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RELATIVISTIC SYSTEMS WITH A CONSERVED PROBABILITY CURRENT

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Abstract

Relativistic wave equations describing localizable relativistic systems have been investigated by many authors /see e.g. [4 - 7] /, however no sufficiently general and systematic their investigation is available. Usually one considers only linear Hamiltonians without giving any proof of imposobility of other forms /see e.g. Gelfand etal. [1]/. It is our aim to start such a systematic treatment based on the assumption that the system is localizable and admits a conserved probability current. A most general form of the Hamiltonian corresponding to two partially overlapping sets of assumptions is given. The finite - dimensional representation of the Hamiltonian relativistic algebra /HRA/ are classified, and a particular infinite - dimensional representation is described. Finally we give some comments concerning the relation with the Lagrangean approach.

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1. Restriction on Hamiltonians coming from the probability current conservation

Let us consider a localizable quantum system /either Galilean or relativistic one/. In particular, for each instant of time x^0, there is a spectral measure E_{x^0} on $\mathbb{R}^3$ that defines position operators $\underline{X}(x^0)$:

$$\underline{X}(x^0) = \int \underline{x} \, dE_{x^0}(\underline{x}) \qquad 1.1$$

We assume that we deal with a reversible system, so that for every pair $(x^0, x^{0'})$ there is a unitary operator $U(x^0, x^{0'})$ such that

$$\underline{X}(x^0) = U(x^0, x^{0'}) \underline{X}(x^{0'}) U(x^0, x^{0'})^* \qquad 1.2$$

and $U(x^0, x^{0'})$ satisfy:

i/ $(x^0, x^{0'}) \longmapsto U(x^0, x^{0'})$ is strongly continuous,

ii/ $U(x^0, x^{0'})^* = U(x^{0'}, x^0)$

iii/ $U(x^0, x^{0'}) U(x^{0'}, x^{0''}) = U(x^0, x^{0''})$

It follows from i/ - iii/ that the Hamiltonian $H(x^0)$:

$$H(x^0) = -i \frac{\partial}{\partial x^0} U(x^0, x^{0'}) \Big|_{x^{0'} = x^0} \qquad 1.3$$

is selfadjoint. For the velocity operator $\dot{\underline{X}}(x^0)$ we then get:

$$\dot{\underline{X}}(x^0) = \frac{d}{dx^0} \underline{X}(x^0) = i \left[H(x^0), \underline{X}(x^0) \right] \qquad 1.4$$

It is also convenient to introduce time operator $\mathcal{X}^0(x^0) = x^0 1$ so that $\dot{\mathcal{X}}^0(x^0) = 1$.

Now, instead of spectral measures E_{x^0} it is convenient to introduce an operator valued distribution $\varrho(x^0, \underline{x})$ such that for a function $f: \mathbb{R}^3 \to \mathbb{R}$

$$\int \varrho(x^0, \underline{x}) f(\underline{x})\, d^3x = \int f(\underline{x})\, dE_{x^0}(\underline{x}) = f(\underline{\mathcal{X}}(x^0)) = \hat{f}_{x^0} \qquad 1.5$$

In other words, $\varrho(x^0, \cdot)$ may be considered as an operator-valued Radon-Nikodym derivative of E_{x^0} with respect to the Lebesque measure on $\mathbb{R}^3$. In order to make this definition mathematically admissible, we must assume that E_{x^0} is absolutely continuous with respect to the Lebesque measure. In fact, in the course of our argument we shall need a stronger assumption:

Assumption 1 For every x^0, the spectral measure E_{x^0} and the Lebesque measure are mutually continuous, i.e: $E_{x^0}(\Delta) = 0$ if and only if $\Delta \subset \mathbb{R}^3$ is of the Lebesque measure zero. A.1

Physically, it means that there are no forbidden regions for the system and that we are not dealing with too singular external fields /like surface δ - potentials etc./.

Owing to the assumed unitary evolution, it is enough to satisfy the above assumption at $x^0 = 0$.

