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SMOOTH POINTS IN ORLICZ SEQUENCE SPACES AND GEOMETRY OF THE DUAL AND THE BIDUAL OF ORLICZ SPACES

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ABSTRACT. Smooth points of Orlicz sequence spaces equipped with the Orlicz norm are characterized. Criteria for smoothness of h_0^Φ and l_0^Φ are deduced. Next, geometry of the dual and the bidual of Orlicz space for both (the Luxemburg and the Orlicz) norms is considered.

0. INTRODUCTION

In the sequel $I\!N, I\!R$ and $I\!R_+$ denote, the set of natural numbers, reals and nonnegative reals, respectively. The triple (T, Σ, μ) stands for a nonatomic σ-finite complete nontrivial measure space or for the space of the counting measure with $T = I\!N, \Sigma = 2^{I\!N}$ and $\mu(A) =$Card (A) for any $A \subset I\!N$. $L^o = L^o(\mu)$ (resp. l^o) stands for the space of all (equivalence classes of) Σ-measurable functions defined on T (resp. of all real sequences). A map $\Phi: I\!R \to [0, \infty]$ is said to be an Orlicz function if Φ is vanishing and continuous at 0, left continuous on the whole $I\!R_+$, even, convex and not identically equal to 0. Given any Orlicz function Φ, we define on L^o (resp. l^o) a convex functional I_Φ by

$$I_\Phi(x) = \int_T \Phi(x(t))d\mu \qquad (\text{resp. } I_\Phi(x) = \sum_{i=1}^{\infty} \Phi(x_i))$$

for every $x \in L^0$ (resp. $x \in l^0$). Then I_Φ is a convex modular on L^o (resp. l^o) , i.e. I_Φ is even, convex, $I_\Phi(0) = 0$ and $x = 0$ whenever $x \in L^o$ (resp. l^0) and $I_\Phi(\lambda x) = 0$ for any $\lambda > 0$ (see [26]). The Orlicz space L^Φ (resp. l^Φ) is defined as the set of all $x \in L^o$ (resp. $x \in l^o$) such that $I_\Phi(\lambda x) < \infty$ for some $\lambda > 0$ depending on x. When μ is non-atomic, the subspace E^Φ of order continuous elements is nontrivial if and only if Φ is finitely valued. Then we have $x \in E^\Phi$ if and only if $I_\Phi(\lambda x) < \infty$ for any $\lambda > 0$. In the case of the counting measure the subspace of order continuous elements is denoted by h^Φ and it is always nontrivial and it equals to the set of all $x \in l^0$ such that for any $\lambda > 0$ there is $m \in N$ satisfying $\sum_{n=m}^{\infty} \Phi(\lambda x_n) < \infty$. We have $h^\Phi \neq \{0\}$ for any Φ and $E^\Phi \neq \{0\}$ iff Φ is finitely valued. We will consider the spaces L^Φ, E^Φ, l^Φ and h^Φ with the Luxemburg norm

$$\|x\|_\Phi = \inf\{\lambda > 0: I_\Phi(x/\lambda) \leq 1\}$$

as well as with the Orlicz norm

$$\|x\|_\Phi^o = \sup\{|\int_T x(t)y(t)d\mu|: y \in L^o \ (\text{resp. } l^o) \text{ and } I_{\Phi^*}(y) \leq 1\},$$

where Φ^* denotes the Orlicz function conjugate to Φ in the sense of Young, i.e.

$$\Phi^*(u) = \sup_{v>0}\{|u|v - \Phi(v)\} \quad (\forall\, u \in I\!R)$$

(see [22], [23], [24], [26], [29] and [33]). In the sequel L^Φ, l^Φ, E^Φ and h^Φ stand for Orlicz spaces (and their subspaces of order continuous elements) equipped with the Luxemburg norm and $L_0^\Phi, l_0^\Phi, E_0^\Phi$ and h_0^Φ stand for the respective spaces equipped with the Orlicz norm. There holds the following Amemiya formula for the Orlicz norm (see [22], [29] and [33]):

$$\|x\|_\Phi^o = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx)) \quad (\forall\, x \in L^\Phi \quad (\text{resp. } l^\Phi)).$$

