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*FUNCTIONAL ANALYSIS*

# Some Rotundities of Orlicz Spaces with Orlicz Norm

by

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**Summary.** This paper deals with local uniform rotundity (LUR), weak local uniform rotundity (WLUR), and weak uniform rotundity (WUR) of Orlicz spaces equipped with Orlicz norm under the case of finite and atomless measure. It is proved here that (I) both LUR and WLUR coincide with that the Orlicz spaces are reflexive and rotund, (II) WUR coincides with UR. Finally, a sufficient condition for uniform rotundity in every direction (URED) is given.

**1. Introduction.** With Luxemburg norm, most rotundities of Orlicz spaces have been investigated [1–8]. But with Orlicz norm, no rotundity except R, UR and HR seemed to be discussed before. This paper considers some rotundities with Orlicz norm, one will find that the corresponding rotundities between the two equivalent norms are of much difference.

Let us introduce some definitions and notations first.

Let $(X, \|\cdot\|)$ be a Banach space. $X$ is said to be (WLUR) LUR provided that for any $x_n$, $x_0$ in $X$ with $\|x_n\| \leqslant 1$, $\|x_0\| \leqslant 1$ $(n = 1, 2, \ldots)$, $\lim_{n\to\infty} \|x_n + x_0\| = 2$ implies $(x_n - x_0 \xrightarrow{w} 0$ as $n \to \infty)$ $\lim_{n\to\infty} \|x_n - x_0\| = 0$. $X$ is said to be (WUR) UR provided that $x_n$, $y_n$ in $X$ with $\|x_n\| \leqslant 1$, $\|y_n\| \leqslant 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$ implies $(x_n - y_n \xrightarrow{w} 0$ as $n \to \infty)$ $\lim_{n\to\infty} \|x_n - y_n\| = 0$. $X$ is said to be URED provided that $x_n$, $z$ in $X$ with $\|x_n\| = 1$, $\|x_n + z\| \leqslant 1$ $(n = 1, 2, \ldots)$ and $\operatorname{Lim}_{n\to\infty} \left\| x_n + \frac{1}{2} z \right\| = 1$ implies $z = 0$.

Throughout this paper, we denote by $G$ a measurable space with finite atomless measure, $M(u)$, $N(v)$ a pair of complementary $N$-functions and $p(u)$, $q(v)$ their derivatives on the right, respectively. For a measurable function $u(t)$ defined on $G$, we introduce the modular of $u$ by $R_M(u) = \int_G M(u(t))\, dt$. The Orlicz space $L_M^*$ generated by $M(u)$ is defined by $L_M^* = \{u(t) : \exists a > 0.\ R_M(au) < \infty\}$. The Orlicz norm and Luxemburg norm of an element $u$ in $L_M^*$ are defined, respectively, as follows

$$\|u\|_M = \sup_{R_N(v)\leqslant 1} \int_G u(t)\cdot v(t)\,dt, \qquad \|u\|_{(M)} = \inf\{\lambda > 0: R_M(u/\lambda) \leqslant 1\}.$$

Sometimes, we also use the following subspace of Orlicz space $L_M^*$: $E_M = \{u(t): \forall\lambda > 0,\ R_M(\lambda u) < \infty\}$. We always express by $M \in \Delta_2$ that $M(u)$ satisfies condition $\Delta_2$ for large $u$, by $M \in \nabla_2$ that $N \in \Delta_2$.

LEMMA 1. [9] *For $u$ in $L_M^*$, if there exists $k > 0$ such that $\int_G N\big(p(k|u(t)|)\big)\,dt = 1$, then*

$$\|u\|_M = \int_G |u(t)|\,p(k|u(t)|)\,dt = \frac{1}{k}\big(1 + R_M(ku)\big).$$

LEMMA 2. [2] *For any nonzero $u$ in $L_M^*$, there exists $k_0 > 0$ such that*

$$\|u\|_M = \inf \frac{1}{k}\big(1 + R_M(ku)\big) = \frac{1}{k_0}\big(1 + R_M(k_0 u)\big).$$

LEMMA 3. [3] *Suppose $M \in \Delta_2$, $x_n$ in $L_M^*$ with $\|x_n\| \leqslant 1$ $(n = 1, 2, \dots)$, then* (i) *$\|x_n\|_{(M)} \to 1$ implies $R_M(x_n) \to 1$;* (ii) *$R_M(x_n) \to 0$ implies $\|x_n\|_{(M)} \to 0$.*

LEMMA 4. [4] *Suppose $M \in \Delta_2$, $x_0$, $x_n$ in $L_M^*$ $(n = 1, 2, \dots)$. Then $R_M(x_n) \to R_M(x_0)$ and $x_n(t) \xrightarrow{\mu} x_0(t)$ (convergence in measure) implies $x_n \to x_0$ in norm.*

We say $M(u)$ is strictly convex if $M\big(au + (1-a)v\big) < aM(u) + (1-a)M(v)$ whenever $a \in (0, 1)$ and $u \neq v$.