Let us proceede to define a probability current $j^\mu(x)$. In classical hydrodynamics we find an expression

$$j^0(x) = \varrho(x) \quad , \quad \underline{j}(x) = \underline{v}(x)\varrho(x),$$

where ρ is a density and v - velocity of particles at x. In quantum theory $v(x)$ is of a rather obscure meaning, so we replace $v(x)$ by the velocity operator $\dot{\underline{X}}(x^0)$. It gives us information concerning velocity distribution in space by its expectation values $\langle \psi, \dot{\underline{X}}(x^0)\psi \rangle$ in localized states ψ. However, the operator $\dot{\underline{X}}(x^0)\rho(x)$ is, in general, not Hermitian. Therefore we replace it by its Hermitian part, and define

$$j^{\mu}(x) = \frac{1}{2}\left(\dot{X}^{\mu}(x^0)\rho(x) + \rho(x)\dot{X}^{\mu}(x^0) \right) \qquad 1.6$$

It is our second assumption that the current j^{μ} is conserved:

<u>Assumption 2</u> The current j^{μ} defined in 1.6 is conserved:

$$\partial_{\mu} j^{\mu}(x) = 0 \qquad A.2$$

<u>Theorem 1.1</u> With the above assumptions the Hilbert space $\mathcal{H}$ of the system can be identified with $L^2(R^3, \mathcal{K}, d^3x)$, $\mathcal{K}$ being a Hilbert space, in such a way that $X^i(0)$ become multiplication operators

$$(X^i(0)\psi)(\underline{x}) = x^i \psi(\underline{x}) \qquad 1.7$$

and $H(0)$ is of the form:

$$H(0) = \frac{1}{4}\left(A^{ij}P_iP_j + P_iA^{ij}P_j + P_iP_jA^{ij} \right) + \frac{1}{2}\left(A^iP_i + P_iA^i \right) + A$$

where

$$(P_i\psi)(\underline{x}) = -i\frac{\partial}{\partial x^i}\psi(\underline{x}) \qquad 1.7a$$

and A^{ij}, A^i, A are multiplications by functions of $\underline{x}$, which values are Hermitian operators in $\mathcal{H}$.

Proof /During the proof we dismiss everywhere the argument $x^0 = 0$. /By 1.1-5 and A2 we have:

$$i[H, \varrho(\underline{x})] + \partial_i j^i(\underline{x}) = 0$$

and smearing out with a test function f :

$$i[H, \hat{f}] = \int j^i(\underline{x})\partial_i f(\underline{x})d^3x$$

Then 1.6 leads to

$$2[H, \hat{f}] = [H, x^i]\widehat{\partial_i f} + \widehat{\partial_i f}[H, x^i] \qquad 1.8$$

The condition 1.8 restrict a possible form of H. In order to find a most general form of H compatible with 1.7, it is convenient to analyse 1.8 for a particular f. With $f(\underline{x}) = x^i x^j x^k$ we get from 1.8 :

$$[x^k, [\dot{x}^j, x^i]] = 0 \qquad 1.9$$

Introducing

$$A^{ij} = i[\dot{x}^i, x^j]$$

we get, from the Jacobi identity,

$$A^{ij} = A^{ji} = A^{ij*}$$

and 1.9 gives

$$[X^k, A^{ij}] = 0 \qquad 1.10$$

Now, we make use of the Assumption 1 and deduce [2] that there is a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ can be identified with $L^2(\mathbb{R}^3, \mathcal{K}, d^3x)$ in such a way, that E becomes a canonical spectral measure /i.e. $E(\Delta)$ becomes multiplication by a characteristic function of the set $\Delta \subset \mathbb{R}^3$ / or, in other words, that 1.7 holds. With P_i : defined as in 1.7a let

$$V^i = \frac{1}{2}(A^{ij} P_j + P_j A^{ij})$$

It follows that $V^i = V^{i*}$ and $A^{ij} = i[V^i, X^j]$. Therefore

$$A^i = \dot{X}^i - V^i$$

satisfy

$$[X^k, A^i] = 0 \qquad , \qquad A^i = A^{i*}$$

Let now

$$H_0 = \frac{1}{2}(P_i \dot{X}^i + \dot{X}^i P_i)$$

Then $[H_0, X^k] = [H, X^k]$ and so, with $A = H - H_0$ we find that

$$[A, X^k] = 0 \qquad , \qquad A = A^*$$

Finally, since A^{ij}, A^i and A are Hermitian and commute with X^k, it follows that there exist functions $A^{ij}(x)$, $A^i(x)$, $A(x)$ with values in Hermitian operators in $\mathcal{K}$ such that

$$(A^{ij}\psi)(\underline{x}) = A^{ij}(x)\psi(\underline{x}) \qquad \text{etc.} \quad \Box$$

Remark 1 It is easy to see that with H as in the above Theorem, 1.8 is satisfied for every f / and not only for a particular choice of f we have used/.