If additionally $\Phi(u)/u \to \infty$ as $u \to \infty$, then for any $x \in L^\Phi$ (resp. l^Φ) there is $k = k(x) > 0$ such that (see [7] and [33])

$$\|x\|_\Phi^o = \frac{1}{k}(1 + I_\Phi(kx)). \tag{0.1}$$

Denote by $K_\Phi(x)$ the set of all $k > 0$ satisfying (0.1). It is known (see [29] and [33]) that defining

$$k^*(\Phi) = \inf\{k > 0 : I_{\Phi^*}(\varphi(k|x|)) \geq 1\},$$

$$k^{**}(\Phi) = \sup\{k > 0 : I_{\Phi^*}(\varphi(k|x|) \leq 1\},$$

where φ denotes the right derivative of Φ, we have $K_\Phi(x) = [k^*(\Phi), k^{**}(\Phi)]$. Denote by φ_- the left derivative of Φ. The subgradient of Φ at $u \in I\!R$ is defined to be

$$\partial\Phi(u) = [\varphi_-(u), \varphi(u)].$$

This also means that if $u > 0$ and $\varphi_-(u) = \infty$ then $\partial\Phi(v) = \{\infty\}$ for $v \geq u$ and if $u < 0$ and $\varphi(u) = -\infty$ then $\partial\Phi(v) = \{-\infty\}$ for $v \leq u$. It is easy to see that

$$\begin{aligned}\partial\Phi(u) &= \{v \in [-\infty, +\infty] : \Phi(u) + \Phi^*) = uv\} \\ &= \{k \in [-\infty, +\infty] : \Phi(v) - \Phi(u) \geq k(v - u) \text{ for all } v \in I\!R\}\end{aligned}$$

for each $u \in I\!R$. If $x \in L^\Phi$ (resp. l^Φ) define by $\theta(x)$ the distance of x from E^Φ (resp. h^Φ). It is known (see [2] and [5]) that

$$\theta(x) = \inf\{\lambda > 0 : I_\Phi(x/\lambda) < \infty\}$$

for a nonatomic measure whenever Φ is finitely valued and we can easily check that

$$\theta(x) = \inf\{\lambda > 0 : \sum_{i=j}^{\infty} \Phi(x_i/\lambda) < \infty \quad \text{for some } j \in I\!N\}$$

for the counting measure and Φ being arbitrary. It is known that $\theta(x)$ is the same for both (the Luxemburg and the Orlicz) norms (see [2]). Recall that an Orlicz function Φ satisfies the Δ_2-condition at zero (at infinity) if there exist positive constants K, u_o and u_1 such that $0 < \Phi(u_i) < \infty$ $(i = 0, 1)$ and the inequality $\Phi(2u) \leq K\Phi(u)$ holds true whenever $|u| \leq u_o$ $(|u| \geq u_1)$.

The suitable Δ_2-condition means the Δ_2-condition at zero for the counting measure space, the Δ_2-condition at infinity when μ is nonatomic finite and the Δ_2-condition at zero and at infinity simultaneously when μ is nonatomic infinite. We indicate this by $\Phi \in \Delta_2$.

We have for finitely valued Φ (see [1], [26], [29] and [33]) that

$$(L^\Phi)^* = L^{\Phi^*} \oplus S,$$

i.e. any $x^* \in (L^\Phi)^*$ is uniquelly represented in the form $x^* = \xi_v + \psi$, where $v \in L^{\Phi^*}$ and $\xi_v(x) =< v, x >= \int_T v(t)x(t)d\mu$ for any $x \in L^\Phi$ and ψ is a singular functional over L^Φ, i.e. $\psi(x) = 0$ for any $x \in E^\Phi$. If L^Φ is equipped with the Luxemburg norm, then $\|x^*\| = \|\xi_v\| + \|\psi\| = \|v\|_{\Phi^*}^o + \|\psi\|$ for $x^* = \xi_v + \psi$ with $v \in L^{\Phi^*}, \psi \in S$. Analogous representation holds true for the dual space of $(l^\Phi)^*$. Of course, in this case we need to replace E^Φ by h^Φ. Such additivity of the norm does not holds in the case of the Orlicz norm. However, for any $\psi \in S, \|\psi\|$ has the same value for both (the Luxemburg and the Orlicz) norms.

