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Duality for Weights on Covariant Systems and its Applications

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Mathematics

by

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ABSTRACT OF THE DISSERTATION

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Professor Masamichi Takesaki, Chairman

The main objectives of this paper are to establish the duality theory for weights on covariant systems in its full generality and to apply this theory on the study of crossed products.

With each faithful, normal, semi-finite weight on a covariant system we associate a "dual" weight on the corresponding crossed product. With the aid of the modular objects associated with this dual weight we obtain the commutation theorem for crossed products, which in turn yields a description of the crossed product as the fixed point algebra of a certain action of the given group. When the group is abelian, this fixed point algebra description is further refined into giving a Galois correspondence between closed subgroups of the dual group and certain subalgebras of the crossed product. Also, when the group is abelian, the relation between the second dual weight and a naturally associated tensor product weight is determined, yielding the socalled "twisted" Plancherel theorem for weights on covariant systems.

Introduction

Crossed products of W^*-algebras by locally compact groups have received increasing attention over the last years as a means of expressing given W^*-algebras in terms of simpler objects. Among important results in this direction we mention: every III_λ-factor, $0 \leq \lambda < 1$, is the crossed product of a type II_∞ W^*-algebra by a single automorphism (Connes [5]); every properly infinite W^*-algebra is the crossed product of a semi-finite W^*-algebra by a 1-parameter group (Takesaki [15]). It is even hoped that any factor can be obtained by a modified version of the usual group measure space construction (that is, with the introduction of an additional 2-cocycle). The group measure space construction, a special case of the crossed product construction, was also used already by Murray and von Neumann to show the existence of non type I factors.

The crossed product construction is based on the notion of a covariant system, that is, a triple $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ of a W^*-algebra $\mathfrak{M}$, a locally compact group $\mathfrak{G}$ and a homomorphism $\alpha : \mathfrak{G} \to \mathrm{Aut}(\mathfrak{M})$, the latter being continuous with respect to the pointwise σ-strong* topology in $\mathrm{Aut}(\mathfrak{M})$ = the automorphism group of $\mathfrak{M}$. If $\mathfrak{M}$ is realized as a von Neumann algebra on a Hilbert space $\mathfrak{H}$, the crossed product $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ is realized on the Hilbert space $L^2(\mathfrak{G}; \mathfrak{H})$ and may be viewed as a generalization of the left regular von Neumann algebra of $\mathfrak{G}$ (see §1). An analogous notion of a C^*-covariant system and C^*-crossed product also exists, but we shall be solely concerned with the W^*-version here.

For the study of W^*-algebras in general, the positive linear functionals, or, more generally, the weights, together with their associated modular objects, have proved to be efficient tools. For example, the algebraic invariants $S(\mathfrak{M})$ and $T(\mathfrak{M})$ of Connes [5] are defined in terms of such objects. For the analysis of the type and structure of crossed products it is therefore of importance to be able to exhibit weights, along with their modular objects, on these algebras. In the first part of this paper we show that given a covariant system $\{\mathfrak{M},\mathfrak{G},\alpha\}$ and a f.n.s.f. (= faithful, normal, semi-finite) weight φ on $\mathfrak{M}$, there is a natural way to obtain a f.n.s.f. weight $\tilde{\varphi}$ on $\mathfrak{M} \otimes_\alpha \mathfrak{G}$, the socalled dual weight of φ. The modular objects of $\tilde{\varphi}$ are expressed in terms of the relative modular objects associated with φ and its translates $\varphi \circ \alpha_g$, $g \in \mathfrak{G}$. This detailed knowledge of the dual weight helps us gain insight into the nature of the crossed products itself. First of all, with the aid of the unitary involution associated with $\tilde{\varphi}$, we obtain the commutation theorem for crossed products, which, in the case of a covariant representation $\{\Phi,\Gamma\}$ of $\{\mathfrak{M},\mathfrak{G},\alpha\}$, says that the commutant of the crossed product is canonically isomorphic to the (right-handed) crossed product of the commutant $\Phi(\mathfrak{M})'$ of $\Phi(\mathfrak{M})$ by the action $g \mapsto \mathrm{Ad}(\Gamma(g))$ of $\mathfrak{G}$ on $\Phi(\mathfrak{M})'$; in general, the commutant of the crossed product is a reduced algebra of such a (right-handed) crossed product (Theorem 3.14 and Remark 3.15). This in turn allows us to identify the crossed product itself with the fixed point algebra of the action $g \mapsto \alpha_g \otimes \mathrm{Ad}(\rho(g))$ of $\mathfrak{G}$ on $\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))$, where ρ is the right regular representation of $\mathfrak{G}$ (Theorem 5.1). When the group $\mathfrak{G}$ is abelian, a Galois type theorem for the correspondence between subgroups of $\hat{\mathfrak{G}}$ and certain subalgebras of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ results (Theorem 5.3). - The formula for the modular automorphism group of the dual weight also gives a sufficient condition for the semi-finiteness of the crossed product (Remark 3.16).

When the group $\mathfrak{G}$ is abelian, an algebraic duality theory for crossed products exists which identifies, in a canonical manner, the second crossed product $(\mathfrak{M} \otimes_\alpha \mathfrak{G}) \otimes_{\hat{\alpha}} \hat{\mathfrak{G}}$, i.e., the crossed product of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ by the dual action $\hat{\alpha}$ of $\hat{\mathfrak{G}}$ on $\mathfrak{M} \otimes_\alpha \mathfrak{G}$, with the tensor product $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$ (see [15; §4]). It is therefore natural to compare the second dual weight $\tilde{\tilde{\varphi}}$ with the tensor product weight $\varphi \otimes \mathrm{Trace}$ on $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$. It turns out that the Radon-Nikodym derivative $(D\tilde{\tilde{\varphi}} : D(\varphi \otimes \mathrm{Trace}))$ is a direct integral of the individual Radon-Nikodym derivatives $(D(\varphi \circ \alpha_g) : D\varphi)$ (Theorem 4.2). This may be viewed upon as a kind of a twisted Plancherel theorem for weights on covariant systems, the twist being measured by the cocycles $(D(\varphi \circ \alpha_g) : D\varphi)$. - As a consequence, the weights $\tilde{\tilde{\varphi}}$ and $\varphi \otimes \mathrm{Trace}$ commute in the sense of [17] if and only if φ and $\varphi \circ \alpha_g$ commute for all $g \in \mathfrak{G}$ (Corollary 4.3).

The results in this article extend some recent results by Takesaki in [15], where the dual weight construction was carried out for a socalled relatively invariant weight, that is, a weight φ such that for each $g \in \mathfrak{G}$ there is a positive number $\chi(g)$ with $\varphi \circ \alpha_g = \chi(g)\varphi$. When this condition is removed, an application of A. Connes' relative modular theory for weights becomes necessary; in particular, we need to perform a spatial analysis of the relative modular objects (see §2).

Some of the results in this article have been announced in [7]. Recently A. Connes and M. Takesaki, among others, have obtained some further applications of the above mentioned results (works to appear).

The organization of the paper is as follows. In §1 we collect some basic facts and results, in particular from the theory of weights and left Hilbert algebras. In §2 we develop the relative modular theory for weights within the spatial framework and provide a spatial analysis of the relative modular objects. Some additional results of general character are also furnished.

In §3 we construct the dual weight. This is done by first constructing a left Hilbert algebra which is dual to that of the given weight in a certain sense, and whose left von Neumann algebra coincides with the crossed product. The dual weight is then defined to be the canonical weight associated with this left Hilbert algebra. The modular objects of the dual weight $\tilde{\varphi}$ are expressed in terms of the modular and relative modular objects associated with the given weight φ and its translates $\varphi \circ \alpha_g$, $g \in \mathfrak{G}$. As an application the commutation theorem for crossed products is shown.

In §4 we consider the case when the group $\mathfrak{G}$ is abelian and investigate the relationship between the second dual weight $\tilde{\tilde{\varphi}}$ and the tensor product weight $\varphi \otimes \mathrm{Trace}$ on $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$. The socalled "twisted Plancherel theorem" for weights results.

Finally, in §5 we apply the commutation theorem from §3 to obtain the description of the crossed product as the fixed point algebra of the action $g \mapsto \alpha_g \otimes \mathrm{Ad}(\rho(g))$ of $\mathfrak{G}$ on $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$ and next, when the group $\mathfrak{G}$ is abelian, to show the Galois type correspondence between closed subgroups of $\hat{\mathfrak{G}}$ and certain subalgebras of $\mathfrak{M} \otimes_{\alpha} \mathfrak{G}$.

A note on notation.

The symbols $\mathbb{R}$, $\mathbb{C}$ stand for the real and complex numbers, respectively. For a locally compact group $\mathfrak{G}$ we use $\mathcal{K}(\mathfrak{G})$ for the space of continuous functions with compact support and δ = modular function of $\mathfrak{G}$. All other notation and terminology is either explained in the text or conforms with standard usage (as set forth in, for instance, [5], [8], [14], [15]).

§1. Preliminaries

In this section we recall some facts from the theory of weights and left Hilbert algebras, give the basic definitions in connection with dynamical systems and provide some lemmas concerning vector-valued integration. Some notation will also be fixed. General references for this section are [1], [2], [3], [4], [8], [14], [15].

A weight on a W^*-algebra $\mathfrak{M}$ is a mapping $\varphi : \mathfrak{M}^+ \to [0,+\infty]$ such that $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(\lambda x) = \lambda\varphi(x)$ for $x,y \in \mathfrak{M}^+$, $\lambda \geq 0$ (with the convention that $0\cdot\infty = 0$). We denote by $\mathfrak{N}_\varphi$ the left ideal $\{x \in \mathfrak{M}; \varphi(x^*x) < \infty\}$ of $\mathfrak{M}$ and set $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^* \cdot \mathfrak{N}_\varphi$. $\mathfrak{M}_\varphi$ is then the complex linear span of the set $\{x \in \mathfrak{M}^+; \varphi(x) < \infty\}$ (see [4]; p. 50). The extension of $\varphi/\mathfrak{M}_\varphi^+$ to a linear functional on $\mathfrak{M}_\varphi$ will also be denoted by φ.

φ is said to be:

(a) <u>Semi-finite</u> if $\mathfrak{M}_\varphi$ (or equivalently $\mathfrak{N}_\varphi$) is σ-weakly dense in $\mathfrak{M}$.

(b) <u>Faithful</u> if $x \in \mathfrak{M}^+$ and $x \neq 0$ imply $\varphi(x) > 0$.

(c) <u>Normal</u> if there is a family $\{\omega_i\}$ of elements from $\mathfrak{M}_*^+$ such that $\varphi(x) = \sup \omega_i(x)$, $x \in \mathfrak{M}^+$.

If φ is normal the family $\{\omega_i\}$ in (c) may be taken to be an increasing net so that $\omega_i(x) \uparrow \varphi(x)$, $x \in \mathfrak{M}_+$ ([3; lemma 1.9]).

Let φ be a f.n.s.f. (= faithful, normal, semi-finite) weight on $\mathfrak{M}$. The mapping $(x,y) \in \mathfrak{N}_\varphi \times \mathfrak{N}_\varphi \mapsto \varphi(y^*x) \in \mathbb{C}$ is an innerproduct on $\mathfrak{N}_\varphi$, and the Hilbert space completion of $\mathfrak{N}_\varphi$ under this inner-product will be denoted by $\mathfrak{H}_\varphi$. Letting $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathfrak{H}_\varphi$ be the canonical injection and denoting the innerproduct in $\mathfrak{H}_\varphi$ by $\langle\cdot,\cdot\rangle$, we have then: $\langle\Lambda_\varphi(x),\Lambda_\varphi(y)\rangle = \varphi(y^*x)$, $x,y \in \mathfrak{N}_\varphi$. A faithful normal representation π_φ of $\mathfrak{M}$ on $\mathfrak{H}_\varphi$ is obtained by defining $\pi_\varphi(x)\Lambda_\varphi(y) = \Lambda_\varphi(xy)$; $x \in \mathfrak{M}$, $y \in \mathfrak{N}_\varphi$. The linear subspace $\mathfrak{A}_\varphi = \Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi)$ of $\mathfrak{H}_\varphi$ becomes a $\#$- algebra under the operations $\Lambda_\varphi(x)\cdot\Lambda_\varphi(y) = \Lambda_\varphi(xy)$, $\Lambda_\varphi(x)^\# = \Lambda_\varphi(x^*)$, $x,y \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi$. Furthermore, the conjugate linear operator: $\Lambda_\varphi(x) \mapsto \Lambda_\varphi(x^*)$, $x \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi$, is pre-closed in $\mathfrak{H}_\varphi$, and $\mathfrak{A}_\varphi$ turns out to be a full (achieved) left Hilbert algebra with these operations and we have $\mathfrak{R}_\ell(\mathfrak{A}_\varphi) = \pi_\varphi(\mathfrak{M})$ ($\mathfrak{R}_\ell(\mathfrak{A})$: left-regular von Neumann algebra of the left Hilbert algebra $\mathfrak{A}$). The closure of the operator $\xi \in \mathfrak{A} \mapsto \xi^\# \in \mathfrak{A}$ in $\mathfrak{H}_\varphi$ will be denoted by S_φ, and we let $S_\varphi = J_\varphi\Delta_\varphi^{1/2}$ be its polar decomposition, where J_φ is a conjugate linear unitary involution ($J_\varphi^2 = I$) and Δ_φ is positive, selfadjoint, non-singular. We have $J_\varphi\Delta_\varphi J_\varphi = \Delta_\varphi^{-1}$, $J_\varphi\pi_\varphi(\mathfrak{M})J_\varphi = \pi_\varphi(\mathfrak{M})'$ and $J_\varphi\pi_\varphi(x)J_\varphi = \pi_\varphi(x)^*$ for $x \in Z(\mathfrak{M})$, thus the representation π_φ of $\mathfrak{M}$ is standard. Also, $\Delta_\varphi^{it}\pi_\varphi(\mathfrak{M})\Delta_\varphi^{-it} = \pi_\varphi(\mathfrak{M})$ for all $t \in \mathbb{R}$, hence a continuous one-parameter group of automorphisms $\{\sigma_t^\varphi\}$ on $\mathfrak{M}$ is obtained by setting $\pi_\varphi(\sigma_t^\varphi(x)) = \Delta_\varphi^{it}\pi_\varphi(x)\Delta_\varphi^{-it}$. We have $\varphi(\sigma_t^\varphi(x)) = \varphi(x)$, $x \in \mathfrak{M}^+$, and $\{\sigma_t^\varphi\}$ satisfies the KMS*-condition with respect to φ, that is:

> for $x,y \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi$ there is a complex function $F = F_{x,y}$ defined on the strip $B = \{z \in \mathbb{C};\ 0 \leq \text{Im } z \leq 1\}$ which is analytic in the interior of B and continuous and bounded on all of B such that: $F(t) = \varphi(\sigma_t^\varphi(x)y)$

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*KMS stands for Kubo-Martin-Schwinger, after the physicists who first introduced this condition in theoretical physics.

and

$$F(t + i) = \varphi(y\sigma_t^{\varphi}(x)) \ .$$

A function such as F above will be referred to as a KMS-function and the set B will be called "the strip." The one-parameter automorphism group $\{\sigma_t^{\varphi}\}$ is called the modular automorphism group of φ at it is characterized by the two above conditions: namely, if $\{\beta_t\}$ is a continuous 1-parameter automorphism group of $\mathfrak{M}$ such that $\varphi(\beta_t(x)) = \varphi(x)$ for $x \in \mathfrak{M}^+$ and if $\{\beta_t\}$ satisfies the KMS-condition with respect to φ, then $\beta_t = \sigma_t^{\varphi}$, $t \in \mathbb{R}$ (the continuity of $t \mapsto \sigma_t^{\varphi}$ refers to the pointwise σ-strong* topology in $\mathrm{Aut}(\mathfrak{M})$ = the group of $*$-automorphisms of $\mathfrak{M}$, namely: the function $t \in \mathbb{R} \mapsto \sigma_t^{\varphi}(x) \in \mathfrak{M}$ is σ-strong* continuous for all $x \in \mathfrak{M}$).

Conversely, a full left Hilbert algebra $\mathfrak{A}$ gives rise to a f.n.s.f weight $\varphi_{\mathfrak{A}}$ on $\mathcal{R}_{\ell}(\mathfrak{A})$ by, for $x \in \mathcal{R}_{\ell}(\mathfrak{A})^+$, defining $\varphi_{\mathfrak{A}}(x) = \|\xi\|^2$ if $x = \pi^{\ell}(\xi)^*\pi^{\ell}(\xi)$ for some $\xi \in \mathfrak{A}$ and $\varphi_{\mathfrak{A}}(x) = +\infty$ otherwise. (Here and elsewhere π^{ℓ} denotes the left regular representation of $\mathfrak{A}$ on the completion of $\mathfrak{A}$, viz. $\pi^{\ell}(\xi)\eta = \xi\eta$ for $\xi, \eta \in \mathfrak{A}$. If $\mathfrak{A} = \mathfrak{A}_{\varphi}$ for some weight φ we will also sometimes write π_{φ}^{ℓ} for π^{ℓ}, thus $\pi_{\varphi} = \pi_{\varphi}^{\ell} \circ \Lambda_{\varphi}$ on $\mathfrak{N}_{\varphi}$). Taking $\mathfrak{A}$ to be the $\mathfrak{A}_{\varphi}$ above we have: $\varphi_{\mathfrak{A}} \circ \pi_{\varphi} = \varphi$ and $\pi^{\ell}(\mathfrak{A}_{\varphi}) = \pi_{\varphi}(\mathfrak{N}_{\varphi}^* \cap \mathfrak{N}_{\varphi}) = \mathfrak{N}_{\varphi_{\mathfrak{A}}}^* \cap \mathfrak{N}_{\varphi_{\mathfrak{A}}}$. Thus we have a canonical one-to-one correspondence between the f.n.s.f. weights and the full left Hilbert algebras associated with a W^*-algebra $\mathfrak{M}$.

In what follows we shall make frequent use of vector valued integration. As regards the notions of measurability and integrals of vector valued functions, we use the definitions of Bourbaki ([1; IV, §5, def. 1] and [2; VI, §1, no. 1]). If $\mathfrak{M}$ has separable predual, the Hilbert space $\mathfrak{H}_\varphi$ of a f.n.s.f. weight φ is separable. Hence if $\mathfrak{G}$ is a locally compact group, a function $\xi : \mathfrak{G} \to \mathfrak{H}_\varphi$ is measurable if and only if the functions $g \in \mathfrak{G} \mapsto \langle \xi(g), \eta \rangle$ are measurable for all $\eta \in \mathfrak{H}_\varphi$. We denote by $L^p(\mathfrak{G}; \mathfrak{H}_\varphi)$ the Banach space of all measurable functions $\xi : \mathfrak{G} \to \mathfrak{H}_\varphi$ such that $\int_{\mathfrak{G}} \|\xi(g)\|^p \, dg < \infty$, with the norm $\|\xi\|_p = (\int_{\mathfrak{G}} \|\xi(g)\|^p \, dg)^{1/p}$. The space $L^2(\mathfrak{G}; \mathfrak{H}_\varphi)$, endowed with its usual operations, is then a Hilbert space canonically isomorphic to $\mathfrak{H}_\varphi \otimes L^2(\mathfrak{G})$ under the map:

$$\xi \otimes F \in \mathfrak{H}_\varphi \otimes L^2(\mathfrak{G}) \mapsto F \cdot \xi \in L^2(\mathfrak{G}; \mathfrak{H}_\varphi) ,$$

where $(F \cdot \xi)(g) = F(g)\xi$, $F \in L^2(\mathfrak{G})$, $\xi \in \mathfrak{H}_\varphi$.

An element $\xi \in L^1(\mathfrak{G}; \mathfrak{H}_\varphi)$ is integrable in the sense of [2; VI, §1, no. 1] and the value $\int_{\mathfrak{G}} \xi(g)dg$ lies in $\mathfrak{H}_\varphi$. If A is a bounded linear operator from $\mathfrak{H}_\varphi$ into a locally convex space E, the function $g \in \mathfrak{G} \mapsto A\xi(g) \in E$ is integrable for every $\xi \in L^1(\mathfrak{G}; \mathfrak{H}_\varphi)$ and $A(\int_{\mathfrak{G}} \xi(g)dg) = \int_{\mathfrak{G}} A\xi(g)dg$.

A covariant system is a triple $(\mathfrak{M}, \mathfrak{G}, \alpha)$ where $\mathfrak{M}$ is a W^*-algebra, $\mathfrak{G}$ a locally compact group and $\alpha : \mathfrak{G} \to \mathrm{Aut}(\mathfrak{M})$ a continuous homomorphism, $\mathrm{Aut}(\mathfrak{M})$ being equipped with the pointwise σ-strong* topology. Given a covariant system $(\mathfrak{M}, \mathfrak{G}, \alpha)$ and a faithful normal representation π of $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$, the crossed product $\mathfrak{R}(\mathfrak{M}, \alpha, \pi)$ of $\mathfrak{M}$ by α, based on the representation π, is defined to be the von Neumann algebra on $L^2(\mathfrak{G}; \mathfrak{H})$ generated by the set $\{\pi_\alpha(a), \tilde{\lambda}(g); a \in \mathfrak{M}, g \in \mathfrak{G}\}$ of operators where π_α and $\tilde{\lambda}$ are given by:

$$(\pi_\alpha(a)\xi)(g) = \pi(\alpha_g^{-1}(a))\xi(g)$$

$$(\tilde{\lambda}(h)\xi)(g) = \xi(h^{-1}g), \quad \xi \in L^2(\mathfrak{G}; \mathfrak{H}),\ a \in \mathfrak{M},\ g,h \in \mathfrak{G}\ .$$

If another faithful normal representation π_1 of $\mathfrak{M}$ had been taken, the resulting crossed product $\mathfrak{R}(\mathfrak{M},\alpha,\pi_1)$ would be isomorphic to $\mathfrak{R}(\mathfrak{M},\alpha,\pi)$ (see [13; Prop. 3.4]). Thus we have a well defined crossed product of the abstract W^*-algebra $\mathfrak{M}$ by α, namely the W^*-isomorphism class of all the $\mathfrak{R}(\mathfrak{M},\alpha,\pi)$, and this will be denoted by $\mathfrak{R}(\mathfrak{M},\alpha)$ or by $\mathfrak{M} \otimes_\alpha \mathfrak{G}$. The representation of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ on $L^2(\mathfrak{G}; \mathfrak{H})$ corresponding to the representation π of $\mathfrak{M}$ on $\mathfrak{H}$ will be denoted by $\tilde{\pi}$, thus $\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G}) = \mathfrak{R}(\mathfrak{M},\alpha,\pi)$.

A covariant representation of the covariant system $\{\mathfrak{M},\mathfrak{G},\alpha\}$ is a pair (π,U) where π is a normal representation of $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$ and U is a strongly continuous unitary representation of $\mathfrak{G}$ on $\mathfrak{H}$ such that $\pi(\alpha_g(x)) = U_g\pi(x)U_g^*$, $x \in \mathfrak{M}$, $g \in \mathfrak{G}$. Under the canonical isomorphism between $L^2(\mathfrak{G}; \mathfrak{H})$ and $\mathfrak{H} \otimes L^2(\mathfrak{G})$ the operator $\tilde{\lambda}(h)$ transforms into $I \otimes \lambda(h)$, where λ is the left regular representation of $\mathfrak{G}$ on $L^2(\mathfrak{G})$; hence $h \mapsto \tilde{\lambda}(h)$ is a strongly continuous representation of $\mathfrak{G}$ on $L^2(\mathfrak{G}; \mathfrak{H})$. Also, π_α is a faithful normal representation of $\mathfrak{M}$ on $L^2(\mathfrak{G}; \mathfrak{H})$; the faithfulness is straightforward, and the normality is seen as follows:

Let $\{a_i\}$ be a bounded increasing net from $\mathfrak{M}^+$ such that $\lim a_i = a \in \mathfrak{M}^+$ and let $\xi : \mathfrak{G} \to \mathfrak{H}$ be a continuous function with compact support. Setting $F_i(g) = \langle \pi(\alpha_g^{-1}(a_i))\xi(g), \xi(g)\rangle$ the family

$\{F_i\}$ is an increasing net of positive continuous functions, hence by [1; Ch. IV, §1, Th. 1]:

$$\langle \pi_\alpha(a)\xi,\xi\rangle = \int_{\mathfrak{G}} \langle \pi(\alpha_g^{-1}(a))\xi(g),\xi(g)\rangle dg$$

$$= \lim_i \int_{\mathfrak{G}} \langle \pi(\alpha_g^{-1}(a_i))\xi(g),\xi(g)\rangle dg$$

$$= \lim_i \ \langle \pi_\alpha(a_i)\xi,\xi\rangle \ .$$

The boundedness of the a_i's then implies $\langle \pi_\alpha(a)\xi,\xi\rangle = \lim_i \langle \pi_\alpha(a_i)\xi,\xi\rangle$ for all $\xi \in L^2(\mathfrak{G};\, \mathfrak{H})$, proving the assertion. A simple computation shows that

$$\pi_\alpha(\alpha_g(a)) = \tilde{\lambda}(g)\pi_\alpha(a)\tilde{\lambda}(g)^* \ , \tag{1.1}$$

hence $(\pi_\alpha, \tilde{\lambda})$ is a covariant representation of the dynamical system $\{\mathfrak{M},\mathfrak{G},\alpha\}$ on $L^2(\mathfrak{G};\, \mathfrak{H})$.

Unless otherwise stated we shall consider our W^*-algebra $\mathfrak{M}$ to be equipped with the σ-strong* topology throughout this article. For the convenience of the reader we recall that this topology is the locally convex topology on $\mathfrak{M}$ generated by the set $\{p_\psi;\ \psi \in \mathfrak{M}_*^+\}$ of seminorms, where $p_\psi(a) = (\psi(a^*a) + \psi(aa^*))^{1/2}$, $a \in \mathfrak{M}$. σ-strong* bounded sets are norm bounded, and if $\mathfrak{M}$ is countably decomposable, bounded subsets are σ-strong* metrizable. Furthermore, the $*$-operation is σ-strong* continuous and multiplication is σ-strong* continuous on bounded sets. Hence $\mathcal{K}(\mathfrak{G};\, \mathfrak{M})$, the set of σ-strong* continuous functions from $\mathfrak{G}$ to $\mathfrak{M}$ with compact support, is a $*$-algebra under pointwise operations. The σ-strong* uniform structure on $\mathfrak{M}$ is quasicomplete, i.e. bounded subsets are complete, hence for $x \in \mathcal{K}(\mathfrak{G}; \mathfrak{M})$ we have $\int_{\mathfrak{G}} x(g)dg \in \mathfrak{M}$. The usual laws for commuting continuous linear maps under the integral sign are valid; also, if $p : \mathfrak{M} \to \mathbb{R}^+$ is a σ-strong* lower semicontinuous seminorm we have $p(\int_{\mathfrak{G}} x(g)dg) \leq \int_{\mathfrak{G}} p(x(g))dg$, in particular $\|\int_{\mathfrak{G}} x(g)dg\| \leq \int \|x(g)\|dg$, $x \in \mathcal{K}(\mathfrak{G}; \mathfrak{M})$ (ref: [1], [2] and [3]).

Remark 1.1. Returning to our dynamical system $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ we note that if $x \in \mathcal{K}(\mathfrak{G}; \mathfrak{M})$, then so is the function $g \mapsto \alpha_g(x(g))$; this is immediate from the relation (1.1) and the faithfulness and normality of π_α.