Remark 2. It should be observed that the functions A^{ij}, H^i, H are unique only up to a gauge transformation. In fact, an identification of h with $L^2(R^3, \mathcal{K}, d^3x)$ is not unique, and different identifications lead to P_i -s that differ by a Λ_i :

$$\Lambda_i = W(\underline{x})\,\partial_i W(\underline{x})^*$$

where $W(x)$ is a function on R^3, with values in the unitary group of $\mathcal{K}$.

Remark 3. We notice that in a relativistic case a quadratic term in H should vanish. In fact, if the theory is to be relativistic, the velocity operators should be bounded. By applying unitary transformation $U(\underline{v}) = \exp\{-i\underline{X}\,\underline{v}\}$ to $\dot{X}^i$ we find that $\dot{X}^i$ is unitarily equivalent to $\dot{X}^i + A^{ij}v_j$. It follows, that if $A^{ij}(\underline{x}) \neq 0$, the numerical range of $\{\langle\psi, \dot{X}^i\psi\rangle : \|\psi\| = 1\}$ coincides with R, so cannot be bounded.

2. Relativistic localizable systems

In this section we consider a localizable system with a Poincare symmetry. Let h be a Hilbert space of the system, and let $U(a, \Lambda)$ be a unitary /in general projective/ representation of the Poincare group in h. Selfadjoint

$J_{kl} = -\varepsilon_{kli} M_i$

~~$M_i =$~~

~~$J_{0i} = N_i$~~

$N_i = J_{i0}$

generators P_μ, $\underline{M}$, $\underline{N}$ of the representation satisfy the commutation relations

i/ $[M_i, P_j] = i\varepsilon_{ijk} P_k$

ii/ $[M_i, P_0] = 0$

iii/ $[M_i, N_j] = i\varepsilon_{ijk} N_k$

iv/ $[M_i, M_j] = i\varepsilon_{ijk} M_k$

v/ $[N_i, P_j] = i\delta_{ij} P_0$

vi/ $[N_i, P_0] = i P_i$

vii/ $[N_i, N_k] = -i\varepsilon_{ijk} M_k$

viii/ $[P_\mu, P_\nu] = 0$

/The metric we use is $g_{\mu\nu}$ = diag $(-,+,+,+)$ /

We assume that the system is localizable i.e. there are selfadjoint operators X_i /corresponding to a localization of a distinguished point of the system on $x^0 = 0$ hyperplane/, satisfying

$$\text{ix/}\quad [X_i, X_j] = 0$$

$$\text{x/}\quad [X_i, M_j] = i\varepsilon_{ijk} X_k$$

$$\text{xi/}\quad [P_i, X_j] = i\delta_{ij}$$

With $\varrho(\underline{x})$ defined as before we make the following assumption:

<u>Assumption 2.1.</u> There is a four-vector current j^μ such that

$$\text{/a/}\quad j^0(x)\big|_{x^0=0} = \varrho(\underline{x})$$

$$\text{/aa/}\quad \partial_\mu j^\mu(x) = 0$$

$$\text{/aaa/}\quad U(a,\Lambda)\, j^\mu(x)\, U(a,\Lambda)^* = \Lambda^{-1\mu}{}_\nu\, j^\nu(\Lambda x + a)$$

Under these assumptions we proceede to find a most general form of the Hamiltonian $H = P_0$.