For any Orlicz function Φ we define a function $\Pi_\Phi : [0, \infty) \to [0, \infty)$ by

$$\Pi_\Phi(\alpha) = \inf\{t > 0 : \Phi^*(\varphi(t)) \geq \alpha\} \quad (\inf \Phi^* \overset{\text{def}}{=} \infty).$$

For the geometric notions such as smooth point, smoothness (S) rotundity (R), local uniform rotundity (LUR), midpoint local uniform rotundity $(MLUR)$, unform rotundity in every direction $(URED)$, (H)- property, non-squareness (NSQ) and local iniform non-squareness $(LUNSQ)$ we refer to the monographs [6], [23], [27], [29] and [33] as well as to the papers [2-5], [7-21], [25], [30-32] and [34]. We are mainly interested in these geometric properties which does not imply reflexivity automatically. Hence such properties as uniform rotundity, uniform smoothness, B-convexity and weak uniform rotundity are omitted.

1. AUXILIARY RESULTS

We start with the following theorem.

Theorem 1.1. *Let Φ be an arbitrary finitely valued Orlicz function such that $\Phi(u)/u \to \infty$ as $u \to \infty$. A functional $x^* = \xi_v + \psi$ $(v \in l^{\Phi^*}, \psi \in S)$, $x^* \neq 0$, is norm attainable at $x \in S(l_o^\Phi)$ if and only if for some (equivalently, for every) $k \in K_\Phi(x)$ there hold:*

1°. $I_{\Phi^*}(v/\|x^*\|) + \|\psi\|/\|x^*\| = 1$;
2°. $\|\psi\| = \psi(kx)$;
3°. $< kx, v/\|x^*\| >= I_\Phi(kx) + I_{\Phi^*}(v/\|x^*\|)$ *(equivalently $v_i \in \partial\Phi(kx_i)$ for any $i \in I\!N$).*

The proof proceeds in the same way as the proof of Theorem 2.3 in [3], so it is omitted here.

Corollary 1.2. *Let Φ be a finitely valued Orlicz function such that $\Phi(u)/u \to \infty$ as $u \to \infty$. A functional ξ_v with $v \in l^{\Phi^*}$ is a support functional at $x \in S(l_o^\Phi)$ if and only if the following assertions are satisfied:*

(i) $I_{\Phi^*}(v) = 1$;
(ii) $v_i \in \partial\Phi(x_i)$ *for any $i \in I\!N$, where $k \in K_\Phi(x)$.*

Corollary 1.3. *Let Φ be a finitely valued Orlicz function such that $\Phi(u)/u \to \infty$ as $u \to \infty$ and $x \in S(l_o^\Phi)$. Then* Grad $(x) \subset \{\xi_v : v \in l^{\Phi^*}\}$ *if and only if:*

(i) $\theta(x) < [k_\Phi^{**}(x)]^{-1}$;
(ii) $I_{\Phi^*}(\varphi_-(k_\Phi^*(x)/x)) = 1$.

For the proof see Corollary 1.6 in [3].

2. SMOOTH POINTS AND SMOOTHNESS IN l_0^Φ

Smooth points and smoothness of Orlicz function spaces equipped with the Luxemburg norm were characterized in [5] and [7]. Smoothness of L_0^Φ was characterized in [2]. Criteria for smooth points of L_0^Φ were given in [3].

Theorem 2.1. *Let Φ be a finitely valued Orlicz function vanishing only at zero, smooth at zero and such that $\Phi(u)/u \to \infty$ as $u \to \infty, x \in S(l_0^\Phi)$ and $k = k_\Phi^*$. Then x is smooth if and only if one of the following conditions is satisfied:*

(i) $I_{\Phi^*}(\varphi_-(k|x|)) = 1$;

(ii) $\theta(kx) < 1$ *and either* $I_{\Phi^*}(\varphi(k|x|)) = 1$ *or the set* $A = \{j \in I\!N: \varphi_-(k|x_j|) < \varphi(k|x_j|)\}$ *contains at most one index.*

Proof. Sufficiency. Assume first that condition (i) holds true. Then, by Corollary 1.2, x has a unique support functional ξ_v with $v = \varphi_-(k|x|)$ sgn x, i.e. x is smooth. Assume now that $\theta(kx) < 1$ and $I_{\Phi^*}(\varphi(k|x|)) = 1$. Then $\psi(kx) \le \theta(kx)\|\psi\| < \|\psi\|$, so condition 2° in Theorem 1.1 satisfies no $\psi \in S$. This yields by Theorem 1.1 that x has regular support functionals ξ_v only. Now, the condition $I_{\Phi^*}(\varphi(k|x|)) = 1$ and Corollary 1.2 yield that $v = \varphi(k|x|)$ sgn x, i.e. x is smooth.