The following polarization identity, valid in any *-algebra $\mathfrak{B}$, will be used repeatedly:

$$b^*a = \frac{1}{4} \sum_{k=0}^{3} i^k(a + i^k b)^*(a + i^k b), \quad a,b \in \mathfrak{B} .$$

For the sake of reference we also prove the following:

Lemma 1.2. Let $\mathcal{B}(\mathfrak{M})^+$ denote the cone of positive σ-weakly continuous linear operators on $\mathfrak{M}$, endowed with the pointwise σ-weak topology, and let $\mathfrak{G}$ be a locally compact group. Further let $b : \mathfrak{G} \to \mathcal{B}(\mathfrak{M})^+$ be a continuous mapping such that $g \mapsto \|b(g)\|$ is bounded. Then for any bounded Radon-measure μ on $\mathfrak{G}$ the linear map: $a \in \mathfrak{M} \mapsto \int_{\mathfrak{G}} b_g(a)d\mu(g) \in \mathfrak{M}$ is σ-weakly continuous.

Proof. We may assume μ is positive, hence it suffices to show that the map is normal so let $\{a_i\}$ be a bounded, increasing net from

$\mathfrak{M}^+$ such that $\lim a_i = a \in \mathfrak{M}^+$ and let $\psi \in \mathfrak{M}_*^+$. Setting $F_i(g) = \psi(b_g(a_i))$ the family $\{F_i\}$ is an increasing net of positive continuous functions on $\mathfrak{G}$, and by the boundedness condition on b the functions $g \mapsto b_g(a_i)$ are μ-integrable; hence by [1; Ch. IV, §1, Th. 1] again:

$$\psi\left(\int_{\mathfrak{G}} b_g(a)d\mu(g)\right) = \int_{\mathfrak{G}} \psi(b_g(a))d\mu(g) = \int_{\mathfrak{G}} \lim_i \psi(b_g(a_i))d\mu(g)$$

$$= \lim \psi\left(\int_{\mathfrak{G}} b_g(a_i)d\mu(g)\right) . \qquad \text{Q.E.D.}$$

The next lemma gives a link between integration with respect to various topologies on $\mathfrak{M}$.

<u>Lemma 1.3</u>. Let φ be a f.n.s.f. weight on $\mathfrak{M}$ and assume $\mathfrak{M}$ has separable predual. Further let $x \in \mathcal{K}(\mathfrak{G}; \mathfrak{M})$ be such that $x(g) \in \mathfrak{N}_\varphi$, $g \in \mathfrak{G}$. Then the function $g \in \mathfrak{G} \mapsto \Lambda_\varphi(x(g)) \in \mathcal{H}_\varphi$ is measurable. If in addition the function $g \mapsto \varphi(x(g)^*x(g))^{1/2}$ is integrable, we have $\int_{\mathfrak{G}} x(g)dg \in \mathfrak{N}_\varphi$, and $\Lambda_\varphi(\int_{\mathfrak{G}} x(g)dg) = \int_{\mathfrak{G}} \Lambda_\varphi(x(g))dg$.

<u>Proof</u>. Since $\mathcal{H}_\varphi$ is separable it suffices to show that the functions $g \mapsto \langle\Lambda_\varphi(x(g)),\xi\rangle$ are measurable for all $\xi \in \mathcal{H}_\varphi$. Now, if $a \in \mathfrak{N}_\varphi$, the function $g \mapsto \langle\Lambda_\varphi(x(g)),\Lambda_\varphi(a)\rangle = \varphi(a^*x(g))$ is, by the polarization identity and the normality of φ, a linear combination of lower semicontinuous functions, hence measurable. By taking a sequence $\{a_n\}$ from $\mathfrak{N}_\varphi$ such that $\Lambda_\varphi(a_n) \mapsto \xi$, the first assertion follows. If in addition the function $g \mapsto \varphi(x(g)^*x(g))^{1/2}$ is integrable and setting $a = \int_{\mathfrak{G}} x(g)dg$, we have:

$$a^*a = \int_{\mathfrak{G}} x(g)^* dg \int_{\mathfrak{G}} x(h)dh = \int_{\mathfrak{G}\times\mathfrak{G}} x(g)^* x(h)dgdh ,$$

thus for any $\omega \in \mathfrak{M}_*^+$:

$$\omega(a^*a) = \int_{\mathfrak{G}\times\mathfrak{G}} \omega(x(g)^* x(h))dgdh$$

$$\leq \left[\int_{\mathfrak{G}} \omega(x(g)^* x(g))^{1/2} dg\right]^2 .$$

Letting $\{\omega_i\}$ be an increasing net from $\mathfrak{M}_*^+$ such that $\varphi(x) = \lim \omega_i(x)$, $x \in \mathfrak{M}^+$, we get:

$$\varphi(a^*a) = \lim_i \omega_i(a^*a) \leq \lim_i \left[\int_{\mathfrak{G}} \omega_i(x(g)^* x(g))^{1/2} dg\right]^2$$

$$= \left[\int \lim_i \omega_i(x(g)^* x(g))^{1/2} dg\right]^2$$

$$= \left[\int_{\mathfrak{G}} \varphi(x(g)^* x(g))^{1/2}\right]^2 < \infty ,$$

proving the second assertion.

As for the last assertion, let $b \in \mathfrak{N}_\varphi$. By the polarization identity we may assume $b \cdot x(g) \in \mathfrak{M}^+$ for all $g \in \mathfrak{G}$. Then:

$$\left\langle \Lambda_\varphi\left(\int_{\mathfrak{G}} x(g)dg\right), \Lambda_\varphi(b)\right\rangle = \varphi\left(b^*\left(\int_{\mathfrak{G}} x(g)dy\right)\right) = \varphi\left(\int b^* x(g)dg\right)$$

$$= \int_{\mathfrak{G}} \lim_i \omega_i(b^* x(g))dg = \int \varphi(b^* x(g))dg$$

$$= \int_{\mathfrak{G}} \langle \Lambda_\varphi(x(g)), \Lambda_\varphi(b) \rangle dg$$

$$= \left\langle \int_{\mathfrak{G}} \Lambda_\varphi(x(g)) dg, \Lambda_\varphi(b) \right\rangle.$$

This holding for all $b \in \mathfrak{N}_\varphi$, the last assertion follows. Q.E.D.

§2. Relative modular theory

In this section we give an account of the relative modular theory for weights (see Connes [5, §1]) from the spatial point of view, and prove some additional results needed for the construction of the dual weight. The spatial approach roughly means that we string out Connes' 2×2-matrices to 4-vectors in the associated Hilbert space, thus obtaining the relative modular objects as elements of 4×4-matrices over this space. This allows us to arrive more directly at the results needed for our purposes. For the state case, Connes himself gave a spatial interpretation of the relative modular objects in [6].

We start by recalling the definition of the "mixed" weight (or the sum-weight) of two f.n.s.f. weights on a W^*-algebra (see [5; §I]):

<u>Definition 2.1.</u> For two f.n.s.f. weights φ and ψ on a W^*-algebra $\mathfrak{M}$ the "mixed" weight $\theta = \theta(\varphi, \psi)$ of φ and ψ is the weight on $\mathfrak{P} = \mathfrak{M} \otimes M_2$ (M_2 = the algebra of 2×2-matrices) defined by:

$$\theta(x) = \varphi(x_{11}) + \psi(x_{22}) ,$$

for

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathfrak{P}_+ .$$

It is then shown in [5; Th. 1.2.1] that θ is a f.n.s.f. weight on $\mathfrak{P}$. From the definition of θ it follows that:

$$(2.1) \qquad \mathfrak{N}_\theta = \begin{pmatrix} \mathfrak{N}_\varphi & \mathfrak{N}_\psi \\ \mathfrak{N}_\varphi & \mathfrak{N}_\psi \end{pmatrix},$$

where the matrix notation is self-explanatory. Since the *-operation in $\mathcal{P}$ is given by:

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11}^* & x_{21}^* \\ x_{12}^* & x_{22}^* \end{pmatrix},$$

we have:

$$(2.2) \qquad \mathfrak{N}_\theta^* \cap \mathfrak{N}_\theta = \begin{pmatrix} \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi & \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\psi \\ \mathfrak{N}_\psi^* \cap \mathfrak{N}_\varphi & \mathfrak{N}_\psi^* \cap \mathfrak{N}_\psi \end{pmatrix}$$

From (2.1) it follows that the Hilbert space $\mathfrak{H}_\theta$ of θ may be written as

$$(2.3) \qquad \mathfrak{H}_\theta = \mathfrak{H}_\varphi \oplus \mathfrak{H}_\varphi \oplus \mathfrak{H}_\psi \oplus \mathfrak{H}_\psi$$

The imbedding $\Lambda_\theta : \mathfrak{N}_\theta \to \mathfrak{H}_\theta$ is then given by:

$$(2.4) \quad \Lambda_\theta : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathfrak{N}_\theta \mapsto (\Lambda_\varphi(x_{11}), \Lambda_\varphi(x_{21}), \Lambda_\psi(x_{12}), \Lambda_\psi(x_{22})) \in \mathfrak{H}_\theta$$

Since $\Lambda_\theta(\mathfrak{N}_\theta^* \cap \mathfrak{N}_\theta)$ is dense in $\mathfrak{H}_\theta$, so is $\Lambda_\varphi(\mathfrak{N}_\psi^* \cap \mathfrak{N}_\varphi)$ in $\mathfrak{H}_\varphi$. Furthermore:

$$(2.5) \qquad \mathfrak{M}_\theta = \mathfrak{N}_\theta^* \cdot \mathfrak{N}_\theta = \begin{pmatrix} \mathfrak{M}_\varphi & \mathfrak{N}_\varphi^* \cdot \mathfrak{N}_\psi \\ \mathfrak{N}_\psi^* \cdot \mathfrak{N}_\varphi & \mathfrak{M}_\psi \end{pmatrix},$$

hence $\Lambda_\varphi(\mathfrak{N}_\psi^* \cdot \mathfrak{N}_\varphi)$ is dense in $\mathfrak{H}_\varphi$ (since $\Lambda_\theta(\mathfrak{M}_\theta)$ is dense $\mathfrak{H}_\theta$).

We denote by $S_{\varphi,\psi}$ the densely defined operator from $\mathcal{H}_\psi$ to $\mathcal{H}_\varphi$ given by:

$$S_{\varphi,\psi}\Lambda_\psi(x) = \Lambda_\varphi(x^*), \qquad x \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\psi \tag{2.6}$$

Writing operators on $\mathcal{H}_\theta$ as 4×4-matrices according to the decomposition (2.3), we consider the densely defined operator Q on $\mathcal{H}_\theta$ given by:

$$Q = \begin{bmatrix} S_\varphi & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi,\psi} & 0 \\ 0 & S_{\psi,\varphi} & 0 & 0 \\ 0 & 0 & 0 & S_\psi \end{bmatrix}, \tag{2.7}$$

with domain $\mathfrak{U}_\theta = \Lambda_\theta(\mathfrak{N}_\theta^* \cap \mathfrak{N}_\theta)$. Since Q coincides with S_θ on $\mathfrak{U}_\theta$, Q is preclosed, hence so are the matrix elements $S_{\varphi,\psi}$. A vector $\eta = (\eta_1,\eta_2,\eta_3,\eta_4) \in \mathcal{H}_\theta$ is in the domain of Q^* if and only if the mapping

$$\begin{aligned}\xi = (\xi_1,\xi_2,\xi_3,\xi_4) \in \mathfrak{U}_\theta &\mapsto \langle Q\xi,\eta\rangle \\ &= \langle S_\varphi\xi_1,\eta_1\rangle + \langle S_{\varphi,\psi}\xi_3,\eta_2\rangle + \langle S_{\psi,\varphi}\xi_2,\eta_3\rangle + \langle S_\psi\xi_4,\eta_4\rangle\end{aligned}$$

is a bounded linear functional on $\mathfrak{U}_\theta$. Since the non-zero functionals on the right-hand side of this equation are linearly independent, considered as functionals on $\mathfrak{U}_\theta$, the mapping $\xi \in \mathfrak{U}_\varphi \mapsto \langle Q\xi,\eta\rangle$ is bounded if and only if each of the functionals $\xi_1 \in \Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi) \mapsto \langle S_\varphi\xi_1,\eta_1\rangle$, $\xi_2 \in \Lambda_\varphi(\mathfrak{N}_\psi^* \cap \mathfrak{N}_\varphi) \mapsto \langle S_{\psi,\varphi}\xi_2,\eta_3\rangle$ etc. is bounded. It follows that $\mathrm{dom}(Q^*) = \mathrm{dom}(S_\varphi^*) \oplus \mathrm{dom}(S_{\varphi,\psi}^*) \oplus \mathrm{dom}(S_{\psi,\varphi}^*) \oplus \mathrm{dom}\, S_\psi^*$ and that:

$$(2.8) \qquad Q^* = \begin{bmatrix} S_\varphi^* & 0 & 0 & 0 \\ 0 & 0 & S_{\psi,\varphi}^* & 0 \\ 0 & S_{\varphi,\psi}^* & 0 & 0 \\ 0 & 0 & 0 & S_\psi^* \end{bmatrix}.$$

Denoting the closure of an S by S again and repeating the argument we get:

$$(2.9) \quad S_\theta = Q^{**} = \begin{bmatrix} S_\varphi & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi,\psi} & 0 \\ 0 & S_{\psi,\varphi} & 0 & 0 \\ 0 & 0 & 0 & S_\psi \end{bmatrix}; \ \mathrm{dom}(S\ ) = \mathrm{dom}(S_\varphi) \oplus \mathrm{dom}(S_{\psi,\varphi}) \oplus \mathrm{dom}(S_{\varphi,\psi}) \oplus \mathrm{dom}(S_\psi)$$

since $S_\theta = Q$ on $\mathfrak{A}_\theta$ and $\mathfrak{A}_\theta$ is a core for S_θ.