LEMMA 5. *Assume that $M(u)$ is strictly convex, that $k_n$, $h_n$ are bounded and that $x_n$, $y_n$ in $L_M^*$ satisfies*

$$\|x_n\|_M = \frac{1}{k_n}\big(1 + R_M(k_n x_n)\big) \leqslant 1, \qquad \|y_n\|_M = \frac{1}{h_n}\big(1 + R_M(h_n y_n)\big) \leqslant 1$$

*$n = 1, 2, \dots$, then $\|x_n + y_n\|_M \to 2$ implies $k_n x_n(t) - h_n y_n(t) \xrightarrow{\mu} 0$.*

Proof. Analogous as in Lemma 11 in [4].

LEMMA 6. *Assume $\{x_n\}$ in $L_M^*$ is norm bounded, $\|x_n\|_M = \dfrac{1}{k_n}\big(1 + R_M(k_n x_n)\big)$ $(n = 1, 2, \dots)$, then $k_n \to \infty$ implies $x_n(t) \xrightarrow{\mu} 0$.*

Proof. Note that the $N$-function $M(u)$ has the property $M(u)/u \to \infty$ as $u \to \infty$.

LEMMA 7. *The set $K = \left\{k: \|x\|_M = \dfrac{1}{k}\big(1 + R_M(kx)\big),\ a \leqslant \|x\|_M \leqslant b\right\}$ is bounded for any $b \geqslant a > 0$ if and only if $M \in \nabla_2$.*

Proof. *Sufficiency*. Denote $u_0 = M^{-1}\left(\frac{1}{2\text{mes } G}\right)$. Similarly as the proof of Theorem 4.2 in [9], it is easily verified that $M \in \nabla_2$ iff there exist $p > 1$ and $l > 1$ such that $M(lu) \geqslant plM(u)$ for all $u \geqslant u_0$. For given $b \geqslant a > 0$ and $x$ in $L_M^*$, suppose $k$ satisfies $a \leqslant \|x\|_M = \frac{1}{k}(1 + R_M(kx)) \leqslant b$. Since $\|x\|_{(M)} \geqslant \frac{1}{2}\|x\|_M \geqslant \frac{1}{2}a$, by the definition of $\|\cdot\|_{(M)}$, we have $R_M\left(\frac{3}{a}x\right) > 1$. Therefore

$$\int_{G\left(\frac{3}{a}|x(t)| \geqslant u_0\right)} M\left(\frac{3}{a}x(t)\right)\text{dt} = R_M\left(\frac{3}{a}x\right) - \int_{G\left(\frac{3}{a}|x(t)| < u_0\right)} M\left(\frac{3}{a}x(t)\right)\text{dt} \geqslant$$

$$\geqslant R_M\left(\frac{3}{a}x\right) - M(u_0)\text{ mes } G = R_M\left(\frac{3}{a}x\right) - \frac{1}{2} > 1 - \frac{1}{2} = \frac{1}{2}$$

Now, if $k > \frac{3}{a}l$, then we can choose a natural number $i$ such that $l^i < \frac{I}{3}ak \leqslant l^{i+1}$. By repeated utilizing the formula $M(lu) \geqslant plM(u)$ $(u \geqslant u_0)$, we get $M(l^i u) \geqslant p^i l^i M(u)$ $(u \geqslant u_0)$. Hence,

$$b \geqslant \|x\|_M = \frac{1}{k}(1 + R_M(kx)) \geqslant \frac{1}{k}\int_{G\left(\frac{3}{a}|x(t)| \geqslant u_0\right)} M\left(\frac{1}{3}ak\frac{3}{a}x(t)\right)\text{dt} >$$

$$> \frac{1}{k}\int_{G\left(\frac{3}{a}|x(t)| \geqslant u_0\right)} M\left(l^i\frac{3}{a}x(t)\right)\text{dt} \geqslant \frac{1}{k}p^i l^i \int_{G\left(\frac{3}{a}|x(t)| \geqslant u_0\right)} M\left(\frac{3}{a}x(t)\right)\text{dt} >$$

$$> \frac{1}{k}p^i l^i \cdot \frac{1}{2} \geqslant p^i l^i \Big/ \left(\frac{6}{a}l^{i+1}\right) = \frac{a}{6l}p^i$$

Therefore, $i < \log_p(6lb/a)$ implying $k \leqslant \frac{3}{a}l^{[1+\log_p(6lb/a)]}$.