First of all, let us observe that analogously as in

Sec.1, can be identified with $L^2(\mathbb{R}^3, \mathcal{H}, d^3x)$ in such a way that

$$(P_i\,\psi)(\underline{p}) = p_i\,\psi(\underline{p})$$

$$(X_i\,\psi)(\underline{p}) = i\frac{\partial}{\partial p_i}\psi(\underline{p})$$

Moreover, if a Hermitian operator A commutes with P_i - s then

$$(A\,\psi)(\underline{p}) = A(\underline{p})\,\psi(\underline{p})$$

where $A(\underline{p})$ is Hermitian in $\mathcal{H}$ for /almost/ all $\underline{p}$

In particular, if A commutes with the X_i - s also, then $A(\underline{p}) \equiv A$ is a constant Hermitian operator in $\mathcal{H}$.
To demonstrate this kind of arguments let us define $\underline{m}$ by

$$\underline{m} = \underline{M} - \underline{X} \times \underline{P}$$

It follows then from /i/, /vii - xi/ that $[\underline{m}, \underline{X}] = [\underline{m}, \underline{P}] = 0$ and so, m_i can be considered as Hermitian operatoes in $\mathcal{H}$.
By /ix/, we get

$$[m_i, m_j] = i\,\varepsilon_{ijk}\,m_k \tag{2.1}$$

To obtain further restrictions let us write the relevant equations /a - aaa/ in an infinitesimal form:

$$[H, \varrho(\underline{x})] - [P_i, j^i(\underline{x})] = 0 \quad . \tag{2.2}$$

$$-[N_i, \varrho(\underline{x})] = i\,j^i(\underline{x}) + [H, \varrho(\underline{x})]x_i \tag{2.3}$$

$$-[N_k , j^i(x)] = i\delta_{ik}\,\rho(\underline{x}) + [H, j^i(\underline{x})]x_k \qquad 2.4$$

By substituting $j^i(\underline{x})$ from 2.3 into 2.2 and smearing out with a function f we get

$$[H,\hat{f}] - i[P_i,[N_i,\hat{f}]] - i[P_i,[H,\hat{f}X_i]] = 0 \qquad 2.5$$

Let us define $\underline{n}$ by

$$\underline{N} = -\tfrac{1}{2}(\underline{X}H + H\underline{X}) + \underline{n} \qquad 2.6$$

It follows than from /v/ that $[n_i , P_j] = 0$ so that $n_i = n_i(\underline{p})$ and substituting 2.6 into 2.5 we have

$$[H,\hat{f}] - \tfrac{1}{2}\{[H,X_i]\widehat{\partial_i f} + \widehat{\partial_i f}[H,X_i]\} = [n_i,\widehat{\partial_i f}] \qquad 2.7$$

A particular choise $f(x) = x_i x_j$ leads to

$$[n_i , X_j] + [n_j , X_i] = 0$$

and so

$$\frac{\partial}{\partial p_i} n_j(\underline{p}) + \frac{\partial}{\partial p_j} n_i(\underline{p}) = 0$$

A general solution of these equations has the form:

$$n_i = a_{ij}P_j + a_i \qquad 2.8$$

where a_{ij} and a_i are Hermitian operators in $\mathcal{H}$, with $a_{ij} = -a_{ji}$. As the next step we take $f(\underline{x}) = x_i x_j x_k$. The left-hand side of 2.7 becomes then $\frac{1}{2}[X_k[X_j,[X_i,H]]]$

and the right-hand side vanishes. Since, by /Vii/, H is a function of the momenta, it follows that a dependence of H on P_i - s is at most quadratic:

$$H = \frac{1}{2} A_{ij} P_i P_j + A_i P_i + A \qquad 2.9$$

where A_{ij}, A_i and A are Hermitian in $\mathcal{K}$, and $A_{ij} = A_{ji}$. Its is easy to see that with $\underline{N}$ as in 2.6 and 2.8, and with H given by 2.9, the relation 2.7 is satisfied automatically for all f. As a next step we shall demonstrate that, owing to 2.4, the coefficients A_{ij} and a_{ij} must vanish. To show this, let us notice that by 2.3, 2.6 and 2.8

$$j_k(\underline{x}) = \frac{1}{2}(\dot{X}_k \varrho(\underline{x}) + \varrho(\underline{x}) \dot{X}_k) + i a_{kj} [P_j, \varrho(\underline{x})]$$

or, after smearing out with a test function f:

$$j_k(f) = \frac{1}{2}(\dot{X}_i \hat{f} + \hat{f} \dot{X}_i) - a_{kj} \widehat{\partial_j f} \qquad 2.10$$