Letting $S_{\varphi,\psi} = J_{\varphi,\psi}(\Delta_\psi^\varphi)^{1/2}$ be the polar decomposition of $S_{\varphi,\psi}$ where $J_{\varphi,\psi} : \mathfrak{H}_\psi \to \mathfrak{H}_\varphi$ is conjugate unitary and $\Delta_\psi^\varphi : \mathfrak{H}_\psi \to \mathfrak{H}_\psi$ is positive, selfadjoint, nonsingular, we get the following matrix expression for the modular operator Δ_θ of θ:

$$(2.10) \quad \Delta_\theta = S_\theta^* S_\theta = \begin{bmatrix} S_\varphi^* & 0 & 0 & 0 \\ 0 & 0 & S_{\psi,\varphi}^* & 0 \\ 0 & S_{\varphi,\psi}^* & 0 & 0 \\ 0 & 0 & 0 & S_\psi^* \end{bmatrix} \begin{bmatrix} S_\varphi & 0 & 0 & 0 \\ 0 & 0 & S_{\varphi,\psi} & 0 \\ 0 & S_{\psi,\varphi} & 0 & 0 \\ 0 & 0 & 0 & S_\psi \end{bmatrix}$$

$$= \begin{bmatrix} \Delta_\varphi & 0 & 0 & 0 \\ 0 & \Delta_\varphi^\psi & 0 & 0 \\ 0 & 0 & \Delta_\psi^\varphi & 0 \\ 0 & 0 & 0 & \Delta_\psi \end{bmatrix}.$$

Since $S_{\varphi,\psi} = (S_{\psi,\varphi})^{-1} = (\Delta_\varphi^\psi)^{-1}(J_{\psi,\varphi})^{-1}$, we have $J_{\psi,\varphi}S_{\varphi,\psi} =$

$J_{\psi,\varphi}(\Delta_{\varphi}^{\psi})^{-1}(J_{\psi,\varphi})^{-1}$, a positive operator ($J_{\psi,\varphi}$ being conjugate unitary); hence by the uniqueness of polar decomposition:

$$J_{\psi,\varphi}(\Delta_{\varphi}^{\psi})^{-1}(J_{\psi,\varphi})^{-1} = \Delta_{\psi}^{\varphi}, \tag{2.11}$$

and thus:

$$J_{\psi,\varphi} = (J_{\varphi,\psi})^{-1}. \tag{2.12}$$

From the relation

$$\pi_\theta\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{21} \end{bmatrix}\Lambda_\theta\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \Lambda_\theta\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}\right)$$

$$= \Lambda_\theta\begin{bmatrix} x_{11}y_{11} + x_{12}y_{21}, & x_{11}y_{12} + x_{12}y_{22} \\ x_{21}y_{11} + x_{22}y_{21}, & x_{21}y_{12} + x_{22}y_{22} \end{bmatrix}$$

for

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in \mathfrak{M} \otimes M_2, \quad \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \in \mathfrak{N}_\theta,$$

it follows that in the 4×4-matrix notation the representation π_θ associated with θ is given by:

$$\pi_\theta\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} \pi_\varphi(x_{11}) & \pi_\varphi(x_{12}) & 0 & 0 \\ \pi_\varphi(x_{21}) & \pi_\varphi(x_{22}) & 0 & 0 \\ 0 & 0 & \pi_\psi(x_{11}) & \pi_\psi(x_{12}) \\ 0 & 0 & \pi_\psi(x_{21}) & \pi_\psi(x_{22}) \end{bmatrix}. \tag{2.13}$$

Setting

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} = \sigma_t^\theta \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$$

and using the fact that

$$\pi_\theta\left(\sigma_t^\theta \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}\right) = \Delta_\theta^{it}\pi_\theta \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \Delta_\theta^{-it} ,$$

we get by the above formulas for Δ_θ and π_θ:

$$(2.14) \quad \begin{bmatrix} \pi_\varphi(y_{11}) & \pi_\varphi(y_{12}) & 0 & 0 \\ \pi_\varphi(y_{21}) & \pi_\varphi(y_{22}) & 0 & 0 \\ 0 & 0 & \pi_\psi(y_{11}) & \pi_\psi(y_{12}) \\ 0 & 0 & \pi_\psi(y_{21}) & \pi_\psi(y_{22}) \end{bmatrix}$$

$$= \begin{bmatrix} \Delta_\varphi^{it}\pi_\varphi(x_{11})\Delta_\varphi^{-it}, & \Delta_\varphi^{it}\pi_\varphi(x_{12})(\Delta_\varphi^\psi)^{-it} & 0 & 0 \\ (\Delta_\varphi^\psi)^{it}\pi_\varphi(x_{21})\Delta_\varphi^{-it}, & (\Delta_\varphi^\psi)^{it}\pi_\varphi(x_{22})(\Delta_\varphi^\psi)^{-it} & 0 & 0 \\ 0 & 0 & (\Delta_\psi^\varphi)^{it}\pi_\psi(x_{11})(\Delta_\psi^\varphi)^{it}, & (\Delta_\psi^\varphi)^{it}\pi_\psi(x_{12})\Delta_\psi^{-it} \\ 0 & 0 & \Delta_\psi^{it}\pi_\psi(x_{12})(\Delta_\psi^\varphi)^{-it}, & \Delta_\psi^{it}\pi_\psi(x_{22})\Delta_\psi^{-it} \end{bmatrix} .$$

This gives $\pi_\varphi(y_{11}) = \Delta_\varphi^{it}\pi_\varphi(x_{11})\Delta_\varphi^{-it} = \pi_\varphi(\sigma_t^\varphi(x_{11})$ and $\pi_\psi(y_{22}) = \Delta_\psi^{-it}\pi_\psi(x_{22})\Delta_\psi^{-it} = \pi_\psi(\sigma_t^\psi(x_{22}))$, hence $y_{11} = \sigma_t^\varphi(x_{11})$ and $y_{22} = \sigma_t^\psi(x_{22})$. It follows that $\pi_\varphi(\sigma_t^\psi(x_{22})) = (\Delta_\varphi^\psi)^{it}\pi_\varphi(x_{22})(\Delta_\varphi^\psi)^{-it}$ and $\pi_\psi(\sigma_t^\varphi(x_{11})) = (\Delta^\varphi)^{it}\pi_\psi(x_{11})(\Delta^\varphi)^{-it}$, hence there is, for all $t \in \mathbb{R}$, a unique unitary $u_t^{\psi,\varphi} \in \mathfrak{M}$ such that $\pi_\varphi(u_t^{\psi,\varphi}) = (\Delta_\varphi^\psi)^{it}\Delta_\varphi^{-it}$. The strongly continuous mapping $t \mapsto u_t^{\psi,\varphi}$ is called the Radon-Nikodym derivative of ψ with respect to φ and is also denoted by $(D_\psi : D_\varphi)$, thus $(D_\psi : D_\varphi)_t = u_t^{\psi,\varphi}$. From the above matrix equation it follows that $\pi_\psi(u_t^{\psi,\varphi}) = \Delta_\psi^{it}(\Delta_\psi^\varphi)^{-it} = ((\Delta_\psi^\varphi)^{it}\Delta_\psi^{-it})^* = \pi_\psi(u_t^{\varphi,\psi})^*$, thus:

$$(2.15) \qquad u_t^{\psi,\varphi} = (u_t^{\varphi,\psi})^*$$

Furthermore, from the definition of $u_t^{\psi,\varphi}$ we get:

$$(2.16) \qquad u_{t_1+t_2}^{\psi,\varphi} = u_{t_1}^{\psi,\varphi}\sigma_{t_1}^\varphi(u_{t_2}^{\psi,\varphi}), \quad t_1, t_2 \in \mathbb{R}$$

and

$$(2.17) \qquad \sigma_t^\psi(x) = u_t^{\psi,\varphi}\sigma_t^\varphi(x)u_t^{\varphi,\psi}, \quad t \in \mathbb{R} .$$

By the matrix equation (2.14) and the definition of $u_t^{\psi,\varphi}$ the modular automorphism group $\{\sigma_t^\theta\}$ of the mixed weight θ is given by:

$$(2.18) \qquad \sigma_t^\theta \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} \sigma_t^\varphi(x_{11}) & u_t^{\varphi,\psi}\sigma_t^\psi(x_{12}) \\ u_t^{\psi,\varphi}\sigma_t^\varphi(x_{21}) & \sigma_t^\psi(x_{22}) \end{bmatrix} .$$

Finally, if χ is a third weight on $\mathfrak{M}$, then the following chain rule holds:

$$(2.19) \qquad (D\chi : D\varphi)_t = (D\chi : D\psi)_t(D\psi : D\varphi)_t .$$

This is easily verified by considering the weight $\bar{\theta}$ on $\mathfrak{M} \otimes M_3$ defined by: $\bar{\theta}(x) = \varphi(x_{11}) + \psi(x_{22}) + \chi(x_{33})$ for

$$x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in (\mathfrak{m} \otimes M_3)^+ .$$

Proposition 2.2. A σ-strongly continuous mapping u from the real line $\mathbb{R}$ into the unitary group of $\mathfrak{m}$ which satisfies relations (2.16) and (2.17) above, coincides with the Radon-Nikodym derivative $(D\psi : D\varphi)$ of ψ with respect to φ if and only if the following KMS-condition is satisfied:

For $x \in \mathfrak{N}_\psi^* \cap \mathfrak{N}_\varphi$, $y \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\psi$ there is a bounded complex function $F = F_{\varphi,\psi}^{x,y}$, holomorphic in and continuous on the strip $B = \{z \in \mathbb{C};$ $0 \leq \operatorname{Im} z \leq 1\}$ such that:

$$F(t) = \psi(u_t \sigma_t^\varphi(x)y)$$

$$F(t + i) = \varphi(yu_t\sigma_t^\varphi(x)) .$$

Proof. If $u_t = (D\psi : D\varphi)_t$, then

$$\sigma_t^\theta \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} \sigma_t^\varphi(x_{11}) & u_t^*\sigma_t^\psi(x_{12}) \\ u_t\sigma_t^\varphi(x_{21} & \sigma_t(x_{22}) \end{bmatrix} ,$$

hence since θ satisfies the KMS-condition with respect to σ_t^θ, there is for $x \in \mathfrak{N}_\psi^* \cap \mathfrak{N}_\varphi$, $y \in \mathfrak{N}_\varphi^* \cap \mathfrak{N}_\psi$, a complex function F with the stated properties such that:

$$F(t) = \theta\left(\sigma_t^\theta\left(\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right)\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}\right) = \psi(u_t\sigma_t^\varphi(x)y)$$

and

$$F(t + i) = \theta\left(\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} \sigma_t^\theta\left(\begin{bmatrix} 0 & 0 \\ x & 0 \end{bmatrix}\right)\right) = \varphi(yu_t\sigma_t^\varphi(x)) .$$

Conversely, if $\{u_t\}$ satisfies (2.16) and (2.17) and if we define β_t by

$$\beta_t \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} \sigma_t^\varphi(x_{11}) & u_t^*\sigma_t^\psi(x_{12}) \\ u_t\sigma_t^\varphi(x_{21}) & \sigma_t^\psi(x_{22}) \end{bmatrix} ,$$

then $t \mapsto \beta_t$ is a strongly continuous 1-parameter automorphism group on $\mathfrak{M} \otimes M_2$ under which θ is invariant, and the KMS-condition imposed on $\{u_t\}$ implies that $\{\beta_t\}$ satisfies the KMS-condition with respect to θ, thus $\beta_t = \sigma_t^\theta$, $t \in \mathbb{R}$. It follows that $u_t = (D\psi : D\varphi)_t$.
Q.E.D.

<u>Corollary 2.3</u>. If β is an automorphism of $\mathfrak{M}$ and φ, ψ are as above, then

$$(D(\psi \circ \beta) : D(\varphi \circ \beta))_t = \beta^{-1}((D\psi : D\varphi)_t) .$$

<u>Proof</u>. Since $\sigma_t^{\chi\circ\beta} = \beta^{-1} \circ \sigma_t^\chi \circ \beta$ for any f.n.s.f. weight χ (as is readily seen by checking the KMS-condition), we have for $x \in \mathfrak{N}_{\psi\circ\beta}^* \cap \mathfrak{N}_{\varphi\circ\beta} = \beta^{-1}(\mathfrak{N}_\psi^* \cap \mathfrak{N}_\varphi)$ and $y \in \mathfrak{N}_{\varphi\circ\beta}^* \cap \mathfrak{N}_{\psi\circ\beta} = \beta^{-1}(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\psi)$:

$$\psi \circ \beta(\beta^{-1}(u_t^{\psi,\varphi})\sigma_t^{\varphi\circ\beta}(x)y) = \psi(u_t^{\psi,\varphi}\sigma_t^\varphi(\beta(x))\beta(y))$$

and

$$\varphi \circ \beta(y\beta^{-1}(u_t^{\psi,\varphi})\sigma_t^{\varphi\circ\beta}(x)) = \varphi(\beta(y)u_t^{\psi,\varphi}\sigma_t^\varphi(\beta(x))) ,$$

thus taking a KMS-function corresponding to $\beta(x)$, $\beta(y)$ for $u_t^{\psi,\varphi}$, the assertion follows. Q.E.D.

If φ_i is a f.n.s.f. weight on a W^*-algebra $\mathfrak{M}_i$, $i = 1,2$, the tensor product weight $\varphi_1 \otimes \varphi_2$ is the unique f.n.s.f. weight on φ on $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ satisfying:

(a) $\varphi(x_1 \otimes x_2) = \varphi_1(x_1)\varphi_2(x_2)$, $x_i \in \mathfrak{M}_{\varphi_i}$, $i = 1,2$,

(b) $\sigma_t^{\varphi} = \sigma_t^{\varphi_1} \otimes \sigma_t^{\varphi_2}$

(see [5; 1.1.2]).