*Necessity*. If $M \notin \nabla_2$, then there exist $l_n \uparrow_n \infty$ with $l_1 \geqslant 2$ and $M(u_1)\text{mes } G \geqslant \frac{1}{4}$ such that $M(l_n u_n) < \left(1 + \frac{1}{n}\right)l_n M(u_n)$ $(n = 1, 2, \ldots)$. For each $n = 1, 2, \ldots,$

select a subset $G_n$ of $G$ such that $M(u_n)\operatorname{mes} G_n=\frac{1}{4}$ and define $x_n(t)=$ $=u_n\chi_{G_n}(t)$ where $\chi_{G_n}(t)$ expresses the characteristic function of $G_n$, then

$$\frac{1}{4}=R_M(x_n)\leqslant\|x_n\|_{(M)}\leqslant\|x_n\|_M\leqslant\frac{1}{l_n}(1+R_M(l_n x_n))=$$

$$=\frac{1}{l_n}+\frac{1}{l_n}M(l_n u_n)\operatorname{mes} G_n\leqslant\frac{1}{l_n}+\frac{1}{l_n}\left(1+\frac{1}{n}\right)l_n M(u_n)\operatorname{mes} G_n=$$

$$=\frac{1}{l_n}+\frac{1}{4}\left(1+\frac{1}{n}\right)\to\frac{1}{4}$$

If $\{k_n\}$ satisfies $\|x_n\|_M=\frac{1}{k_n}(1+R_M(k_n x_n))$, then $k_n>1$ and so

$$\frac{1}{l_n}+\frac{1}{4}\left(1+\frac{1}{n}\right)>\|x_n\|_M=\frac{1}{k_n}+\frac{1}{k_n}M(k_n u_n)\operatorname{mes} G_n\geqslant$$

$$\geqslant\frac{1}{k_n}+M(u_n)\operatorname{mes} G_n=\frac{1}{k_n}+\frac{1}{4}.$$

Let $n\to\infty$, we see $k_n\to\infty$ completing the proof.

## 2. Local rotundity.

THEOREM 1. *The following are equivalent*

(i) $L_M^*$ *is locally uniformly rotund,*

(ii) $L_M^*$ *is weakly locally uniformly rotund,*

(iii) $M\in\Delta_2$, $M\in\nabla_2$ *and* $M(u)$ *is strictly convex.*

Proof. (i)$\Rightarrow$(ii) Trivial.

(ii)$\Rightarrow$(iii) Since WLUR$\Rightarrow$R, by [1, 2], $M(u)$ is strictly convex. If $M\notin\Delta_2$, there exists $x_0\in L_M^*\backslash E_M$. Define

$$x_n(t)=\begin{cases}x_0(t), & \text{when } |x_0(t)|\leqslant n\\ 0, & \text{when } |x_0(t)|>n\end{cases}$$

then $x_n\in E_M$ $(n=1,2,\ldots)$ and $\|x_n\|_M\uparrow_n\|x_0\|_M$, therefore

$$\left\|\frac{x_n}{\|x_n\|_M}+\frac{x_0}{\|x_0\|_M}\right\|_M\geqslant\left(\frac{1}{\|x_n\|_M}+\frac{1}{\|x_0\|_M}\right)\|x_n\|_M\to 2.$$

On the other hand, since $x_0\notin E_M$, there exists $f\in(L_M^*)^*$ (the dual of $L_M^*$) such that $f(x_0)\neq 0$ and such that $f(u)=0$ for all $u$ in $E_M$. Thus, $f(x_0/\|x_0\|_M-x_n/\|x_n\|_M)=f(x_0)/\|x_0\|_M\neq 0$ contradicting (ii).

If $M \notin \nabla_2$, then there exist $a_k \uparrow \infty$ and a sequence $\{G_n\}$ of pairwise disjoint subsets of $G$ such that $E \overset{\text{def}}{=} G \setminus \bigcup_{k=1}^{\infty} G_k$ is not a null set and such that

$$N\left(\left(1+\frac{1}{k}\right)a_k\right) > 2^k N(a_k) \quad \text{and} \quad N(a_k) \operatorname{mes} G_k = 1/2^k$$

$k = 1, 2, \ldots$. It follows that for any $n \geqslant 1$, $R_N(a_n \chi_{G_n}) = 1/2^n < 1$ and $R_N\left(\left(1+\frac{1}{n}\right)a_n \chi_{G_n}\right) > 2^n R_N(a_n \chi_{G_n}) = 1$, hence, by the definition of $\|\cdot\|_{(N)}$, $\|a_n \chi_{G_n}\|_{(N)} \geqslant 1 \Big/ \left(1+\frac{1}{n}\right)$.

