in any system of local coordinates that we may express $D = -(h^{ij}\partial_i\partial_j + A^i\partial_i + B)$ where we adopt the Einstein convention and sum over repeated indices $1 \le i, j \le m = \dim M$.*

c) *We follow the sign-convention of [8] for the Ricci tensor $\mathrm{Ric}(\nabla)$ of ∇. Let $\rho(\nabla) := \frac{1}{2}(\mathrm{Ric}(\nabla)(u, v) + \mathrm{Ric}(\nabla)(v, u))$ be the symmetrization of $\mathrm{Ric}(\nabla)$. We say that ∇ is Ricci-symmetric if $\mathrm{Ric}(\nabla)$ is symmetric on M. We say ω is a parallel volume form if and only if $\nabla\omega = 0$; ∇ is Ricci-symmetric if and only if ∇ locally admits a parallel volume form ω.*

d) Let $\nabla \to \nabla^{\#}$ be a projective transformation as in equation (2.1). We use [3; p.159] to relate the Ricci tensors:

$$-(\mathrm{Ric}^{\#}_{ij} - \mathrm{Ric}_{ij}) = m(\nabla_j \tau_i - \tau_i \tau_j) - (\nabla_i \tau_j - \tau_j \tau_i);$$

note that the sign conventions for the Ricci tensor differ in [3] from those used in our paper.

3.2 Lemma. *Consider a projective transformation $\nabla \mapsto \nabla^{\#}$ as in equation (2.1).*

a) We have that $\rho(\nabla^{\#})_{ij} = \rho(\nabla)_{ij} - (m-1)\{(\nabla_j \tau_i + \nabla_i \tau_j)/2 - \tau_i \tau_j\}$.

b) If $\tau = d\log(\beta)$ for some function $0 < \beta \in C^\infty(M)$, then

$$\rho(\nabla^{\#}) = \rho(\nabla) + (m-1)\beta H_\nabla(\beta^{-1}).$$

3.3 Differential operators of Laplace type. Let ∇ be a torsion free connection and let h be semi-Riemannian metric. Let $f \in C^\infty(M)$. We define the second order operators $H(\nabla)$ and $D(h,\nabla)$:

$$H(\nabla)f := (\mathrm{Hess}_\nabla + (m-1)^{-1}\rho_\nabla)f \mathrm{and} Df := D(h,\nabla)f = -\mathrm{Tr}_h(H(\nabla)f).$$

Clearly the operator D is of Laplace type if and only if h is a Riemann metric. We refer to [9] for the proof of the following Lemma; there the proof is given for Ricci-symmetric connections.

3.4 Transformation Lemma. *Let $\tau = d\log(\beta)$. We use equation (2.1) to define a projective change $\nabla \mapsto \nabla^{\#}$. Then $H(\nabla^{\#})(\beta f) = \beta H(\nabla)(f)$. If $h^{\#} = \beta h$, then $D(h^{\#},\nabla^{\#})(\beta f) = D(h,\nabla)(f)$.*

3.5 Heat equation asymptotics. Let M be a closed manifold and let h be Riemannian. Let $\{\lambda_\nu\}$ be the eigenvalues of $D = D(h,\nabla)$ where each eigenvalue is repeated according to its multiplicity and where $\lambda_1 \le \lambda_2 \le \ldots$. If $t > 0$, then e^{-tD} is trace class on $L^2(M)$ and

$$\mathrm{Tr}_{L^2}(e^{-tD}) = \Sigma_\nu e^{-t\lambda_\nu};$$

this series converges uniformly on compact subsets of $(0,\infty)$ since the eigenvalues satisfy the growth estimate $\lambda_\nu \sim \nu^{2/m}$ for ν large. As $t \downarrow 0$, there is an asymptotic series of the form:

$$\mathrm{Tr}_{L^2}(e^{-tD}) \sim \Sigma_{n \ge 0} a_n(D) t^{(n-m)/2}.$$

The coefficients $a_n(D)$ are spectral invariants which are locally computable in the following sense. Let $d\nu = d\nu_h$ be the Riemannian measure on M. There exist local invariants $a_n(x, D)$ of the geometry of $\{\nabla, h\}$ defined for $x \in M$ so that $a_n(D) = \int_M a_n(x, D) d\nu_h$.

3.6 Application to Weyl manifolds. Let $\mathcal{W}$ be a Weyl structure and let $\{\mathcal{P}^*, \mathcal{C}\}$ be its canonical Codazzi structure. Note that in general the projective class is not Ricci-symmetric; i.e. $\nabla^* \in \mathcal{P}^*$ may have a non-symmetric Ricci tensor. We use $\{\mathcal{P}^*, \mathcal{C}\}$ to define the class of operators

$$\{D(h, \nabla^*) \mid \{\nabla^*, h\} \in \mathcal{P}^* \times \mathcal{C}\}.$$

A gauge transformation as described in equation (1.2) of $\mathcal{W}$ induces a Codazzi transformation as defined in equation (2.3); the induced transformation of operators $D(h, \nabla^*) \mapsto D(h^\#, \nabla^{*\#})$ is described by the transformation Lemma 3.4.