Lemma 2.4. If φ_1, ψ_1 (resp. φ_2, ψ_2) are f.n.s.f. weights on a W^*-algebra $\mathfrak{M}_1$ (resp. $\mathfrak{M}_2$), then:

$$(D(\varphi_1 \otimes \varphi_2) : D(\psi_1 \otimes \psi_2))_t = (D\varphi_1 : D\psi_1)_t \otimes (D\varphi_2 : D\psi_2)_t .$$

Proof. Let $\varphi = \varphi_1 \otimes \varphi_2$, $\psi = \psi_1 \otimes \psi_2$ and set $v_t = (D\varphi_1 : D\psi_1)_t \otimes (D\varphi_2 : D\psi_2)_t$. It is then clear that $t \mapsto v_t$ is a cocycle with respect to $\{\sigma_t^{\psi}\}$ and that $\sigma_t^{\varphi}(x) = v_t\sigma_t^{\psi}(x)v_t^*$, $x \in \mathfrak{M}_1 \otimes \mathfrak{M}_2$ (using (b) above). It remains to show that $\{v_t\}$ satisfies the KMS-condition with respect to φ and ψ (Proposition 2.2). For this it suffices to consider elements of the form: $x = x_1 \otimes x_2$, $y = y_1 \otimes y_2$ where $x_i \in \mathfrak{N}_{\varphi_i}^* \cap \mathfrak{N}_{\psi_i}$, $y_i \in \mathfrak{N}_{\psi_i}^* \cap \mathfrak{N}_{\varphi_i}$, $i = 1,2$ (since $(\mathfrak{N}_{\varphi_1}^* \cap \mathfrak{N}_{\psi_1}) \otimes (\mathfrak{N}_{\varphi_2}^* \cap \mathfrak{N}_{\psi_2}) = (\mathfrak{N}_{\varphi_1}^* \otimes \mathfrak{N}_{\varphi_2}^*) \cap (\mathfrak{N}_{\psi_1} \otimes \mathfrak{N}_{\psi_2})$ with algebraic tensor products). We have:

$$\varphi([v_t\sigma_t^{\psi}(x_1 \otimes x_2)][y_1 \otimes y_2]) = \varphi_1(u_t^{\varphi_1\psi_1}\sigma_t^{\psi_1}(x_1)y_1)\varphi_2(u_t^{\varphi_2\psi_2}\sigma_t^{\psi_2}(x_2)y_2)$$

and:

$$\psi([y_1 \otimes y_2][v_t\sigma_t^{\psi}(x_1 \otimes x_2)]) = \psi_1(y_1u_t^{\varphi_1\psi_1}\sigma_t^{\psi_1}(x_1))\psi_2(y_2u_t^{\varphi_2\psi_2}\sigma_t^{\psi_2}(x_2)) ,$$

thus, if F_1 and F_2 are KMS-functions for $\{u_t^{\varphi_1\psi_1}\}$ and $\{u_t^{\varphi_2\psi_2}\}$, respectively, $F = F_1 \cdot F_2$ is a KMS-function for $\{v_t\}$. Q.E.D.

We shall need some results concerning the natural cone P_φ associated with a f.n.s.f. weight φ, which plays an important role in the spatial description of W^*-algebras (see [6]). We recall the definition:

$$P_\varphi = \{\pi^\ell(\xi)J_\varphi\xi; \quad \xi \in \mathfrak{A}_\varphi = \Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi)\}^- ,$$

where the closure is taken in the Hilbert space $\mathfrak{H}_\varphi$. The cone P_φ is self dual in the sense that an element $\xi \in \mathfrak{H}_\varphi$ is in P_φ if and only if $\langle\xi,\eta\rangle \geq 0$ for all $\eta \in P_\varphi$. Moreover, if φ and ψ are f.n.s.f. weights on $\mathfrak{M}$ and if $\Phi : \pi_\varphi(\mathfrak{M}) \to \pi_\psi(\mathfrak{M})$ is an isomorphism, there is a unique unitary $U_\Phi : \mathfrak{H}_\varphi \to \mathfrak{H}_\psi$ which preserves the natural cones and implements Φ (i.e. $U_\Phi P_\varphi = P_\psi$ and $\Phi(x) = U_\Phi x U_\Phi^*$, $x \in \pi_\varphi(\mathfrak{M})$), the so called canonical unitary implementation of Φ. For a covariant system $\{\mathfrak{M},\mathfrak{G},\alpha\}$ we also have that the mapping $g \mapsto U_g$ is strongly continuous, where U_g is the unique unitary on $\mathfrak{H}_\varphi$ such that $U_g P_\varphi = P_\varphi$ and $\pi_\varphi(\alpha_g(x)) = U_g\pi_\varphi(x)U_g^*$, $g \in \mathfrak{G}$, $x \in \mathfrak{M}$. This continuous homomorphism $U : \mathfrak{G} \to \mathfrak{B}(\mathfrak{H}_\varphi)$ will be referred to as the canonical unitary implementation of α on $\mathfrak{H}_\varphi$ (see [11; Corollary 3.11]).

For the analysis of the conjugate unitary operator $J_{\psi,\varphi}$ appearing in the polar decomposition of $S_{\psi,\varphi}$ we shall need the fact that P_φ coincides with the following apparently larger set, namely $Q_\varphi = \{\pi^\ell(\xi)J_\varphi\xi;\ \xi \in \mathfrak{B}_\varphi = \Lambda_\varphi(\mathfrak{N}_\varphi)\}$. For this we invoke a result of

Haagerup [11] concerning the possibility of approximating, in a suitable manner, elements in the full left Hilbert algebra $\mathfrak{A}_\varphi$ by elements in the maximal Tomita algebra $\mathfrak{A}_{\varphi,0}$ contained in $\mathfrak{A}_\varphi$ (we recall that $\mathfrak{A}_{\varphi,0} = \{\xi \in \mathfrak{H}_\varphi;\ \xi \in \mathrm{dom}(\Delta^z)$ and $\Delta^z\xi \in \mathfrak{A}_\varphi$, all $z \in \mathbb{C}\}$ is a Tomita algebra which is a sub-Hilbert algebra of $\mathfrak{A}_\varphi$ and is equivalent to $\mathfrak{A}_\varphi$). It is shown in [11; Lemma 1.4] that for each $\xi \in \mathfrak{A}_\varphi = \Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi)$ there is a sequence $\{\xi_n\}$ from $\mathfrak{A}_{\varphi,0}$ such that $\xi_n \to \xi$, $\xi_n^\# \to \xi^\#$, $\|\pi^\ell(\xi_n)\| \leq \|\pi^\ell(\xi)\|$, $\pi^\ell(\xi_n) \to \pi^\ell(\xi)$ and $\pi^\ell(\xi^\#) \to \pi^\ell(\xi^\#)$ (the latter two in the strong topology), and hence $P_\varphi = \{\xi \cdot J_\varphi\xi;\ \xi \in \mathfrak{A}_{\varphi,0}\}^-$. In our case we only know that $\xi \in \Lambda_\varphi(\mathfrak{N}_\varphi)$, thus the statements concerning the #-convergence do not make sense. However, we still can find a sequence $\{\xi_n\}$ from $\mathfrak{A}_{\varphi,0}$ such that $\xi_n \to \xi$, $\|\pi^\ell(\xi_n)\| \leq \|\pi^\ell(\xi)\|$ and $\pi^\ell(\xi_n) \to \pi^\ell(\xi)$ strongly. For the proof of this we proceed exactly as in the proof of [11; Lemma 1.4]: for each $n \in \mathbb{Z}^+$, we let $F_n(t) = e^{-t^2/2n^2}$, $t \in \mathbb{R}$, and set $\xi_n = F_n(\log \Delta)\xi$. Then by [12; Theorem 13.24b)] $\xi_n \in \mathrm{dom}(\Delta_\varphi^z)$ for all $z \in \mathbb{C}$; also, $\Delta^z\xi_n = G_{n,z}(\log \Delta)\xi$ where $G_{n,z}(t) = \exp(zt - t^2/2n^2)$ is linear combination of continuous positive definite functions (as is readily seen by Fourier transforming $G_{n,z}$), hence there is a bounded Radon measure $\mu = \mu_{n,z}$ on $\mathbb{R}$ such that $G_{n,z}(t) = \int_{\mathbb{R}} e^{ist}d\mu(s)$. Since Δ_φ^{it} leaves $\Lambda_\varphi(\mathfrak{N}_\varphi)$ invariant and $\pi^\ell(\Delta_\varphi^{it}\xi) = \sigma_t^\varphi(\pi^\ell(\xi))$ for $\xi \in \Lambda_\varphi(\mathfrak{N}_\varphi)$, we have for $\eta \in \mathfrak{A}'$ = the right Hilbert algebra associated with $\mathfrak{A}$ (see [14], Definition 3.1):

$$\|(\Delta_\varphi^z\xi_n)\eta\| = \|[G_{n,z}(\log \Delta)\xi]\eta\|$$

$$= \left\|\int_{\mathbb{R}} (\Delta^{is}\xi)\eta d\mu(s)\right\| \leq \int \|\pi^\ell(\xi)\| \cdot \|\eta\| d|\mu|(s)$$

$$= \|\mu\|\cdot\|\pi^{\ell}(\xi)\|\cdot\|\eta\| ,$$

where $\|\mu\|$ is the total variation of μ. It follows that $\Delta_\varphi^z\xi_n$ is left-bounded for all $z \in \mathbb{C}$ and $n \in \mathbb{Z}^+$, hence $\xi_n \in \mathfrak{A}_{\varphi,0}$ for all $n \in \mathbb{Z}^+$. Since F_n is positive definite and the Radon measure corresponding to F_n has total variation = 1, we get by a similar argument that $\|\pi^{\ell}(\xi_n)\| \leq \|\pi^{\ell}(\xi)\|$, $n \in \mathbb{Z}^+$. Since $F_n(\log\Delta)$ converges strongly to the identity, we have $\xi_n \to \xi$, and since the $\pi^{\ell}(\xi_n)$'s are uniformly bounded, we also have $\pi^{\ell}(\xi_n) \to \pi^{\ell}(\xi)$ strongly (by testing on the dense set $\mathfrak{A}'$ of right-bounded elements). Combining these three facts we get $\pi^{\ell}(\xi_n)J_\varphi\xi_n \to \pi^{\ell}(\xi)J_\varphi\xi$, thus

$$\{\pi^{\ell}(\xi)J_\varphi\xi \ ; \ \xi \in \Lambda_\varphi(\mathfrak{N}_\varphi)\} \subseteq \{\pi^{\ell}(\xi)J_\varphi\xi; \ \xi \in \mathfrak{A}_{\varphi,0}\}^- \subseteq P_\varphi ,$$

and we have proved:

Lemma 2.5.

$$P_\varphi = \{\pi^{\ell}(\xi)J_\varphi\xi; \ \xi \in \Lambda_\varphi(\mathfrak{N}_\varphi)\}^- .$$

The next proposition relates the unitary part $J_{\psi,\varphi}$ in the polar decomposition of the mixed #-operation $S_{\psi,\varphi}$ to the pure unitary involutions J_φ and J_ψ and will be used in the computation of the dual unitary involution in the next chapter.

Proposition 2.6. Let φ, ψ be f.n.s.f. weights on $\mathfrak{M}$ and let $V_{\psi,\varphi} : \mathcal{H}_\varphi \to \mathcal{H}_\psi$ be the canonical unitary implementation of the isomorphism: $\pi_\varphi(x) \in \pi_\varphi(\mathfrak{M}) \mapsto \pi_\psi(x) \in \pi_\psi(\mathfrak{M})$, $x \in \mathfrak{M}$. Then with notations as above:

$$J_{\psi,\varphi} = V_{\psi,\varphi}J_\varphi = J_\psi V_{\psi,\varphi} .$$

<u>Proof.</u> Since $\Lambda_\theta(\mathfrak{M}_\theta)$ is a core for the involution S_θ of the mixed weight $\theta = \theta(\varphi,\psi)$ of φ and ψ on $\mathcal{P} = \mathfrak{M} \otimes M_2$, it follows from (2.5) and (2.9) that $\Lambda_\varphi(\mathfrak{N}_\psi^*\mathfrak{N}_\varphi)$ is a core for $S_{\psi,\varphi}$. By the uniqueness of the polar decomposition it therefore suffices to show that $J_\varphi V_{\varphi,\psi}S_{\psi,\varphi}$ is positive on $\Lambda_\varphi(\mathfrak{N}_\psi^*\mathfrak{N}_\varphi)$. So let $x \in \mathfrak{N}_\varphi$, $y \in \mathfrak{N}_\psi$. We have:

$$\begin{aligned}
\langle J_\varphi V_{\varphi,\psi}S_{\psi,\varphi}\Lambda_\varphi(y^*x),\Lambda_\varphi(y^*x)\rangle &= \langle J_\varphi V_{\varphi,\psi}\Lambda_\psi(x^*y),\Lambda_\varphi(y^*x)\rangle \\
&= \langle J_\varphi V_{\varphi,\psi}\pi_\psi(x^*)\Lambda_\psi(y),\pi_\varphi(y^*)\Lambda_\varphi(x)\rangle \\
&= \langle \pi_\varphi(y)V_{\varphi,\psi}J_\psi\pi_\psi(x^*)\Lambda_\psi(y),\Lambda_\varphi(x)\rangle \\
&= \langle V_{\varphi,\psi}\pi_\psi(y)(J_\psi\pi_\psi(x^*)J_\psi)J_\psi\Lambda_\psi(y),\Lambda_\varphi(x)\rangle \\
&= \langle V_{\varphi,\psi}(J_\psi\pi_\psi(x^*)J_\psi)\pi_\psi(y)J_\psi\Lambda_\psi(y),\Lambda_\varphi(x)\rangle \\
&= \langle (J_\varphi\pi_\varphi(x^*)J_\varphi)V_{\varphi,\psi}\pi_\psi(y)J_\psi\Lambda_\psi(y),\Lambda_\varphi(x)\rangle \\
&= \langle V_{\varphi,\psi}\pi_\psi(y)J_\psi\Lambda_\psi(y),J_\varphi\pi_\varphi(x)J_\varphi\Lambda_\varphi(x)\rangle \\
&= \langle V_{\varphi,\psi}\pi_\psi(y)J_\psi\Lambda_\psi(y),\pi_\varphi(x)J_\varphi\Lambda_\varphi(x)\rangle \geq 0 ,
\end{aligned}$$

where the last inequality follows from the cone-preserving property of $V_{\varphi,\psi}$. The relation $J_\psi V_{\psi,\varphi} = V_{\psi,\varphi}J_\varphi$ has been used already and follows from the fact that $V_{\psi,\varphi}P_\varphi = P_\psi$ (see [11; p. 34]). Q.E.D.