Denote $c = N^{-1}(1/\operatorname{mes} E)$, $c_n = N^{-1}\left(\left(1-\frac{1}{2^n}\right)/\operatorname{mes} E\right)$ and $v_n(t) = c_n \chi_E(t) + a_n \chi_{G_n}(t)$, then $c_n \uparrow_n c$, $R_N(c\chi_E) = 1$ implies $\|c\chi_E\|_{(N)} = 1$ and

$$R_N(v_n) = N(c_n) \operatorname{mes} E + N(a_n) \operatorname{mes} G_n = 1 - \frac{1}{2^n} + \frac{1}{2^n} = 1$$

Recall that $(E_N, \|\ \|_{(N)})^* = (L_M^*, \|\ \|_M)$ [9] and $c\chi_E$, $a_n \chi_{G_n} \in E_N$, there exist $x_0, x_n \in L_M^*$ with $\|x_0\|_M = \|x_n\|_M = 1$ and $x_0(t) = x_0(t)\chi_E(t)$, $x_n(t) = x_n(t)\chi_{G_n}(t)$ such that

$$1 = \|c\chi_E\|_{(N)} = \int_G c\chi_E(t)\, x_0(t)\, dt = \int_E c \cdot x_0(t)\, dt$$

$$1 \Big/ \left(1+\frac{1}{n}\right) \leqslant \|a_n \chi_{G_n}\|_{(N)} = \int_G a_n \chi_{G_n}(t)\, x_n(t)\, dt = \int_{G_n} a_n x_n(t)\, dt$$

$(n = 1, 2, \ldots)$. Now, observing $R_N(v_n) = 1$ and $\chi_E \in E_N \subset (L_M^*)^*$, we have

$$\|x_0 + x_n\|_M \geqslant \int_G (x_0(t) + x_n(t))\, v_n(t)\, dt = c_n \int_E x_0(t)\, dt + a_n \int_{G_n} x_n(t)\, dt \geqslant$$

$$\geqslant c_n/c + 1 \Big/ \left(1+\frac{1}{n}\right) \to 2$$

and

$$\chi_E(x_0 - x_n) = \int_G (x_0(t) - x_n(t))\, \chi_E(t)\, dt = \int_E x_0(t)\, dt = 1/c > 0$$

also contradicting (ii).

(iii) $\Rightarrow$ (i) Suppose (iii) holds. For given $x_0$, $x_n \in L_M^*$ with $\|x_0\|_M = \|x_n\|_M = 1$ $(n = 1, 2, \ldots)$ and satisfying $\|x_n + x_0\| \to 2$, we have to show $\|x_n - x_0\|_M \to 0$ which reduces to that $\{x_n\}$ contains a subsequence convergent to $x_0$ in norm, since above $x_0$ and $x_n$ are arbitrarily given.

For each $n = 0, 1, 2, \ldots$, choose $k_n > 1$ such that $1 = \|x_n\|_M =$

$= \frac{1}{k_n}(1+R_M(k_n x_n))$. By Lemma 7, $\{k_n\}$ is bounded, therefore, from Lemma 5, $k_n x_n(t) \xrightarrow{\mu} k_0 x_0(t)$. If we are able to show that $\{k_n\}$ contains a subsequence $\{k_{n_i}\}$ convergent to $k_0$, then by the choice of $\{k_n\}$, $R_M(k_{n_i} x_{n_i}) = k_{n_i} - 1 \to k_0 - 1 = R_M(k_0 x_0)$, it immediately follows by Lemma 4 that $\| k_{n_i} x_{n_i} - k_0 x_0 \|_M \to 0$ therefore $\{x_{n_i}\}$ converges to $x_0$ in norm, completing the proof.

For any $\varepsilon > 0$, since $M \in \nabla_2$, by Lemma 3, there exist $a > 0$ and $\varepsilon' > 0$ such that $R_N(v) < a$ implies $\| v \|_{(N)} < \varepsilon$, and that $R_N(v) \leqslant 1 - a$ implies $\| v \|_{(N)} < 1 - 2\varepsilon'$. Moreover, since $M \in \Delta_2$, by [9], there exists $\delta > 0$ such that $\| x_0 \chi_F \|_M < \min\{\varepsilon', \varepsilon k_n/k_0\}$ for all $n \geqslant 1$ whenever $F \subset G$ with $\operatorname{mes} F < \delta$.

Without loss of generality, we may assume $k_n x_n(t) \xrightarrow{\text{a.e.}} k_0 x_0(t)$, (since a sequence convergent in measure contains an a.e. convergent subsequence), therefore, there exists a subset $G_0$ of $G$ such that $\operatorname{mes} G_0 < \delta$ and that $\{k_n x_n(t)\}$ uniformly converges to $k_0 x_0(t)$ on $G \backslash G_0$.