Assume now that $A = \{i\}$ and $\theta(kx) < 1$. Then we get again that at x there exist regular support functionals only. Let ξ_v $\in$Grad (x), where $v \in L^{\Phi^*}$. By Corollary 1.2, $I_{\Phi^*}(v) = 1$. Clearly, $v_j = \varphi(k|x_j|)$ sgn x_j for any $j \ne i$, and so, $I_{\Phi^*}(v) = 1$ and Corollary 1.2 imply

$$v_i = (\Phi^*)^{-1}(1 - \sum_{j \ne i} \Phi^*(k|x_j|)) \text{ sgn } x_i.$$

This means that v is uniquely determined, and so x is smooth.

Necessity. Assume that (i) is not satisfied, i.e. $I_{\Phi^*}(\varphi_-(k|x|)) < 1$. Then we need to prove that condition (ii) must be satisfied. Assume the contrary, i.e.

a) $\theta(kx) = 1$ or

b) $I_{\Phi^*}(\varphi(k|x|)) \ne 1$ and Card $(A) \ge 2$.

Consider first case a). Since $\|x\|_\Phi^0 = \frac{1}{k}(1 + I_\Phi(kx)) = 1$, we have $I_\Phi(kx) = k - 1 < \infty$. Therefore, $|x_i| \to 0$ as $i \to \infty$. Consequently, there is i_o such that $\sup_i |x_i| = |x_{i_o}|$. Hence it follows that there exists a permutation (i_k) of $I\!N$ such that

$$(1.1) \qquad |x_{i_1}| \ge |x_{i_2}| \ge \cdots \ge |x_{i_k}| \ge |x_{i_{k+1}}| \ge \cdots$$

Define

$$A = \{i_1, i_3, \ldots, i_{2k-1}, \ldots\},$$
$$B = \{i_2, i_4, \ldots, i_{2k}, \ldots\}$$

for $k = 1, 2, \ldots$ and put $x_1 = x1_A, x_2 = x1_B$, where 1_A denotes the characteristic sequence of A, i.e. 1 stands on any place of A and 0 on any place outside A. Then conditions a) and (1.1) yield that

$$\theta(kx_1) \doteq \theta(kx_2) = \theta(kx) = \theta(k(x_1 - x_2)) = 1.$$

Therefore, there exist linear continuous singular functionals ψ_1, ψ_2 such that $\|\psi_1\| = \|\psi_2\| = 1$ and $\psi_i(kx_i) = \theta(kx_i) = 1$. So, we have

$$1 \pm \psi_1(kx_2) = \psi_1(kx_1) \pm \psi_1(kx_2) = \psi_1(k(x_1 \pm x_2))$$
$$\le \theta(k(x_1 \pm x_2))\|\psi_1\| \le \|\psi_1\| = 1,$$

whence it follows that $\psi_1(kx_2) = 0$. We can prove by the same way that $\psi_2(kx_1) = 0$. This means that $\psi_1 \ne \psi_2$ and $\psi_1(kx) = \psi_2(kx) = \theta(kx) = 1$. Denote

$$\alpha = I_{\Phi^*}(\varphi_-(k|x|)).$$

We have by the assumption that $0 < \alpha < 1$. Define two functionals $x_i^* = \xi_v + (1 - \alpha)\psi_i$ $(i = 1, 2)$, where $v = \varphi_-(k|x|)$ sgn x. Then $x_1^* \ne x_2^*$ and, in virtue of Theorem 1.1, we have x_1^*, x_2^* $\in$Grad (x). So, x is not a smooth point.