Now, since $\dot{X}_k$ is linear in the momenta, we have

$$\frac{1}{2}(j_k(f) X_i + X_i j_k(f)) + a_{ki} \hat{f} = j_k(f x_i) \qquad 2.11$$

By substituting 2.10 and 2.11 into 2.4 we obtain:

$$-\frac{i}{2}[X_i, \{H, j_k(f)\}] + i[H, a_{ki}\hat{f}] + i[N_i, j_k(f)] = \qquad 2.12$$
$$= \delta_{ik} \hat{f}.$$

For a particular choice $f \equiv 1$, we get $j_k(f) = A_{kj} P_j + A_k = \dot{X}_k$ and 2.12 reads

$$-\frac{1}{2}[x_i,[x_k,H^2]] + i[H,a_{ki}] + i[n_i, A_{kj}P_j + A_k] = \delta_{ik} \qquad 2.13$$

The terms: quadratic, linear, and constant in the momenta, must vanish separately. For a quadratic term one gets

$$\frac{1}{4}\{A_{ij},A_{kl}\} + \frac{1}{4}\{A_{il},A_{kj}\} + \frac{1}{4}\{A_{ik},A_{jl}\} +$$

$$+\frac{i}{2}[a_{ij},A_{kl}] + \frac{i}{2}[a_{il},A_{kj}] + \frac{i}{2}[a_{ik},A_{jl}] = 0$$

With $k=i$, $l=j$, and taking into account that $A_{ij}=A_{ji}$ and $a_{ij}=-a_{ji}$, we deduce that:

$$\frac{1}{2}A_{ij}^2 + \frac{1}{2}\{A_{ii},A_{jj}\} = 0$$

and so, $A_{ij}=0$. Now, since $H = A_jP_j + A$ so $j_k(f) = A_k\hat{f} + a_{kj}\partial_j f$ and substitution of these expressions into 2.12 leads to

$$\frac{1}{2}\{A_i, j_k(f)\} + i[H,a_{ki}]\hat{f} + i[n_i, j_k(f)] = \delta_{ik}\hat{f} \qquad 2.14$$

The /symmetric/ coefficient of $\delta_{ik}f$ is proportional to $a_{kl}a_{ij} + a_{kj}a_{il}$ and so,

$$a_{kl}a_{ij} + a_{kj}a_{il} = 0$$

or, with $i=k$, $j=l$, $a_{kl}^2=0$. Therefore $n_i = a_i = const$, and 2.14 reduces to

$$\frac{1}{2}\{A_i,A_k\}\hat{f} + i[n_i,A_k]\hat{f} = \delta_{ik}\hat{f}$$

or

$$[n_i,A_k] = \frac{i}{2}\{A_i,A_k\} - i\delta_{ik} \qquad 2.15$$

Now, with $\underline{H} = A_i P_i + A$ and $\underline{N} = -\frac{1}{2}\{\underline{X}, \underline{H}\} + n$ the relations /iii/, /ii/, /vi/ and /vii/ lead to

$$
\begin{aligned}
&[m_i, A] = 0 \\
&[m_i, A_j] = i\,\varepsilon_{ijk} A_k \\
&[m_i, n_j] = i\,\varepsilon_{ijk} n_k \\
&[n_i, A] = \tfrac{i}{2}\{A_i, A\} \\
&[n_i, n_j] = -i\,\varepsilon_{ijk} m_k + \tfrac{i}{4}[A_i, A_j]
\end{aligned}
\qquad 2.16
$$

The above considerations can be summarized in the following:

<u>Theorem 2.1.</u> Let $\underline{m}, \underline{n}, \underline{A}, A$ be Hermitian operators in a Hilbert space $\mathcal{K}$ satisfying 2.1 , 2.15 and 2.16 , let $P_i = p_i$, $X_i = i\frac{\partial}{\partial p_i}$, $H = A_i P_i + A$ and $\underline{M} = \underline{X} \times \underline{P} + m$, $\underline{N} = -\frac{1}{2}\{\underline{X}, H\} + \underline{n}$. Then /i - xi/ and the Assumption 2.1 are satisfied. Conversely, every solution of the relations satisfying the Assumption 2.1 is of this form . $\square$

The vectors in $\mathcal{K}$ correspond to internal degrees of freedom of the system. According to the above theorem, relativistic systems with a conserved probability current are in 1 - 1 correspondence with representations of the commutation relations 2.1 , 2.15 , 2.16 by Hermitian operators acting on $\mathcal{K}$. This set of commutation relations is not a Lie algebra, since we have anticommutators as well as commutators, nevertheless we shall call it the Hamiltonian Relativistic Algebra /HRA/. If $\mathcal{K}$ is infinite - dimensional one has, strictly speaking, to deal with domains of unbounded operators etc. In the finite - dimensional case the problem of finding representations of HRA is pure algebraic.