Since $\| x_n + x_0 \|_M \to 2$, we can select $v_n$ in $L_N^*$ with $R_N(v_n) \leqslant 1$ $(n = 1, 2, \dots)$ such that

$$\int_G [x_n(t) + x_0(t)]\, v_n(t)\, dt \to 2.$$

It immediately follows that

$$\int_G x_n(t)\, v_n(t)\, dt \to 1, \qquad \int_G x_0(t)\, v_n(t)\, dt \to 1 \qquad (n \to \infty).$$

Thus, there exists $N > 0$ such that $\int_G x_0(t)\, v_n(t)\, dt > 1 - \varepsilon'$ $(n > N)$. Furthermore, by Hölder's inequality, when $n > N$, we have

$$1 - \varepsilon' < \int_G x_0(t)\, v_n(t)\, dt = \int_{G \backslash G_0} x_0(t)\, v_n(t)\, dt + \int_{G_0} x_0(t)\, v_n(t)\, dt \leqslant$$

$$\leqslant \| x_0 \|_M \, \| v_n \chi_{G \backslash G_0} \|_{(N)} + \| x_0 \chi_{G_0} \|_M < \| v_n \chi_{G \backslash G_0} \|_{(N)} + \varepsilon'$$

Recalling the choice of $a$ and $\varepsilon'$, for all $n > N$, we have $R_N(v_n \chi_{G \backslash G_0}) > 1 - a$, therefore

$$R_N(v_n \chi_{G_0}) = R_N(v_n) - R_N(v_n \chi_{G \backslash G_0}) < 1 - (1 - a) = a$$

which implies $\| v_n \chi_{G_0} \|_{(N)} < \varepsilon$. Hence, for all $n > N$,

$$\left| \int_G x_n(t)\, v_n(t)\, dt - 1 \right| = \left| \int_{G \backslash G_0} \left[ x_n(t) - \frac{k_0}{k_n} x_0(t) \right] v_n(t)\, dt + \right.$$

$$\left. + \int_G \frac{k_0}{k_n} x_0(t)\, v_n(t)\, dt - 1 + \int_{G_0} \left[ x_n(t) - \frac{k_0}{k_n} x_0(t) \right] v_n(t)\, dt \right| \geqslant$$

$$\geqslant \left| \int_{G \setminus G_0} \left[ x_n(t) - \frac{k_0}{k_n} x_0(t) \right] v_n(t)\,dt + \frac{k_0}{k_n} \int_G x_0(t)\, v_n(t)\,dt - 1 \right| -$$

$$- \| v_n \chi_{G_0} \|_{(N)} - \frac{k_0}{k_n} \| x_0 \chi_{G_0} \|_M$$

Let $n \to \infty$, we obtain $\overline{\lim\limits_{n\to\infty}} |k_0/k_n - 1| - 2\varepsilon = 0$ therefore $k_n \to k_0$ since $\varepsilon$ is arbitrary.

**3. Weak uniform rotundity.** It is shown in [3] that $L_M^*$ is UR if and only if the following two conditions are satisfied:

(I) $M \in \Delta_2$ and $M(u)$ is strictly convex,

(II) $M(u)$ is uniformly convex for large $u$, i.e., for any $\varepsilon > 0$, there exist $u_0 > 0$ and $\delta > 0$ such that $p((1+\varepsilon)u) \geqslant (1+\delta)\, p(u)$ for all $u \geqslant u_0$.

For WUR, we present

THEOREM 2. *$L_M^*$ is weakly uniformly rotund iff it is uniformly rotund.*

Proof. By Theorem 1 and [3], conditions (I) and (II) $\Rightarrow$ UR $\Rightarrow$ WUR $\Rightarrow$ $\Rightarrow$ WLUR $\Rightarrow$ (I) and $M \in \nabla_2$. It only remains to verify the necessity of condition (II).

Select $a > 0$ and $A \subset G$ with mes $A <$ mes $G$ such that $N(p(a))$ mes $A = \frac{1}{2}$. If (II) does not hold, then there exist $\varepsilon > 0$ and $u_n > 0$ such that

$$p((1+\varepsilon)u_n) < \left(1 + \frac{1}{n}\right) p(u_n), \qquad N(p(u_n)) \text{ mes } (G \setminus A) \geqslant \frac{1}{2},$$