Suppose now that $I_{\Phi^*}(\varphi_-(k|x|)) < 1$ and b) holds. If additionally $\theta(kx) = 1$, then as we just proved, x is not a smooth point. So, we can assume that $\theta(kx) < 1$. But, in this case we can argue as in the proof of the sufficiency that any support functional at x is regular. This means by the Hahn-Banach theorem and Corollary 1.2 that there is $v \in l^{\Phi^*}$ such that

$$\varphi_-(k|x_i|) \le |v_i| \le \varphi(k|x_i|) \quad (\forall i \in I\!N)$$

and $I_{\Phi^*}(v) = 1$. Therefore, assuming that $I_{\Phi^*}(\varphi(k|x|)) \ne 1$, we have $I_{\Phi^*}(\varphi(k|x|)) > 1$. Since $K_\Phi(x) \ne \emptyset$ and A contains two different numbers $j, k \in I\!N$, we can find two sequences $v = (v_i)$ and $w = (w_i)$ of nonnegative numbers with $v_j \ne w_j, v_k \ne w_k$ and $v_i, w_i \in [\varphi_-(k|x_i|), \varphi(k|x_i|)]$ for every $i \in I\!N$ and $I_{\Phi^*}(v) = I_{\Phi^*}(w) = 1$. Then, the sequences $\widetilde{v} = (\widetilde{v}_i)$ and $\widetilde{w} = (\widetilde{w}_i)$ with $\widetilde{v}_i = v_i$ sgn x_i and $\widetilde{w}_i = w_i$ sgn x_i determine, by Corollary 1.2, two different support functionals at x. So, x is not a smooth point. This finishes the proof.

Theorem 2.2. *Let Φ be a finitely valued Orlicz function vanishing only at zero, smooth at zero and such that $\Phi(u)/u \to \infty$ as $u \to \infty$. Then:*

(i) h_0^Φ *is smooth if and only if Φ is smooth on the interval* $[0, \Pi_\Phi(1/2))$ *and* $\Phi^*(\varphi_-(\Pi_\Phi(1/2))) = \frac{1}{2}$;

(ii) l_0^Φ *is smooth if and only if: (a) Φ satisfies the Δ_2-condition at zero, (b) Φ is smooth on the interval* $[0, \Pi_\Phi(1/2))$ *and (c)* $\Phi^*(\varphi_-(\Pi_\Phi(1/2))) = \frac{1}{2}$.

Proof. If $\Phi \in \Delta_2$ at zero then $l_0^\Phi = h_0^\Phi$ and if $\Phi \notin \Delta_2$ at zero, then l_0^Φ is not order continuous, i.e. there is a sequence $(x_n), 0 \le x_n \searrow 0$ such that $\|x_n\|_\Phi^0 \ge \delta > 0$ for any $n \in I\!N$.. Consequently l_0^Φ cannot be smooth (see [6]). So, we only need to prove (i).

Sufficiency. Assume that Φ is smooth on the interval $[0, \Pi_\Phi(1/2))$ and $\Phi^*(\varphi_-(\Pi_\Phi(1/2)) = \frac{1}{2}$. Take an arbitrary $x \in S(h_0^\Phi)$ and define $k = \inf K_\Phi(x)$. If $I_{\Phi^*}(\varphi_-(k|x|)) = 1$, then in virtue of Theorem 2.1, x is a smooth point. So, consider the case when $I_{\Phi^*}(\varphi_-(k|x|)) < 1$. Then there is at most one index $j \in I\!N$ such that $\Phi^*(\varphi_-(k|x_j|)) \ge \frac{1}{2}$, i.e. for all $i \in I\!N, i \ne j$, we have $\Phi^*(\varphi_-(k|x_i|)) < \frac{1}{2}$. By the definition of the function Π_Φ it follows that φ is continuous at $k|x_i|$ for $i \ne j$, so in view of Theorem 2.1, x is a smooth point. By the arbitrariness of x in $S(h_0^\Phi)$ we conclude that h_0^Φ is smooth.