Returning to our covariant system $\{\mathfrak{M},\mathfrak{G},\alpha\}$ we know that for a fixed $g \in \mathfrak{G}$, the mapping $t \in \mathbb{R} \mapsto (D(\varphi \circ \alpha_g) : D\varphi)_t$ is a continuous cocycle with respect to $\{\sigma_t^\varphi\}$. If we instead fix t and consider the mapping $g \in \mathfrak{G} \mapsto (D\varphi : D(\varphi \circ \alpha_g^{-1}))_t$, another continuous cocycle result, this time with respect to α. We prove:

<u>Proposition 2.7</u>. For a fixed $t \in \mathbb{R}$ the mapping $g \in \mathfrak{G} \mapsto (D\varphi : D(\varphi \circ \alpha_g^{-1}))_t$ is a σ-strong* continuous cocycle with respect to α; that is, setting $U_t(g) = (D\varphi : D(\varphi \circ \alpha_g^{-1}))_t$, we have:

$$U_t(g_1 g_2) = U_t(g_1)\alpha_{g_1}(U_t(g_2)), \quad g_1, g_2 \in \mathfrak{G},\ t \in \mathbb{R} .$$

<u>Proof</u>. To prove the continuity we consider the weight $\psi = \varphi \otimes \mu$ on $\mathfrak{M} \otimes L^\infty(\mathfrak{G}) = L^\infty(\mathfrak{G}; \mathfrak{M})$ under the action $g \in \mathfrak{G} \mapsto \beta_g = \alpha_g \otimes \tau_g$ where μ is Haar-measure on $\mathfrak{G}$ and τ is left translation on $L^\infty(\mathfrak{G})$, i.e. $(\tau_g F)(h) = F(g^{-1}h)$, $F \in L^\infty(\mathfrak{G})$, $g,h \in \mathfrak{G}$. Define an automorphism γ on $L^\infty(\mathfrak{G},\mathfrak{M})$ by:

$$(\gamma x)(g) = \alpha_g^{-1}(x(g); \quad x \in L^\infty(\mathfrak{G}; \mathfrak{M}) .$$

We have:

$$(\gamma\beta_h\gamma^{-1}x)(g) = \alpha_g^{-1}\alpha_h\alpha_h^{-1}\alpha_g(x(h^{-1}g)) = x(h^{-1}g)$$

$$= ((I \otimes \tau_h)x)(g), \quad x \in L^\infty(\mathfrak{G}; \mathfrak{M}) ,$$

Thus $\gamma \circ \beta_h = (I \otimes \tau_h) \circ \gamma$, $h \in \mathfrak{G}$, where I denotes the identity automorphism on $\mathfrak{M}$. Setting $\chi = \psi \circ \gamma$, we get:

$$\chi \circ \beta_h = \psi \circ \gamma \circ \beta_h = \psi \circ (I \otimes \tau_h) \circ \gamma$$

$$= (\varphi \otimes \mu) \circ (I \otimes \tau_h) \circ \gamma$$

$$= (\varphi \otimes \mu) \circ \gamma = \chi ,$$

hence χ is β-invariant. By the chain rule (2.19) we get:

$$\begin{aligned}(D(\psi \circ \beta_g) : D\psi)_t &= (D(\psi \circ \beta_g) : D(\chi \circ \beta_g))_t \cdot (D(\chi \circ \beta_g) : D\chi)_t \\ &\qquad\qquad \cdot (D\chi : D\psi)_t \\ &= \beta_g^{-1}[(D\psi : D\chi)_t] \cdot (D\chi : D\psi)_t ,\end{aligned}$$

which by the continuity assumption on α is a σ-strong* continuous function of g. By Lemma 2.4:

$$\begin{aligned}(D(\psi \circ \beta_g) : D\psi)_t &= (D(\varphi \circ \alpha_g \otimes \mu \circ \tau_g) : D(\varphi \otimes \mu))_t \\ &= (D(\varphi \circ \alpha_g) : D\varphi)_t \otimes (D(\mu \circ \tau_g) : D\mu)_t \\ &= (D(\varphi \circ \alpha_g) : D\varphi)_t \otimes I ,\end{aligned}$$

hence $g \in \mathfrak{G} \mapsto (D(\varphi \circ \alpha_g) : D\varphi)_t$ is continuous, for all $t \in \mathbb{R}$.

The cocycle identity is an immediate consequence of Corollary 2.3 and the chain rule. Q.E.D.

§3. Dual weight and the commutation theorem

In this section we construct the dual weight $\tilde{\varphi}$ on $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ of a given weight φ on $\mathfrak{M}$. The duality between $\tilde{\varphi}$ and φ is expressed in a formula giving the value of $\tilde{\varphi}$ on certain elements of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ (Corollary 3.6) and also in the formulas relating the modular objects associated with φ and $\tilde{\varphi}$. The formula for the unitary involution associated with $\tilde{\varphi}$ (Corollary 3.12) enables us to give a set of generators for the commutant of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ on $L^2(\mathfrak{G}; \mathfrak{H}_\varphi)$, and this in turn gives the general commutation theorem for crossed products, i.e. the description of the commutant of the crossed product, based on an arbitrary covariant representation, as a crossed product of the commutant of $\mathfrak{M}$ in the given representation (Theorem 3.14).

Before we go into the construction of the dual weight, we fix some notation. Throughout this section $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ will be a fixed dynamical system as defined in §1, and φ will be a fixed f.n.s.f. weight on $\mathfrak{M}$. We shall assume that the locally compact group $\mathfrak{G}$ and the predual $\mathfrak{M}_*$ of $\mathfrak{M}$ are separable (these assumptions are only used in a couple of measurability arguments and may very well turn out to be unnecessary). We set $\varphi_g = \varphi \circ \alpha_g$, and objects pertaining to the weight φ_g will be indexed with g rather than with φ_g; thus: $\mathfrak{N}_g\ (= \mathfrak{N}_{\varphi_g}) = \{x \in \mathfrak{M};\ \varphi \circ \alpha_g(x^*x) < \infty\}$, $\mathfrak{M}_g = \mathfrak{N}_g^* \cdot \mathfrak{N}_g$; $\Lambda_g : \mathfrak{N}_g \to \mathfrak{H}_g$ canonical injection into the Hilbert space $\mathfrak{H}_g$ of φ_g; $S_g = J_g \Delta_g^{1/2}$; $\pi_g(\sigma_t^g(x)) = \Delta_g^{it} \pi_g(x) \Delta_g^{-it}$, $x \in \mathfrak{M}$, etc. (the index e, the neutral element of $\mathfrak{G}$, will be suppressed; thus the symbols $\mathfrak{N}$, $\mathfrak{M}$, Λ, $\mathfrak{H}$, $S = J\Delta^{1/2}$, σ_t, π etc. without index pertain to the original weight

φ). Similarly for the relative modular objects we shall write: $S_{g,h} = J_{g,h}(\Delta^g_h)^{1/2}$, $u^{h,g}_t = (D(\varphi \circ \alpha_h) : D(\varphi \circ \alpha_g))_t$ etc. (here, of course, the neutral element e will not be dropped as an index).

The construction of the dual weight goes via the construction of a left Hilbert algebra whose left von Neumann algebra coincides with the realization of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ on $L^2(\mathfrak{G}; \mathfrak{H})$ (Theorem 3.4) (this left Hilbert algebra may be viewed as a W^*-analog of the covariance algebra for C^*-covariant systems [9], although it lacks the corresponding universal property).

Definition 3.1. We denote by $\tilde{\mathfrak{M}}$ the linear space of continuous functions $x : \mathfrak{G} \to \mathfrak{M}$ with compact support such that:

(a) $x(g) \in \mathfrak{N}^*_g \cap \mathfrak{N}$

(b) the functions $g \mapsto \varphi(x(g)^*x(g))$ and $g \mapsto \varphi_g(x(g)x(g)^*)$ are integrable.

For $x,y \in \tilde{\mathfrak{M}}$ we define:

(i) Product (convolution):

$$(x * y)(g) = \int_{\mathfrak{G}} \alpha_h(x(gh))y(h^{-1})dh, \quad g \in \mathfrak{G}$$

(ii) Involution:

$$x^{\#}(g) = \delta(g)^{-1}\alpha_g^{-1}(x(g^{-1})^*), \quad g \in \mathfrak{G} .$$

(Note: the word "continuous" without qualification always refers to the σ-strong* topology of $\mathfrak{M}$).

Proposition 3.2. If $x,y \in \tilde{\mathfrak{M}}$, then $x * y$ and $x^{\#}$ are again in $\tilde{\mathfrak{M}}$, and $\tilde{\mathfrak{M}}$ becomes an involutive algebra under these operations.

Proof. We show that $\widetilde{\mathfrak{M}}$ is closed under the operations $*$ and $\#$, the verification of the usual algebraic axioms being left to the reader.

Let $x,y \in \widetilde{\mathfrak{M}}$ and set $z = x * y$. From §1 it follows that z is a well defined function from $\mathfrak{G}$ into $\mathfrak{M}$. To prove the continuity, let $\{g_n\}$ be a sequence from $\mathfrak{G}$ converging to $g \in \mathfrak{G}$, let $\omega \in \mathfrak{M}_*^+$ and let p_ω be the σ-strong* seminorm corresponding to ω (see §1). Since the function of two variables: $(g,h) \in \mathfrak{G} \times \mathfrak{G} \mapsto \alpha_h(x(gh)y(h^{-1})) \in \mathfrak{M}$ is continuous, there is a constant $c \geq 0$, independent of h and n, such that $p_\omega(\alpha_h(x(g_nh))y(h^{-1})) \leq c$, $h \in \mathfrak{G}$, $n \in Z^+$. Hence by Lebesgues theorem:

$$\begin{aligned}\lim_n p_\omega(z(g_n) - z(g)) &\leq \lim_n \int_{\mathfrak{G}} p_\omega(\alpha_h(x(g_nh))y(h^{-1}) - \alpha_h(x(gh))y(h^{-1}))dh \\ &= \int_{\mathfrak{G}} \lim_n p_\omega(\alpha_h(x(g_nh)y(h^{-1}) - \alpha_h(x(gh))y(h^{-1}))dh \\ &= 0\end{aligned}$$

proving the continuity.

The fact that z has compact support follows by the usual arguments for convolution. To verify properties (a) and (b) of Definition 3.1 we consider

$$z(g)^*z(g) = \int_{\mathfrak{G}\times\mathfrak{G}} (\alpha_h(x(gh))y(h^{-1}))^*\alpha_k(x(gk))y(k^{-1})dhdk ,$$

hence for any $\omega \in \mathfrak{M}_*^+$:

$$\omega(z(g)^*z(g)) = \int_{\mathfrak{G}\times\mathfrak{G}} \omega[(\alpha_h(x(gh))y(h^{-1}))^*\alpha_k(x(gk))g(k^{-1})]dhdk$$

$$\leq \left(\int_{\mathfrak{G}} [\omega((\alpha_h(x(gh))y(h^{-1}))^* \alpha_h(x(gh))y(h^{-1}))]^{1/2} dh\right)^2$$

$$\leq \left(\int_{\mathfrak{G}} \|x(gh)\| [\omega(y(h^{-1})^* y(h^{-1}))]^{1/2} dh\right)^2$$

Taking an increasing net $\{\omega_i\}$ from $\mathfrak{M}_*^+$ such that $\omega_i(x) \uparrow \varphi(x)$, $x \in \mathfrak{M}^+$, the above inequality persists, thus:

$$\varphi(z(g)^* z(g)) \leq \left(\int_{\mathfrak{G}} \|x(gh)\| \cdot [\varphi(y(h^{-1})^* y(h^{-1}))]^{1/2} dh\right)^2$$

$$= \left(\int_{\mathfrak{G}} \|x(h)\| \cdot [\varphi(y(h^{-1}g)^* y(h^{-1}g))]^{1/2} dh\right)^2 ,$$

which by the assumption on y is finite $(g \mapsto \|x(g)\|$ being bounded), hence $z(g) \in \mathfrak{N}$. Furthermore, by Fubini's theorem:

$$\int_{\mathfrak{G}} \varphi(z(g)^* z(g)) dg \leq \int_{\mathfrak{G}} \left(\int_{\mathfrak{G}} \|x(h)\| \cdot [\varphi(y(h^{-1}g)^* y(h^{-1}g))]^{1/2} dh\right)^2 dg$$

$$= \int_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}} \|x(h)\| \|x(k)\| \cdot [\varphi(y(h^{-1}g)^* y(h^{-1}g))]^{1/2}$$

(3.1) $$\cdot [\varphi(y(k^{-1}g)^* y(k^{-1}g))]^{1/2} dg dh dk$$

$$\leq \int_{\mathfrak{G}\times\mathfrak{G}} \|x(h)\| \|x(k)\| \left[\left(\int_{\mathfrak{G}} \varphi(y(k^{-1}g)^* y(h^{-1}g)) dg\right)^{1/2} \cdot \left(\int_{\mathfrak{G}} \varphi(y(k^{-1}g)^* y(k^{-1}g)) dg\right)^{1/2}\right] dh dk$$

$$= \left(\int_{\mathfrak{G}} \|x(h)\| dh\right)^2 \int_{\mathfrak{G}} \varphi(y(g)^* y(g)) dg ,$$

which again is finite $(g \mapsto \|x(g)\|$ being lower semicontinuous, bounded and with compact support), thus $g \mapsto \varphi(z(g)^* z(g))$ is integrable. Similarly we get:

$$\int_{\mathfrak{G}} \varphi_g(z(g)z(g)^*)dg \leq \left(\int_{\mathfrak{G}} \|y(h)\|dh\right)^2 \int_{\mathfrak{G}} \varphi_g(x(g)x(g)^*)dg ,$$

hence $z = x * y \in \widetilde{\mathfrak{M}}$.