$(n = 1, 2, \ldots)$. Hence, for each $n = 1, 2, \ldots$, we may choose $G_n \subset G \setminus A$ such that $N(p(u_n))$ mes $G_n = \frac{1}{2}$. Denote

$$k_n = (1+\varepsilon)\, u_n\, p((1+\varepsilon)u_n) \text{ mes } G_n + a p(a) \text{ mes } A$$

$$h_n = u_n\, p(u_n) \text{ mes } G_n + a p(a) \text{ mes } A$$

and define

$$x_n(t) = \frac{1}{k_n}(1+\varepsilon)\, u_n\, \chi_{G_n}(t) + \frac{1}{k_n} a \chi_A(t),$$

$$y_n(t) = \frac{1}{h_n} u_n\, \chi_{G_n}(t) + \frac{1}{h_n} a \chi_A(t),$$

$n = 1, 2, \ldots$. We estimate the norm of $x_n$, $y_n$ and $\frac{1}{2}(x_n + y_n)$. Since

$$\int_G N\left(p\left(k_n x_n(t)\right)\right) dt \geqslant \int_G N\left(p\left(h_n y_n(t)\right)\right) dt = N\left(p\left(u_n\right)\right) \text{mes } G_n +$$

$$+ N\left(p(a)\right) \text{mes } A = \frac{1}{2} + \frac{1}{2} = 1$$

by Lemma 1, the definition of $k_n$, $h_n$ and Young's inequality,

$$\| y_n \|_M = \int_G y_n(t)\, p\left(h_n y_n(t)\right) dt = \frac{u_n}{h_n} p\left(u_n\right) \text{mes } G_n + \frac{a}{h_n} p(a) \text{mes } A = 1$$

and

$$\| x_n \|_M \leqslant \frac{1}{k_n}\left(1 + R_M\left(k_n x_n\right)\right) \leqslant \frac{1}{k_n}\left[\int_G N\left(p\left(k_n x_n(t)\right)\right) dt + \int_G M\left(k_n x_n(t)\right) dt\right] =$$

$$= \frac{1}{k_n} \int_G k_n x_n(t)\, p\left(k_n x_n(t)\right) dt =$$

$$= \frac{1}{k_n}(1+\varepsilon) u_n\, p\left((1+\varepsilon) u_n\right) \text{mes } G_n + \frac{1}{k_n} a p(a) \text{mes } A = 1,$$

$n = 1, 2, \ldots$. Since $M \in \nabla_2$, there exists $k > 2$ such that $N(2v) \leqslant k N(v)$ for all $v \geqslant p(u_1) > 0$. Combine the convexity of $N(v)$, we have

$$N\left(p\left(\frac{k_n h_n}{k_n + h_n}\left(\frac{1+\varepsilon}{k_n} + \frac{1}{h_n}\right) u_n\right)\right) = N\left(p\left(\left(1 + \frac{\varepsilon h_n}{k_n + h_n}\right) u_n\right)\right) \leqslant$$

$$\leqslant N\left(p\left((1+\varepsilon) u_n\right)\right) < N\left(\left(1 + \frac{1}{n}\right) p\left(u_n\right)\right) \leqslant \left(1 - \frac{1}{n}\right) N\left(p\left(u_n\right)\right) +$$

$$+ \frac{1}{n} N\left(2p\left(u_n\right)\right) < N\left(p\left(u_n\right)\right) + \frac{k}{n} N\left(p\left(u_n\right)\right) = \left(1 + \frac{k}{n}\right) N\left(p\left(u_n\right)\right)$$

therefore

$$\int_G N\left(p\left(\frac{2k_n h_n}{k_n + h_n} \frac{x_n(t) + y_n(t)}{2}\right)\right) dt =$$

$$= N\left(p\left(\frac{k_n h_n}{k_n + h_n}\left(\frac{1+\varepsilon}{k_n} + \frac{1}{h_n}\right) u_n\right)\right) \text{mes } G_n +$$

$$+ N\left(p\left(\frac{k_n h_n}{k_n + h_n}\left(\frac{a}{k_n} + \frac{a}{h_n}\right)\right)\right) \text{mes } A <$$

$$< \left(1 + \frac{k}{n}\right)\left[N\left(p\left(u_n\right)\right) \text{mes } G_n + N\left(p(a)\right) \text{mes } A\right] = 1 + \frac{k}{n}.$$

It follows that

$$\int_G N\left(\frac{1}{1+\frac{k}{n}}\, p\left(\frac{2k_n h_n}{k_n+h_n}\,\frac{x_n(t)+y_n(t)}{2}\right)\right) \mathrm{dt} \leqslant 1$$

hence, by the definition of $k_n$, $h_n$ and $\|\cdot\|_M$

$$\left\|\frac{x_n+y_n}{2}\right\|_M \geqslant \int_G \frac{x_n(t)+y_n(t)}{2}\,\frac{1}{1+\frac{k}{n}}\, p\left(\frac{2k_n h_n}{k_n+h_n}\,\frac{x_n(t)+y_n(t)}{2}\right) \mathrm{dt} \geqslant$$

$$\geqslant \frac{1}{1+\frac{k}{n}}\left\{\frac{(1+\varepsilon)\,h_n+k_n}{2k_n h_n} u_n\, p(u_n) \operatorname{mes} G_n + \frac{h_n+k_n}{2k_n h_n} a p(a) \operatorname{mes} A\right\} =$$