Necessity. Assume first that Φ is not smooth at some point $\alpha \in [0, \Pi_\Phi(1/2))$. Let $a = \frac{1}{2}(\varphi_-(\alpha + \varphi(\alpha))$. There is $u > 0$ and $s \in \partial\Phi(v)$ such that $2\Phi^*(a) + \Phi^*(s) = 1$. Define $x = (\alpha, \alpha, u, 0, \ldots)$. Then $I_{\Phi^*}(\varphi(|x|)) > 1$ and $I_{\Phi^*}(\varphi_-(|x|)) < 1$. This shows that $K_\Phi(x) = \{1\}$, whence $\|x\|_\Phi^0 = 1 + I_\Phi(x) > 1$. Defining $y = x/\|x\|_\Phi^0$, we have $\|y\|_\Phi^0 = 1$ and

$K_\Phi(y) = \|x\|_\Phi^0 K_\Phi(x) = \{\|x\|_\Phi^0\}$. In view of Theorem 2.1, y is not a smooth point, so h_0^Φ is not a smooth space.

Assume now that $\Phi^*(\varphi_-(\Pi_\Phi(1/2))) < \frac{1}{2}$. Then it follows by the definition of $\Pi_\Phi(1/2)$ that it is a point of discontinuity of φ. Take $s > 0$ such that $\Phi^*(\varphi(s)) > 0$ and

$$2\Phi^*(\varphi_-(\Pi_\Phi(1/2))) + \Phi^*(\varphi(s)) < 1 \tag{2.2}$$

and define

$$x = (\Pi_\Phi(1/2), \Pi_\Phi(1/2), s, 0, \dots).$$

We have by (2.2) that $I_{\Phi^*}(\varphi_-(x)) < 1$. Moreover, by the definition of $\Pi_\Phi(1/2)$, we get

$$\begin{aligned} I_{\Phi^*}(\varphi(x)) &= 2\Phi^*(\varphi(\Pi_\Phi(1/2))) + \Phi^*(\varphi(s)) \\ &\geq 1 + \Phi^*(\varphi(s)) > 1. \end{aligned}$$

So, we have $K_\Phi(x) = \{1\}$ and $\|x\|_\Phi^0 = 1 + I_\Phi(x) > 1$. Defining $y = x/\|x\|_\Phi^0$, we get $\|y\|_\Phi^0 = 1$ and $K_\Phi(y) = \{\|x\|_\Phi^0\}$. Hence, in virtue of Theorem 2.1, y is not a smooth point, and consequently h_0^Φ is not a smooth space. This finishes the proof.

3. GEOMETRY OF THE DUAL AND THE BIDUAL OF ORLICZ SPACES

Theorem 3.1. *For any finitely valued Orlicz function Φ when μ is nonatomic and any Orlicz function Φ when μ is the counting measure if Φ does not satisfy the suitable Δ_2-condition, then the dual spaces $(L^\Phi)^*, (l^\Phi)^*, (L_o^\Phi)^*$ and $(l_o^\Phi)^*$ are neither smooth nor non-square.*

Proof. Since the singular parts of the dual of Orlicz spaces for both the Luxemburg norms and the Orlicz norm are isometric (see [1]) and by the assumption that $\Phi \notin \Delta_2$ these parts are nontrivial we can restrict the proof to the Luxemburg norm only. Taking two functionals $\psi_2, \psi_2 \in S$ of norm 1 and with disjoint supports, we have by Lemma 6 in [4]

$$\|\psi_1 + \psi_2\| = \|\psi_1 - \psi_2\| = 2,$$

which means that $(L^\Phi)^*$ and $(l^\Phi)^*$ are not non-square. If $\Phi \notin \Delta_2$ then L^Φ, L_0^Φ (resp. l^Φ, l_0^Φ) contain an isomorphic copy of l^∞ (see [11], [20] and [30]), whence it follows that the duals $(L^\Phi)^*, (L_0^\Phi)^*$ (resp. $(l^\Phi)^*, (l_0^\Phi)^*$) contains isomorphic copies of $(l^\infty)^*$. Since it is well known that $(l^\infty)^*$ can not be renormed to be smooth, we get that our duals are not smooth.