Finally, if $x \in \widetilde{\mathfrak{M}}$ and $z = x^{\#}$, it is clear that $z(g) \in \mathfrak{N}_g^* \cap \mathfrak{N}$ for all $g \in \mathfrak{G}$; also, the functions $g \mapsto \varphi(z(g)^* z(g)) = \delta(g)^{-2}\varphi_{g^{-1}}(x(g^{-1})x(g^{-1})^*)$ and $g \mapsto \varphi_g(z(g)z(g)^*) = \delta(g)^{-2}\varphi(x(g^{-1})^* x(g^{-1}))$ are integrable since

$$\int_{\mathfrak{G}} \delta(g)^{-2}\varphi_{g^{-1}}(x(g^{-1})x(g^{-1})^*)dg = \int_{\mathfrak{G}} \delta(g)\varphi_g(x(g)x(g)^*)dg < \infty$$

and

$$\int_{\mathfrak{G}} \delta(g)^{-2}\varphi(x(g^{-1})^* x(g^{-1}))dg = \int_{\mathfrak{G}} \delta(g)\varphi(x(g)^* x(g))dg < \infty$$

(x having compact support). Q.E.D.

We define a mapping $\widetilde{\Lambda} : \widetilde{\mathfrak{M}} \to L^2(\mathfrak{G}; \mathfrak{H})$ by:

$$(\widetilde{\Lambda}x)(g) = \Lambda(x(g)) .$$

By Lemma 1.3 $\widetilde{\Lambda}x$ is indeed an element of $L^2(\mathfrak{G}; \mathfrak{H})$. We set $\widetilde{\mathfrak{A}} = \widetilde{\Lambda}(\widetilde{\mathfrak{M}})$ and introduce a $\#$-algebra structure in $\widetilde{\mathfrak{A}}$ by transporting the $\#$-algebra structure of $\widetilde{\mathfrak{M}}$ to $\widetilde{\mathfrak{A}}$ via the mapping $\widetilde{\Lambda}$. If for each $g,h \in \mathfrak{G}$ we define a unitary operator $U_{h,g} : \mathfrak{H}_g \to \mathfrak{H}_h$ by

$U_{h,g}\Lambda_g(a) = \Lambda_h(\alpha_{h^{-1}g}(a))$, $a \in \mathfrak{N}_g$, and if we let $W(g) : \mathcal{H} \to \mathcal{H}$ be the canonical unitary implementation α_g in the representation π (see §2), the operations in $\tilde{\mathfrak{A}}$ may be written:

$$(3.2)\quad \begin{aligned} \xi^{\#}(g) &= \delta(g)^{-1}U_{e,g^{-1}}S_{g^{-1},e}\xi(g^{-1}) = \delta(g)^{-1}S_{e,g}U_{g,e}\xi(g^{-1}) \\ (\xi * \eta)(g) &= \int_{\mathfrak{G}} W(h)\pi^{\ell}(\xi(gh))W(h^{-1})\eta(h^{-1})dh, \quad \xi,\eta \in \tilde{\mathfrak{A}}\ , \end{aligned}$$

where the last identity follows from Lemma 1.3.

We want to show that $\tilde{\mathfrak{A}}$ is a left Hilbert algebra with the above operations (and the innerproduct inherited from $L^2(\mathfrak{G}; \mathcal{H})$).

<u>Proposition 3.3</u>. $\mathfrak{A}$ is a dense subspace of the Hilbert space $L^2(\mathfrak{G}; \mathcal{H})$, and it becomes a left Hilbert algebra with the above operations and the inner product inherited from $L^2(\mathfrak{G}; \mathcal{H})$.

<u>Proof</u>. We first show that $\tilde{\mathfrak{A}} * \tilde{\mathfrak{A}}$ (and hence $\tilde{\mathfrak{A}}$) is dense in $L^2(\mathfrak{G}; \mathcal{H})$. Take $a_i, b_i \in \mathfrak{N}$, $F_i \in \mathcal{K}(\mathfrak{G})$ and set $x_i(g) = F_i(g)\alpha_g^{-1}(b_i^*)a_i$, $i = 1,2$. Then $x_i \in \tilde{\mathfrak{M}}$ because:

$$\varphi(x_i(g)^* x_i(g)) \leq |F_i(g)|^2 \|b_i\|^2 \varphi(a_i^* a_i)$$

and

$$\varphi(x_i(g)x_i(g)^*) \leq |F_i(g)|^2 \|a_i\|^2 \varphi(b_i^* b_i), \quad i = 1,2\ .$$

We have:

$$(x_1 * x_2)(g) = \alpha_g^{-1}(b_1^*)\left[\int_{\mathfrak{G}} F_2(h^{-1})F_1(gh)\alpha_h(a_1 b_2^*)dh\right]\cdot a_2$$

hence:

$$(\tilde{\Lambda}(x_1) * \tilde{\Lambda}(x_2))(g) = \pi(\alpha_g^{-1}(b_1^*))\left[\int_{\mathfrak{G}} F_1(gh)F_2(h^{-1})\pi(\alpha_h(a_1b_2^*))dh\right]\Lambda(a_2) .$$

Substituting for a_1 and b_2 sequences $\{a_1^{(n)}\}$ and $\{b_2^{(n)}\}$ from $\mathfrak{M}$ such that $\|a_1^{(n)}\| \leq 1$, $\|b_2^{(n)}\| \leq 1$, $\lim_n a_1^{(n)} = \lim_n b_2^{(n)} = 1$ (σ-strongly*) and setting:

$$B_n(g) = \int_{\mathfrak{G}} F_1(gh)F_2(h^{-1})\pi(\alpha_h(a_1^{(n)}b_2^{(n)*}))dh ,$$

we have $B_n(g) \to (F_1 * F_2)(g)\cdot I$ σ-weakly by lemma 1; also from §1: $\|B_n(g)\| \leq (|F_1|*|F_2|)(g)$, an integrable function. Now assume $\xi \in L^2(\mathfrak{G}; \mathcal{H})$ is such that $\langle \xi, \tilde{\Lambda}(x_1 * x_2)\rangle = 0$ for all x_1, x_2 of the above type. We have:

$$0 = \langle \xi, \tilde{\Lambda}(x_1 * x_2)\rangle = \int_{\mathfrak{G}} \langle \xi(g), \pi(\alpha_g^{-1}(b_1^*))B_n(g)\Lambda(a_2)\rangle dg ,$$

for all $n \in \mathbb{Z}^+$ hence by Lebesgue's theorem:

$$0 = \lim_n \int_{\mathfrak{G}} \langle \xi(g), \pi(\alpha_g^{-1}(b_1^*))B_n(g)\Lambda(a_2)\rangle dg$$

$$= \int_{\mathfrak{G}} \lim_n \langle \xi(g), \pi_e(\alpha_g^{-1}(b_1^*))B_n(g)\Lambda(a_2)\rangle dg$$

$$= \int_{\mathfrak{G}} \langle \xi(g), \pi(\alpha_g^{-1}(b_1^*))(F_1 * F_2)(g)\Lambda(a_2)\rangle dg$$

for all $F_1, F_2 \in \mathcal{K}(\mathfrak{G})$. Since $\mathcal{K}(\mathfrak{G}) * \mathcal{K}(\mathfrak{G})$ is dense in $L^2(\mathfrak{G})$ it follows that $\langle \xi(g), \Lambda(\alpha_g^{-1}(b_1^*)a_2)\rangle = 0$ for almost all $g \in \mathfrak{G}$ and for all $a_2, b_1 \in \mathfrak{N}$. But for all $g \in \mathfrak{G}$ the set $\{\Lambda(\alpha_g^{-1}(b_1^*)a_2);\ a_2, b_1 \in \mathfrak{N}\}$ is total in $\mathfrak{H}$ (§2), hence $\xi(g) = 0$ a.e., that is $\xi = 0$. It follows that $\tilde{\mathfrak{A}} * \tilde{\mathfrak{A}}$, and hence $\tilde{\mathfrak{A}}$, is dense in $L^2(\mathfrak{G}; \mathfrak{H})$.

The boundedness of the map $\eta \mapsto \xi * \eta$ for all $\xi \in \tilde{\mathfrak{A}}$ has already been shown in the previous proposition; namely, setting $\xi = \tilde{\Lambda}(x)$ $(x \in \tilde{\mathfrak{M}})$ it follows from inequality (3.1) that

$$\|\xi * \eta\| \leq \left[\int_{\mathfrak{G}} \|x(g)\| dg\right] \cdot \|\eta\| = \left[\int_{\mathfrak{G}} \|\pi^{\ell}(\xi(g))\| dg\right] \cdot \|\eta\| ,$$

thus $\|\tilde{\pi}^{\ell}(\xi)\| \leq \int_{\mathfrak{G}} \|\pi^{\ell}(\xi(g))\| dg$, where $\tilde{\pi}^{\ell}$ denotes the left regular representation associated with $\tilde{\mathfrak{A}}$.

The verification of the identity $\langle \xi * \eta, \zeta\rangle = \langle \eta, \xi^{\#} * \zeta\rangle$ is a straightforward application of Fubini's theorem and is left to the reader.

Finally we show that the $\#$-operation is preclosed. Let $\{\xi_n\}$ be a sequence from $\mathfrak{A}$ such that $\xi_n \to 0$, set $\eta_n = \xi_n^{\#}$ and assume $\eta_n \to \eta$. By passing to a subsequence, if necessary, we may assume $\xi_n(g) \to 0$ and $\eta_n(g) \to \eta(g)$ a.e. Then $\xi_n(g^{-1}) \to 0$ a.e., and since $S_{g^{-1},e}$ is preclosed:

$$S_{g^{-1},e}\, \xi_n(g^{-1}) \to 0 \quad \text{a.e.},$$

thus:

$$\eta_n(g) = \delta(g)^{-1} U_{e,g^{-1}} S_{g^{-1},e}\, \xi_n(g^{-1}) \to 0 \quad \text{a.e.},$$

that is: $\eta(g) = 0$ a.e., and preclosedness follows. Q.E.D.

We are now in a position to obtain a description of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ on $L^2(\mathfrak{G}; \mathfrak{H})$ as the left von Neumann algebra of $\tilde{\mathfrak{A}}$. But first we fix some notation. We denote by $\tilde{S}$ the closure of the #-operation of $\tilde{\mathfrak{A}}$ and let $\tilde{S} = \tilde{J}\tilde{\Delta}^{1/2}$ be its polar docomposition, where as usual $\tilde{J}$ is conjugate unitary and $\tilde{\Delta}$ is positive, selfadjoint, non-singular. In general we qualify with a tilda "~" objects pertaining to $\tilde{\mathfrak{A}}$; thus: $\tilde{\pi}^\ell$ = left regular representation of $\tilde{\mathfrak{A}}$; $\tilde{\sigma}_t$: the associated modular automorphism group; $\tilde{\varphi}$: the f.n.s.f. weight on $\mathfrak{R}_\ell(\tilde{\mathfrak{A}})$ associated with the fulfilment of $\tilde{\mathfrak{A}}$ etc. We also define a mapping $\tilde{\pi} : \tilde{\mathfrak{M}} \to \mathfrak{B}(L^2(\mathfrak{G}; \mathfrak{H}))$ by:

$$\tilde{\pi}(x) = \int_{\mathfrak{G}} \tilde{\lambda}(g)\pi_\alpha(x(g))dg, \qquad x \in \tilde{\mathfrak{M}},$$

where $(\pi_\alpha, \tilde{\lambda})$ is the covariant representation of $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ on $L^2(\mathfrak{G}; \mathfrak{H})$ based on π (see §1). This notation is consistent with the previous use of the symbol $\tilde{\pi}$ (namely as the representation of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ on $L^2(\mathfrak{G}; \mathfrak{H})$), as the next theorem shows:

<u>Theorem 3.4</u>. The mapping $\tilde{\pi} : \tilde{\mathfrak{M}} \to \mathfrak{B}(L^2(\mathfrak{G}; \mathfrak{H}))$ is a #-representation of $\tilde{\mathfrak{M}}$ on $L^2(\mathfrak{G}; \mathfrak{H})$ and $\tilde{\pi}(\tilde{\mathfrak{M}})'' = \tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})$. We have $\tilde{\pi} = \tilde{\pi}^\ell \circ \tilde{\Lambda}$, hence $\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})$ coincides with the left von Neumann algebra $\mathfrak{R}_\ell(\tilde{\mathfrak{A}})$ of $\tilde{\mathfrak{A}}$.