$$= \frac{1}{2\left(1+\frac{k}{n}\right)}\left\{\left[\frac{1+\varepsilon}{k_n} u_n\, p(u_n) \operatorname{mes} G_n + \frac{a}{k_n} p(a) \operatorname{mes} A\right] + \right.$$

$$\left. + \left[\frac{1}{h_n} u_n\, p(u_n) \operatorname{mes} G_n + \frac{a}{h_n} p(a) \operatorname{mes} A\right]\right\} \geqslant$$

$$\geqslant \frac{1}{2\left(1+\frac{k}{n}\right)}\left\{\left[\frac{1+\varepsilon}{\left(1+\frac{1}{n}\right)k_n} u_n\, p((1+\varepsilon)\,u_n) \operatorname{mes} G_n + \right.\right.$$

$$\left.\left. + \frac{ap(a)}{\left(1+\frac{1}{n}\right)k_n} \operatorname{mes} A\right] + \frac{1}{1+\frac{1}{n}}\right\} = \frac{1}{\left(1+\frac{k}{n}\right)\left(1+\frac{1}{n}\right)}.$$

This shows that $\left\|\frac{1}{2}(x_n+y_n)\right\|_M \to 1$ as $n \to \infty$.

Finally, observe $\frac{\chi_A(t)}{a \cdot \operatorname{mes} A} \in E_N \subset (L_M^*)^*$ and

$$\int_G [y_n(t)-x_n(t)]\,\frac{\chi_A(t)}{a \cdot \operatorname{mes} A}\, \mathrm{dt} = \frac{1}{h_n} - \frac{1}{k_n}$$

to show that $\{y_n - x_n\}$ does not weakly converge to zero finishing the proof, it is sufficient to verify that $\frac{1}{h_n} - \frac{1}{k_n}$ does not converge to zero. We may reduce this to showing that $\{k_n\}$ is bounded since

$$k_n - h_n = [(1+\varepsilon)\,u_n\, p((1+\varepsilon)\,u_n) - u_n\, p(u_n)] \operatorname{mes} G_n \geqslant$$

$$\geqslant \varepsilon u_n p(u_n) \text{ mes } G_n > \varepsilon N(p(u_n)) \text{ mes } G_n = \frac{1}{2}\varepsilon,$$

$n = 1, 2, \ldots$. From the inequalities in [9], for every $n = 1, 2, \ldots$,

$$u_n p(u_n) \text{ mes } G_n \leqslant q(p(u_n)) p(u_n) \text{ mes } G_n \leqslant N(2p(u_n)) \text{ mes } G_n \leqslant$$
$$\leqslant kN(p(u_n)) \text{ mes } G_n = \frac{1}{2}k.$$

hence

$$k_n \leqslant (1+\varepsilon)\left(1+\frac{1}{n}\right) u_n p(u_n) \text{ mes } G_n + ap(a) \text{ mes } A \leqslant (1+\varepsilon)k +$$
$$+ ap(a) \text{ mes } A$$

completing the proof.

### 4. Uniform rotundity in every direction.

THEOREM 3. *If $M \in \Delta_2$ and $M(u)$ is strictly convex, then $L_M^*$ is uniformly rotund in every direction.*

Proof. Suppose that $M \in \Delta_2$ and that $M(u)$ is strictly convex. For given $z, x_n$ in $L_M^*$ with $\|x_n\|_M = 1$, $\|x_n + z\|_M \leqslant 1$ $(n = 1, 2, \ldots)$ and $\left\|x_n + \frac{1}{2}z\right\|_M \to 1$, we need to show $z = 0$.

Assume $z \neq 0$.

First, we show that $\{x_n\}$ contains a subsequence convergent to $az$ in measure for some real $a$. If $\{x_n\}$ or $\{x_n + z\}$ contains a subsequence convergent to zero in measure, then the statement is true while we take $a = 0$ or $-1$. If not, then $\{k_n, h_n\}$ is bounded where $\{k_n, h_n\}$ satisfies

$$1 = \|x_n\|_M = \frac{1}{k_n}(1 + R_M(k_n x_n)), \quad 1 \geqslant \|x_n + z\|_M = \frac{1}{h_n}(1 + R_M(h_n(x_n + z)))$$

$(n = 1, 2, \ldots)$, therefore, by Lemma 5, $k_n x_n(t) - h_n(x_n(t) + z(t)) \overset{\mu}{\to} 0$. Without loss of generality, we may assume that $k_n \to k_0$ and that $h_n \to h_0$. Hence, $x_n(t) \overset{\mu}{\to} \frac{h_0}{k_0 - h_0} z(t)$ $(z \neq 0$ implies $k_0 \neq h_0)$.