We will also present here a short direct proof for the necessity of $\Phi \in \Delta_2$. If $\Phi \notin \Delta_2$ take $\psi_1, \psi_2 \in S$ with disjoint supports and $\|\psi_1\| = \|\psi_2\| = 1$. There exist $f_1, f_2 \in (L^\Phi)^{**}$ such that supp $f_1 \subset$supp ψ_2, supp $f_2 \subset$supp ψ_2 and $f_1(\psi_1) = f_2(\psi_2) = 1$. Then we have $f_1(\psi_2) = f_2(\psi_1) = 0$. Let $L = \text{Span}(\psi_1, \psi_2)$ and the functional f over L be defined by

$$f(\alpha\psi_1 + \beta\psi_2) = \alpha + \beta \quad (\forall \alpha, \beta \in I\!R).$$

Let us extend f by the Hahn-Banach theorem to the whole $(L^\Phi)^*$ in the norm preserving way, denoting this extension again by f. Since supp $\psi_1 \cap$ supp $\psi_2 = \emptyset$, we have by Lemma 6 in [4] that $\|\alpha\psi_1 + \beta\psi_2\| = |\alpha| + |\beta|$ for every $\alpha, \beta \in I\!R$. So, assuming that $\|\alpha\psi_1 + \beta\psi_2\| \leq 1$, we have $|f(\alpha\psi_1 + \beta\psi_2)| = |\alpha + \beta| \leq |\alpha| + |\beta| \leq 1$. Moreover $f(\psi_2) = 1$ and $\|\psi_1\| = 1$ whence $\|f\| = 1$. This means that ψ_1 is not a smooth point of $(L^\Phi)^*$.

As an immediate consequence of this theorem, we get the following result.

Theorem 3.2. *Under the assumptions of Theorem 3.1 for every geometric property A which implies smoothness or non-squareness we have that $(L^\Phi)^*, (L_0^\Phi)^*, (l^\Phi)^*, (l_0^\Phi)^*$ have property A if and only if $\Phi \in \Delta_2$ and $L_0^{\Phi^*}, L^{\Phi^*}, l_0^{\Phi^*}$ and l^{Φ^*} (respectively) have property A.*

Theorem 3.3. *Let Φ be an Orlicz function satisfying the assumptions of Theorem 3.1 and let L denote one among the Orlicz spaces $L^\Phi, L_o^\Phi, l^\Phi, l_o^\Phi$ and L^*, L^{**} denote respectively the dual and the bidual of L. Let A denote one among the properties $LUR, MLUR, HR, R$, H and S for both the Luxemburg and the Orlicz norms, and in the case of the Luxemburg norm also $URED, LUNSQ$ and NSQ. Assume in the case of the property S for the Orlicz norm that $\Phi(u)/u \to \infty$ as $u \to \infty$ and $\Phi(u)/u \to 0$ as $u \to 0_+$. Then the following assertions are equivalent:*

(i) *L^{**} has property A;*
(ii) *L has property A and L is reflexive;*
(iii) *L has property A and Φ^* satisfies the suitable Δ_2-condition.*

Proof. Consider first the case of the Luxemburg norm. If L^{**} is R then L is R as an isometric subspace of L^{**}, and so (see [10], [14], [21], [25], [32] and [33]) Φ satisfies there suitable Δ_2-condition. Thus, any among the properties $LUR, URED, MLUR$ and HR also implies the suitable Δ_2-condition for Φ. It is known that the H-property (see [7], [12] and [33]) and the property S (see [2], [3], [7], [15], [34] and Theorem 2.1) for L also implies the suitable Δ_2-condition for Φ. Moreover, if L^{**} is $LUNSQ$ then L^{**} is NSQ and consequently L is non-square. Since, we consider only the Luxemburg norm we also have the suitable Δ_2-condition for Φ (see [9]). So, in any case, property A for L^{**} implies the suitable Δ_2-condition for Φ. Hence, we have $L^{**} = (L_o^{\Phi^*})^*$ (resp. $(l_o^{\Phi^*})^*$). Since R of L^{**} implies S of L^*, we have that also everyone among the properties $LUR, URED, MLUR, HR$ for L^{**} implies S of $L^* = L_o^{\Phi^*}$ (resp. $l_o^{\Phi^*}$). But this implies the suitable Δ_2-condition for Φ^* (see [2], [3], [5], [7], [15], [33] and Theorem 2.1). Note that in virtue of Theorem 3.1, Φ^* satisfies the suitable Δ_2-condition if L^{**} is $LUNSQ$ or NSQ. This finishes the proof of the implication $(i) \Rightarrow (ii)$ for the Luxemburg norm, because reflexivity of L means exactly the suitable Δ_2-condition for Φ and Φ^*. The implication $(ii) \Rightarrow (iii)$ is obvious. Since property A for L implies the suitable Δ_2-condition for Φ^*, the implication $(iii) \Rightarrow (i)$ is also obvious.