3. The case of finite - dimensional $\mathcal{K}$.

Here we briefly show how to get the most general representation of HRA by Hermitian operators acting on a finite-dimensional space $\mathcal{K}$. First of all, observe, that every of the operators $\underline{m}$, $\underline{n}$, $\underline{A}$, A has a complet orthonormal set of eigenvectors. If $A_i \psi = \lambda_i \psi$ then $0 = \langle \psi, [n_i, A_i]\psi \rangle = i\langle \psi, (A_i^2 - 1)\psi \rangle = i(\lambda_i^2 - 1)\|\psi\|^2$. It follows that $A_i^2 = 1$. With this in mind, one can first get, for $i \neq j$, $0 = [n_j, A_i^2] = \{A_i, [n_j, A_i]\} =$ $= \frac{i}{2}\{A_i, \{A_j, A_i\}\} = i(A_i A_j A_i + A_j)$. It follows, that $\{A_i, A_j\} = 2\delta_{ij}$ and therefore $[n_k, A_l] = 0$ for all k, l . On the other hand, one easily gets $0 =$ $= Tr([n_i, A A_i A]) = i Tr(A A_i A A_i + A^2) = Tr(\{A_i, A\}^2)$. Since $\{A_i, A\}^2 \geq 0$, it follows that $\{A_i, A\} = 0$. Therefore we get:

$$[n_i, A_j] = [n_i, A] = 0$$

Let us now introduce Hermitian operators s_k by

$$\epsilon_{ijk} s_k = -\frac{i}{4}[A_i, A_j]$$

It is easy to see that $[s_i, s_j] = i\epsilon_{ijk} s_k$ and $\tilde{\underline{m}} = \underline{m} - \underline{s}$ also satisfy $[\tilde{m}_i, \tilde{m}_j] = i\epsilon_{ijk}\tilde{m}_k$, $[\tilde{m}_i, n_j] = i\epsilon_{ijk} n_k$, $[n_i, n_j] = -i\epsilon_{ijk}\tilde{m}_k$

The Hermitian operators $\tilde{\underline{m}}$, $\underline{n}$ satisfy therefore the commutation relations of the Lorentz group. Now, since $\mathcal{K}$ is finite - dimensional, it follows that $\underline{n} = \tilde{\underline{m}} = 0$

We can state therefore the following:

<u>Theorem 3.1</u> Finite - dimensional representations of HRA are in one - to - one correspondence with representations of the algebra:

$$/i/ \qquad \{A_i, A_j\} = 2\delta_{ij}$$

$$/ii/ \qquad \{A_i, A\} = 0$$

Then

$$m_i = -\frac{i}{4}\,\epsilon_{ijk} A_j A_k \quad , \quad n_i = 0 \qquad \Box$$

Now, let $A^2 = \sum_1^N \lambda_i P_i$ be the spectral decomposition of A^2 , $\lambda_i \geq 0$, $\lambda_i < \lambda_j$ for $i < j$. The subspace $\mathcal{K}_i = P_i \mathcal{K}$ are then invariant under A_i and A , so we can reduce the problem to the case of $A^2 = \lambda 1$

If $\lambda = 0$, then, what remains are the anticommutators $\{A_i, A_j\} = 2\delta_{ij}$ The most general representations is of the form:

$$A_i = (\otimes^p \sigma_i) \oplus (\otimes^q (-\sigma_i))$$

In the case of $\lambda \neq 0$, with $\alpha_i = A_i$, $\beta = \sqrt{\lambda} A$ we get:

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}$$

$$\{\alpha_i, \beta\} = 0$$

$$\beta^2 = 1$$

i.e. the Dirac algebra. There exists only one irreducible representation, by standard Dirac Matrices, so we finally get:

<u>Coriollarry 3.2</u> The most general finite dimensional representation of HRA is of the form:

$$\mathcal{K} = \mathcal{K}_0^+ \oplus \mathcal{K}_0^- \oplus (\oplus^r \mathcal{K}_i)$$

where

$$\mathcal{K}_0^+ = \cancel{\otimes}^{p}\, \mathbb{C}^2$$

$$\mathcal{K}_0^- = \cancel{\otimes}^{q}\, \mathbb{C}^2$$

$$\mathcal{K}_i = \cancel{\otimes}^{r_i}\, \mathbb{C}^4$$

$$A_i = (\cancel{\otimes}^{p} \sigma_i) \oplus (\cancel{\otimes}^{q}(-\sigma_i)) \oplus^{i} (\cancel{\otimes}^{r_i} \alpha_i)$$

$$A = 0 \oplus 0 \oplus \Big(\bigoplus_i \lambda_i^{\frac{1}{2}} \cancel{\otimes \beta} \Big)$$

In other words, we get a direct sum $\boxed{\text{of } \cancel{\text{tensor products}}}$ of Pauli and Dirac representations.

4. An infinite - dimensional representation of HRA

In this section we consider a particular solution of the commutation relations 2.1 , 2.15 , 2.16 with commuting velocity components A_i /a similar, but different, case has been considered by Corben [3] /Sec.19//. In this case $\underline{m}$ and $\underline{n}$ are generators of a unitary representation of the Lorentz group. If one assumes that $A^2 \leq 1$, and there are no eigenvectors of A^2 corresponding to the eigenvalue 1 , then, with $\Pi_0 = \mu(1-\underline{A}^2)^{-\frac{1}{2}}$ and $\underline{\Pi} = \mu \underline{A}(1-\underline{A}^2)^{-\frac{1}{2}}$ one easily finds that $\underline{m}$, $\underline{n}$, Π_μ satisfy commutation relations of the Poincare group, with $\Pi^2 = \mu^2$. So, we conclude that for every unitary representation $\underline{m}$, $\underline{n}$, Π_μ of the Poincare group, every solution A of the equations:

$$[\, m_i , A \,] = 0$$

$$[n_i, A] = \frac{i}{2}\left\{\frac{\Pi_i}{\Pi_0}, A\right\}$$

leads to an admissible wave equation. It should be observed that the spectrum of A is an excitation spectrum in the center of the mass system. The simplest representation acts in $\mathcal{K} = L^2(\mathbb{R}^3, d^3p)$ with $\Pi^0 = \sqrt{p^2+\mu^2}$, $\underline{m} = \underline{q} \times \underline{p}$ and $n = -\frac{1}{2}(q\Pi^0 + \Pi^0 q)$ where $q = i\ \partial/\partial\underline{p}$.
It is easy to see that $A = \Pi^{0-1}$ and $A' = \{\Pi^{0-1}, \tilde{n}^2\} - 2\,\underline{m}^2\,\Pi^{0-1}$ where $\tilde{\underline{n}} = \underline{q}\,\Pi^{0-1}$ are admissible solutions.
The transition from A to A' is an example of the following method of generating new solutions from the known ones: suppose that Hermitian operators $\underline{m}, \underline{n}, \underline{A}$ in $\mathcal{K}$ satisfy commutation relations:

$$[m_i, m_j] = i\,\varepsilon_{ijk}\,m_k$$

$$[m_i, n_j] = i\,\varepsilon_{ijk}\,n_k$$

$$[m_i, A_j] = i\,\varepsilon_{ijk}\,A_k$$

$$[n_i, n_j] = -i\,\varepsilon_{ijk}\,m_k + \frac{1}{4}[A_i, A_j] \qquad 4.1$$

$$[n_i, A_j] = \frac{i}{2}\{A_i, A_j\} - i\,\delta_{ij}$$

$$[L_i, A_j] = i\,A_j A_i - i\,\delta_{ij}$$

$$[L_i^*, A_k] = i\,(A_i A_k - \delta_{ik})$$

We will search for all Hermitian operators A in $\mathcal{K}$ fulfilling:

$$[m_i, A] = 0 \qquad 4.2$$

$$[n_i, A] = \frac{i}{2}\{A_i, A\}$$

$$[L_k, A] = i\,A\,A_k$$

$$\left(n_i - \frac{i}{2}A_i\right)A = A\left(n_i + \frac{i}{2}A_i\right)$$

It is convenient to introduce operators $l_i = n_i - \frac{i}{2} A_i$. Operators $\underline{m}$, $\underline{l}$ are now generators of a nonunitary /because $\underline{l}^* = l + A \neq 0$ / representation of the Lorentz group. We can now replace 4.2 by

$$[m_i, A] = 0 \qquad 4.3$$
$$l_i A = A l_i^*$$

Let us denote the real linear space of all solutions of 4.3 by $\mathcal{A}$,

$$\mathcal{A} = \{ A : A^* = A \; , A \text{ satisfies } 4.3 \}$$

Let $(\underline{m}, \underline{l}^*)'$ be the commutant of $(\underline{m}, l^*)$. Then

/i/ for each $A \in \mathcal{A}$ and $\omega \in (\underline{m}, l^*)'$

$$A\omega + \omega^* A \in \mathcal{A}$$

/ii/ if there exists an invertible $A \in \mathcal{A}$, then every element $A' \in \mathcal{A}$ is of the form

$$A' = A\omega + \omega^* A \qquad \text{for some } \omega \in (\underline{m}, \underline{l}^*)'$$

This method of generation of new solutions has been applied above to $A = \Pi^{v-1}$, with $\omega = \underline{l}^{*2} - \underline{m}^2$ being the Casimir operator of the representation $(\underline{l}^*, \underline{m})$.

5. Comparison with relativistic wave equations.

Usually relativistic systems are thought of as being described by relativistic wave equations of the form:

$$(B^\mu p_\mu + C)\,\psi(p) = 0 \qquad 5.1$$

The operators B^μ and C act in a space $\mathcal{K}$ of values of ψ. From the requirement that Poincare group acts on the manifold of all solutions of 5.1 one then finds /see e.g. [1] / that there exists a representation $\underline{K}, \underline{L}$ of the Lorentz group on $\mathcal{K}$ / $\underline{K}$ -rotation, $\underline{L}$ -boost generators/, such that B^μ is a four - vector, and C is a scalar. Let us call this algebra LRA /L for Lagrange'an/ This approach is natural from the point of view of the Lagrange'an formalism.

On the other hand, our algebra HRA given by 2.1 , 2.15 2.16 together with the requirement of hermicity is natural in the framework of relativistic quantum mechanics with the Hamiltonian as a fundamental quantity.

To clarify the relations between the HRA and LRA we observe that the following Theorem holds:

<u>Theorem 5.1</u> Let $(\underline{m}, \underline{n}, \underline{A}, A)$ be a representation of HRA on $\mathcal{K}$. If there exists a Hermitian, invertible operator G such that $[\underline{m}, G] = 0$ and $[\underline{n}, G] = \frac{1}{2}\{\underline{A}, G\}$ then $(\underline{K} = \underline{m},\ \underline{L} = \underline{n} - \frac{1}{2}\underline{A},\ \underline{B} = \underline{A}G^{-1},\ B^0 = G^{-1},\ C = AG^{-1})$ is a representation of LRA, with $G^{-1}\mathcal{O} = \mathcal{O}^{+}G^{-1}$ for all $\mathcal{O} \in$ LRA. /In particular, if A is invertible, we can take $G = A$ /. On the other hand, if $(\underline{K}, \underline{L}, B^\mu, C)$ is a representation of LRA, if B^0 is Hermitian and invertible, and satisfies $B^0\mathcal{O} = \mathcal{O}^{+}B^0$ for all $\mathcal{O} \in$ LRA, then $(\underline{m} = \underline{K},\ \underline{n} = \underline{L} + \frac{1}{2}\underline{B}B_0^{-1},\ \underline{A} = \underline{B}B_0^{-1},\ A = CB_0^{-1})$ is on representation of HRA.

In particular, it follows, that a representation of HRA cannot be reduced to that of LRA only in the case of non-invertible A . Unfortunately, except of the two-component neutrino, no example of such representation is known to the authors.

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