In [1] we studied such operators and their invariants under Codazzi transformations in case of Ricci-symmetric projective classes. The transformation Lemma 3.4 is the key for an extension to general projective classes. This procedure leads to new gauge invariants of Weyl geometries. The following lemma provides the basic formulas that are necessary to calculate $a_n(D)$; we refer to see [5,6] for further details.

3.7 Lemma. *Let h be a Riemannian metric. Let $D = -(h^{\mu\nu}\partial_\mu\partial_\nu + A^\nu\partial_\nu + B)$ be an operator of Laplace type on $C^\infty(M)$. Let $\Gamma_{h,\mu\sigma}{}^\nu$, τ_h, $\|\rho_h\|$, and $\|R_h\|$ be the Christoffel symbols, the scalar curvature, the norm of the Ricci curvature, and norm of full curvature tensor for the Levi-Civita connection defined by h.*

a) *There exists a unique connection ∇_D on $C^\infty(M)$ and a unique function $E_D \in C^\infty(M)$ so that $D = -(\mathrm{Tr}(\nabla_D^2) + E_D)$. If ω_D is the connection 1-form of ∇_D, then we have that $\omega_{D,\delta} = \frac{1}{2}h_{\nu\delta}(A^\nu + h^{\mu\sigma}\Gamma_{h,\mu\sigma}{}^\nu)$ and that $E_D = B - h^{\nu\mu}(\partial_\mu\omega_{D,\nu} + \omega_{D,\nu}\omega_{D,\mu} - \omega_{D,\sigma}\Gamma_{h,\nu\mu}{}^\sigma)$. Let $\Omega_{D,ij}$ be the curvature of the connection defined by D.*

b) $a_0(x, D) = (4\pi)^{-m/2}$.

c) $a_2(x, D) = 6^{-1}(4\pi)^{-m/2}\{\tau_h + 6E_D\}$.

d) $a_4(x, D) = 360^{-1}(4\pi)^{-m/2}\{60(E_D)_{;kk} + 60\tau_h E_D + 180(E_D)^2$
$+30\Omega_{D,ij}\Omega_{D,ij} + 12(\tau_h)_{;kk} + 5(\tau_h)^2 - 2\|\rho_h\|^2 + 2\|R_h\|^2\}$.

We apply these formulas to the setting at hand:

3.8 Lemma. *Let* $D = D(h, \nabla^*) = -\mathrm{tr}_h(\mathrm{Hess}_{\nabla^*} + (m-1)^{-1}\mathrm{Ric}^*)$. *Then*

a) $A^k = -h^{ij}\Gamma^{*k}_{ij}$, $B = (m-1)^{-1}h^{ij}R^*_{ij}$, *and* $\omega_{D,\delta} = -2^{-1}(m+2)\theta_\delta$.

b) $E_D = 4^{-1}\{(m+2)(m-1)^{-1}\mathrm{Tr}_h\tilde{\mathrm{Ric}} - m(m-2)\kappa\}$.

c) $\Omega_{D,ij} = 2^{-1}(m+2)\{\nabla(h)_j\theta_i - \nabla(h)_i\theta_j\}$.

We combine Lemma 3.7 and Lemma 3.8 to prove the following Theorem. Note that $\tau_h = m(m-1)\kappa$.

3.9 Theorem. *Let* $D = D(h, \nabla^*)$ *on* $C^\infty(M)$. *If* M *is closed, then*

a) $a_0(x, D) = (4\pi)^{-m/2}$.

b) $a_2(x, D) = 12^{-1}(4\pi)^{-m/2}\{m(4-m)\kappa + 3(m+2)(m-1)^{-1}\mathrm{Tr}_h\tilde{\mathrm{Ric}}\}$.

c) $a_2(D) = 12^{-1}(4\pi)^{-m/2}\int\{m(4-m)\kappa + 3(m+2)(m-1)^{-1}\mathrm{Tr}_h\tilde{\mathrm{Ric}}\}d\nu_h$.

d) $a_4(D) = 360^{-1}(4\pi)^{-m/2}\int\{5(m(m-1)\kappa + 6E_D)^2 + 30\|\Omega_D\|^2$
$+2(\|R_h\|^2 - \|\rho_h\|^2)\}d\nu_h$.

These results lead to the following theorem

3.10 Theorem. *Let* M *be a closed Weyl manifold of dimension* m .

a) $a_m(D)$ *is a global Weyl invariant. Note that* $a_m(D) = 0$ *if* m *is odd.*

b) *If* $m = 4$, *then the following expressions are global Weyl invariants:* $\int((m(m-1)\kappa + 6E_D)^2 d\nu_h$, $\int\|\Omega_D\|^2 d\nu_h$, $\int\{\|R_h\|^2 - \|\rho_h\|^2\}d\nu_h$, $a_4(D)$.

We are going to study relations between such invariants in [2].

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