Now, we will consider the case of the Orlicz norm. First we will prove that if L^{**} is R then Φ satisfies the suitable Δ_2-condition. In fact, if L^{**} is R then $L^{*}*$ is S. But, if Φ does not satisfy the suitable Δ_2-condition, then in virtue of Theorem 3.1, L^* is not smooth, a contradiction. This means that Φ satisfies the suitable Δ_2-condition. Consequently $L^* = L^{\Phi^*}$ (resp. l^{Φ^*}). Since L^* is smooth this implies that Φ^* satisfies the suitable Δ_2-condition (see [2], [3], [5], [7], [15], [28] and [33]).

Since $LUR \Rightarrow MLUR \Rightarrow R \Leftarrow URED$ and $LUR \Rightarrow HR$, any among these properties for L^{**} implies, by the above considerations, that both Φ and Φ^* satisfy the suitable Δ_2-condition. But then L is isometric to L^{**} and our theorem for the Orlicz norm follows if A means $LUR, URED, MLUR, HR$ or R.

Assume now that L^{**} is S. Then L is S and so (see above) Φ satisfies the suitable Δ_2-condition. Therefore $L^* = L^{\Phi^*}$ (resp. l^{Φ^*}). But, if Φ^* does not satisfy the suitable Δ_2-condition then, in view of Theorem 3.1, L^{**} is not smooth. Therefore, both Φ and Φ^* satisfy the suitable Δ_2-condition. This finishes the proof of the implication $(i) \Rightarrow (ii)$ in the case of the Orlicz norm. The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ follow in the same way as in the case of the Luxemburg norm.

Theorem 3.4. *Let Φ be an arbitrary Orlicz function in the case of the counting measure and a finitely valued Orliez function in the case of a non-atomic measure space. If we consider property S for the Orlicz norm, assume additionally that $\Phi(u)/u \to 0$ as $u \to 0$ and $\Phi(u)/u \to \infty$ as $u \to \infty$. Let L be one among the spaces E^Φ, E_o^Φ, h^Φ and h_o^Φ. Then:*

(i) *L^{**} is LUR, $MLUR$ or HR iff Φ and Φ^* satisfy the suitable Δ_2-condition and L is LUR, $MLUR$ or HR, respectively;*
(ii) *L^{**} is $URED, R, S, LUNSQ$ or NSQ iff Φ^* satisfies the suitable Δ_2-condition and L is $R, S, LUNSQ$ or NSQ, respectively.*

Proof. In (i) we need only to prove the necessity part. If Φ does not satisfy the suitable Δ_2-condition then L has no property among LUR, $MLUR, HR$ (see [4], [7], [12], [17] and [18], [22]). So, we need only to prove that Φ^* satisfies the suitable Δ_2-condition. But this can be proved in the same way as in Theorem 1.7. Also (ii) can be proved as in Theorem 3.3.

Remark 3.5. *If Φ is an arbitrary Orlicz function in the case of the counting measure and a finitely valued Orlicz function in the case if the nonatomic measure space then $(E^\Phi)^{**}$ (resp. $(h^\Phi)^{**}$ is $LUNSQ$ iff both functions Φ and Φ^* satisfy the suitable Δ_2-condition and E^Φ (resp. h^Φ) is $LUNSQ$.*

Proof. This follows by the proof of Theorem 3.3 and by the fact that the suitable Δ_2-condition is necessary in order that E^Φ (resp. h^Φ) be $LUNSQ$ (see [9]).

Remark 3.6. *By the results of this section we can easily get criteria for the geometric properties which occour in Theorems 3.2, 3.3, 3.4 and Remark 3.5 for heigher order duals of Orlicz spaces. For example, denoting the Orlicz spaces by L we get among others:*

(i) *L^{***} is rotund if and only if L is smooth and both Φ and Φ^* satisfy the suitable Δ_2-condition;*
(ii) *L^{***} is smooth if and only if L is rotund and both Φ and Φ^* satisfy the suitable Δ_2-condition.*

Recall that a theorem of Dixmier states that a rotund fourth dual is reflexive. It follows from our results that the duals of Orlicz spaces of order $k \geq 2$ which are rotund or smooth are reflexive.

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