Secondly, we may assume $x_n(t) \xrightarrow{\mu} az(t) \neq 0$ (otherwise, we consider $x_n + z$ instead of $x_n$ below), therefore, $R_M(az) > 0$. It follows by $M \in \Delta_2$ that there exist $\beta > 0$ and $\delta > 0$ such that $\int_{G \setminus e} M(az(t))\,dt > \beta$ and $\| az\chi_e \|_M < \frac{\beta}{2}$ for all $e \subset G$ satisfying $\operatorname{mes} e < \delta$. Since a sequence convergent in measure contains an a.e. convergent subsequence, we may assume $x_n(t) \xrightarrow{\text{a.e.}} az(t)$, therefore, there exists a subset $G_0$ of $G$ with $\operatorname{mes} G_0 < \delta$ such that $\{x_n(t)\}$ uniformly converges to $az(t)$ on $G \setminus G_0$ and that $az(t)$ is bounded on $G \setminus G_0$. Knowing that $k_n > 1$, we see

$$1 = \| x_n \|_M = \frac{1}{k_n}\left[1 + \int_{G \setminus G_0} M(k_n x_n(t))\,\mathrm{dt} + \int_{G_0} M(k_n x_n(t))\,\mathrm{dt}\right] \geqslant$$

$$\geqslant \| x_n \chi_{G_0} \|_M + \frac{1}{k_n} \int_{G \setminus G_0} M(k_n x_n(t))\,\mathrm{dt} \geqslant \| x_n \chi_{G_0} \|_M + \int_{G \setminus G_0} M(x_n(t))\,\mathrm{dt},$$

$(n = 1, 2, \ldots)$. Let $n \to \infty$, we get $\overline{\operatorname{Lim}}_{n \to \infty} \| x_n \chi_{G_0} \|_M \leqslant 1 - \int_{G \setminus G_0} M(az(t))\,\mathrm{dt} < 1 - \beta$.

Finally, since $\left\| x_n + \frac{1}{2} z \right\|_M \to 1$, there exists $v_n$ in $L_N^*$ with $R_N(v_n) \leqslant 1$ $(n = 1, 2, \ldots)$ such that $\int_G \left[ x_n(t) + \frac{1}{2} z(t) \right] v_n(t)\,\mathrm{dt} \to 1$. This immediately implies that $\int_G x_n(t) v_n(t)\,\mathrm{dt} \to 1$ and that $\int_G [x_n(t) + z(t)]\, v_n(t)\,\mathrm{dt} \to 1$ therefore $\int_G z(t) v_n(t)\,\mathrm{dt} \to 0$. Since

$$\int_G x_n(t) v_n(t)\,\mathrm{dt} \leqslant \int_{G \setminus G_0} x_n(t) v_n(t)\,\mathrm{dt} + \| x_n \chi_{G_0} \|_M$$

and

$\overline{\operatorname{Lim}}_{n \to \infty} \| x_n \chi_{G_0} \|_M < 1 - \beta$ and $x_n(t) \xrightarrow{\text{u.c.}} az(t)$ on $G \setminus G_0$, let $n \to \infty$, we obtain

$$\operatorname{Lim}_{n \to \infty} \int_{G \setminus G_0} az(t) v_n(t)\,\mathrm{dt} = \underline{\operatorname{Lim}}_{n \to \infty} \int_{G \setminus G_0} x_n(t) v_n(t)\,\mathrm{dt} > 1 - (1 - \beta) = \beta,$$

Therefore

$$0 = \operatorname{Lim}_{n \to \infty} \int_G z(t) v_n(t)\,\mathrm{dt} \geqslant \underline{\operatorname{Lim}}_{n \to \infty} \int_{G \setminus G_0} z(t) v_n(t)\,\mathrm{dt} - \| z\chi_{G_0} \|_M > \frac{\beta}{\alpha} - \frac{\beta}{2\alpha}$$

This contradiction completes the proof.

Question. In the proof of Theorem 3, $M \in \Delta_2$ is once used to indicate $\operatorname{Lim}_{\operatorname{mes} e \to 0} \| z\chi_e \|_M = 0$. Since URED $\Rightarrow$ R, it is necessary in Theorem 3 that

$M(u)$ is strictly convex. The question here is whether or not the assertion in Theorem 3 still holds withouw condition $M \in \Delta_2$ which is not necessary as far as $I$ know.

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Чен Шутао, **Некоторые округлости пространства Орлича с нормой Орлича**

Настоящая работа посвящена характеризации локально равномерной, слабо локально равномерной, слабо равнояерной и равномерной в произвольном направлении округлостям промеранств Орлича с нормой Орлича. Доказывается, что локально равномерная и слабо локально равномерная округлости совпадают с тем, что пространство является рефлексивным и округлым, и слабо равномерная округлость совпадает с равномерной округлостью. Наконец, даны некоторые достаточные условия для равномерной округлости в произвольном направлении.