<u>Proof</u>. Let $x, y \in \tilde{\mathfrak{M}}$. By Fubini's theorem:

$$\tilde{\pi}(x)\tilde{\pi}(y) = \int_{\mathfrak{G}} \tilde{\lambda}(g)\pi_\alpha(x(g))dg \int_{\mathfrak{G}} \tilde{\lambda}(h)\pi_\alpha(y(h))dh$$

$$= \int_{\mathfrak{G}} \left(\int_{\mathfrak{G}} \tilde{\lambda}(gh)\pi_\alpha(\alpha_h^{-1}(x(g)))dg \right) \pi_\alpha(y(h))dh$$

$$= \int_{\mathfrak{G}} \left(\delta(h)^{-1} \int_{\mathfrak{G}} \tilde{\lambda}(g)\pi_\alpha(\alpha_h^{-1}(x(gh^{-1})))dg \right) \pi_\alpha(y(h))dh$$

$$= \int_{\mathfrak{G}} \tilde{\lambda}(g) \left[\int_{\mathfrak{G}} \delta(h)^{-1}\pi_\alpha(\alpha_h^{-1}(x(gh^{-1}))y(h))dh \right] dg$$

$$= \int_{\mathfrak{G}} \tilde{\lambda}(g)\pi_\alpha \left(\int_{\mathfrak{G}} \alpha_h(x(gh))y(h^{-1})dh \right) dg$$

$$= \int_{\mathfrak{G}} \tilde{\lambda}(g)\pi_\alpha((x * y)(g))dg$$

$$= \tilde{\pi}(x * y),$$

and:

$$\tilde{\pi}(x^{\#}) = \int_{\mathfrak{G}} \tilde{\lambda}(g)\pi_\alpha(x^{\#}(g))dg$$

$$= \int_{\mathfrak{G}} \tilde{\lambda}(g)\delta(g)^{-1}\pi_\alpha(\alpha_g^{-1}(x(g^{-1})))^* dg$$

$$= \int_{\mathfrak{G}} \tilde{\lambda}(g^{-1})\tilde{\lambda}(g)\pi_\alpha(x(g))^*\tilde{\lambda}(g)^* dg$$

$$= \int_{\mathfrak{G}} \pi_\alpha(x(g))^*\tilde{\lambda}(g)^* dg = \left(\int_{\mathfrak{G}} \tilde{\lambda}(g)\pi_\alpha(x(g))dg \right)^*$$

$$= \tilde{\pi}(x)^*,$$

proving the first assertion.

As for the second statement it is clear that $\tilde{\pi}(\tilde{\mathfrak{M}}) \subseteq \tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})$, the latter being generated by elements of the form $\pi_\alpha(a)$, $\tilde{\lambda}(g)$, $a \in \mathfrak{M}$,

$g \in \mathfrak{G}$. On the other hand, if an element $z \in \mathcal{B}(L^2(\mathfrak{G}; \mathcal{H}))$ commutes with $\tilde{\pi}(\tilde{\mathfrak{M}})$, it commutes with all elements of the form

$$(*) \qquad \pi_\alpha(b^*) \int_{\mathfrak{G}} F(g)\pi_\alpha(\alpha_g(a))\tilde{\lambda}(g)dg, \qquad a,b \in \mathfrak{N}, \qquad F \in \mathcal{K}(\mathfrak{G}),$$

hence with all elements of the form:

$$\int_{\mathfrak{G}} F(g)\pi_\alpha(\alpha_g(a))\tilde{\lambda}(g)dg, \qquad a \in \mathfrak{N}, \qquad F \in \mathcal{K}(\mathfrak{G}) .$$

Taking a net $\{a_i\}$ from $\mathfrak{N}$ such that $a_i \to I$, it follows by the polarization identity and Lemma 1.2 that

$$\int_{\mathfrak{G}} F(g)\pi_\alpha(\alpha_g(a_i))\tilde{\lambda}(g)dg \to \int_{\mathfrak{G}} F(g)\tilde{\lambda}(g)dg \qquad \sigma\text{-weakly},$$

thus z commutes with all the $\int_{\mathfrak{G}} F(g)\tilde{\lambda}(g)dg$, $F \in \mathcal{K}(\mathfrak{G})$. But then by standard group representation theory, z commutes with all the $\tilde{\lambda}(g)$'s. Since the representation $F \mapsto \int_{\mathfrak{G}} F(g)\tilde{\lambda}(g)dg$ of $\mathcal{K}(\mathfrak{G})$ on $L^2(\mathfrak{G}; \mathcal{H})$ is non-degenerate it is now clear from $(*)$ above that z also commutes with all the $\pi_\alpha(b)$, $b \in \mathfrak{m}$; thus $\tilde{\pi}(\tilde{\mathfrak{M}})' \subseteq \tilde{\pi}(\mathfrak{m} \otimes_\alpha \mathfrak{G})'$ and hence $\tilde{\pi}(\tilde{\mathfrak{m}})'' = \tilde{\pi}(\mathfrak{m} \otimes_\alpha \mathfrak{G})$.

As for the last statement we have for all $x,y \in \tilde{\mathfrak{M}}$:

$$\tilde{\pi}(x)\tilde{\Lambda}(y) = \int_{\mathfrak{G}} \tilde{\lambda}(h)\pi_\alpha(x(h))\tilde{\Lambda}(y)dh$$

and:

$$(\tilde{\pi}^{\ell} \circ \tilde{\Lambda}(x)\tilde{\Lambda}(y))(g) = (\tilde{\Lambda}(x * y))(g)$$

$$= \int_{\mathfrak{G}} \pi(\alpha_h(x(gh)))\Lambda(y(h^{-1}))dh$$

$$= \int_{\mathfrak{G}} \pi(\alpha^{-1}_{h^{-1}g} (x(h)))\Lambda y(h^{-1}g)dh$$

$$= \int_{\mathfrak{G}} (\pi_{\alpha}(x(h))\tilde{\Lambda}y)(h^{-1}g)dh$$

$$= \int_{\mathfrak{G}} (\tilde{\lambda}_h\pi_{\alpha}(x(h))\tilde{\Lambda}y)(g)dh$$

By a straightforward application of Fubini's theorem we have:

$$\left[\int_{\mathfrak{G}} \tilde{\lambda}(h)\pi_{\alpha}(x(h))\tilde{\Lambda}(y)dh\right](g) = \int_{\mathfrak{G}} [\tilde{\lambda}(h)\pi_{\alpha}(x(h))\tilde{\Lambda}(y)](g)dh \quad \text{a.e.},$$

and the assertion follows. Q.E.D.

Definition 3.5. The canonical weight $\tilde{\varphi}$ associated with the left Hilbert algebra $\tilde{\mathfrak{A}}$ (see §1) is called the dual weight of φ.

Corollary 3.6. For $x,y \in \tilde{\mathfrak{A}}$ we have:

$$\tilde{\varphi}(\tilde{\pi}(x)\tilde{\pi}(y)) = \varphi((x * y)(e)) .$$

Proof. Using the above theorem we get:

$$\tilde{\varphi}(\tilde{\pi}(x)\tilde{\pi}(y)) = \tilde{\varphi}(\tilde{\pi}^{\ell}(\tilde{\Lambda}(x^{\#}))^{*}\tilde{\pi}^{\ell}(\Lambda(y)))$$

$$= \langle \Lambda(y), \Lambda(x^{\#})\rangle = \int_{\mathfrak{G}} \varphi(x^{\#}(g)^{*}y(g))\,dg$$

$$= \int_{\mathfrak{G}} \varphi(\delta(g)^{-1}\alpha_g^{-1}(x(g^{-1}))y(g))dg$$

$$= \int_{\mathfrak{G}} \varphi(\alpha_g(x(g))y(g^{-1}))dg$$

$$= \varphi\left(\int_{\mathfrak{G}} \alpha_g(x(g))y(g^{-1})dg\right)$$

$$= \varphi((x * y)(e)) ,$$

where in the second last equality we once again used the polarization identity and the normality of φ. Q.E.D.

We now set out to compute the modular objects associated with $\tilde{\varphi}$. For this we first define for each $t \in \mathbb{R}$ a mapping $\tau_t : \tilde{\mathfrak{M}} \to \tilde{\mathfrak{M}}$ by:

(3.3) $$(\tau_t x)(g) = \delta(g)^{it} u_t^{g,e} \sigma_t(x(g)), \quad x \in \tilde{\mathfrak{M}} ,$$

where we recall that $u_t^{g,e} = (D(\varphi \circ \alpha_g) : D\varphi)_t$.

Lemma 3.7. The mapping $t \in \mathbb{R} \to \tau_t$ is a 1-parameter group of #-automorphisms on $\tilde{\mathfrak{M}}$.

Proof. We must first show that if $x \in \tilde{\mathfrak{M}}$, then indeed $\tau_t x \in \tilde{\mathfrak{M}}$, for all $t \in \mathbb{R}$. We compute:

$$\varphi((\tau_t x)(g)^*(\tau_t x)(g)) = \varphi(\sigma_t(x(g)^*)u_t^{e,g}u_t^{g,e}\sigma_t(x(g)))$$

$$= \varphi(x(g)^*x(g))$$

and similarly:

$$\varphi_g((\tau_t x)(g)(\tau_t x)(g)^*) = \varphi_g(x(g)x(g)^*) .$$

By Proposition 2.7 the function $g \mapsto (\tau_t x)(g)$ is continuous, thus $\tau_t x \in \widetilde{\mathfrak{M}}$. The group property of the mapping $t \mapsto \tau_t$ is clear, and for $x, y \in \widetilde{\mathfrak{M}}$ we have:

$$(\tau_t x * \tau_t y)(g) = \int_{\mathfrak{G}} \alpha_h(\tau_t x(gh))\tau_t y(h^{-1})dh$$

$$= \int_{\mathfrak{G}} \alpha_h(\delta(gh)^{it} u_t^{gh,e} \sigma_t(x(gh)))\delta(h)^{-it} u_t^{h^{-1},e} \sigma_t(y(h^{-1}))dh$$

$$= \int_{\mathfrak{G}} \delta(g)^{it} u_t^{g,h^{-1}} \sigma_t^{h^{-1}} \circ \alpha_h(x(gh)) u_t^{h^{-1},e} \sigma_t(y(h^{-1}))dh$$

$$= \int_{\mathfrak{G}} \delta(g)^{it} u_t^{g,e} \sigma_t \circ \alpha_h(x(gh))\sigma_t(y(h^{-1}))dh$$

$$= \delta(g)^{it} u_t^{g,e} \sigma_t\left(\int_{\mathfrak{G}} \alpha_h(x(gh))y(h^{-1})dh\right)$$

$$= (\tau_t(x * y))(g);$$

and:

$$(\tau_t x)^{\#}(g) = \delta(g)^{-1}\alpha_g^{-1}((\tau_t x)(g^{-1}))^*$$

$$= \delta(g)^{-1}\alpha_g^{-1}(\delta(g)^{-it} u_t^{g^{-1},e} \sigma_t(x(g^{-1})))^*$$

$$= \delta(g)^{it}\alpha_g^{-1}(\delta(g)^{-1}\sigma_t(x(g^{-1})^*) u_t^{e,g^{-1}})$$

$$= \delta(g)^{-it}\sigma_t^g(\delta(g)^{-1}\alpha_g^{-1}(x(g^{-1})^*)) u_t^{g,e}$$

$$= \delta(g)^{it} u_t^{g,e} \sigma_t(x^{\#}(g))$$

$$= (\tau_t x^{\#})(g) ,$$

which shows the #-automorphism properties. Q.E.D.

The group $\{\tau_t\}_{t\in\mathbb{R}}$ gives rise to a 1-parameter group of #-automorphisms $\{T_t\}_{t\in\mathbb{R}}$ on $\tilde{\mathfrak{A}}$ by setting $T_t = \tilde{\Lambda} \circ \tau_t \circ \tilde{\Lambda}^{-1}$. From the relations in §2 we then get for $\xi = \tilde{\Lambda}(x) \in \tilde{\mathfrak{A}}$:

$$(T_t\xi)(g) = (\tilde{\Lambda} \circ \tau_t(x))(g) = \Lambda(\delta(g)^{it} u_t^{g,e} \sigma_t(x(g)))$$

$$= \delta(g)^{it}(\Delta_e^g)^{it}\xi(g) = H_g^{it}\xi(g), \text{ where } H_g = \delta(g)\Delta_e^g .$$

T_t is thus a decomposable isometry on $\tilde{\mathfrak{A}}$, and its unique extension to a unitary operator on $L^2(\mathfrak{G}; \mathcal{H})$ will also be denoted by T_t. We now identify the modular automorphism group and the modular operator of $\tilde{\varphi}$.

<u>Theorem 3.8</u>. $\{T_t\}_{t\in\mathbb{R}}$ is a continuous 1-parameter group of isometric #-automorphisms of the left Hilbert algebra $\tilde{\mathfrak{A}}$, and the continuous 1-parameter group of automorphisms $\{\tilde{\sigma}_t\}_{t\in\mathbb{R}}$ on $\mathfrak{R}_\ell(\tilde{\mathfrak{A}})$ defined by $\tilde{\sigma}_t(\tilde{\pi}^\ell(\xi)) = \tilde{\pi}^\ell(T_t\xi)$, $\xi \in \tilde{\mathfrak{A}}$, coincides with the modular automorphism group of $\tilde{\varphi}$. The modular operator $\tilde{\Delta}$ of $\tilde{\varphi}$ is thus determined by: $\tilde{\Delta}^{it} = T_t$.

<u>Proof</u>. The algebraic and isometric properties of the T_t's have already been verified in the previous lemma. As for the continuity we have for $\xi, \eta \in L^2(\mathfrak{G}; \mathcal{H})$:

$$\langle (T_t - I)\xi, \eta\rangle = \int_{\mathfrak{G}} \langle (H_g^{it} - I)\xi(g), \eta(g)\rangle dg .$$

Since $|\langle (H_g^{it} - I)\xi(g), \eta(g)\rangle| \leq 2\|\xi(g)\|\cdot\|\eta(g)\|$, Lebesgue's theorem gives:

$$\lim_{t\to 0} \langle (T_t - I)\xi, \eta\rangle = \int_{\mathfrak{G}} \lim_{t\to 0} \langle (H_g^{it} - I)\xi(g), \eta(g)\rangle dg = 0 ,$$

hence the group $\{T_t\}$ is continuous.

We now define a 1-parameter group $\{\tilde{\sigma}_t\}$ of automorphisms on $\mathcal{R}_\ell(\tilde{\mathfrak{A}})$ by setting $\tilde{\sigma}_t(\tilde{\pi}^\ell(\xi)) = \tilde{\pi}^\ell(T_t\xi)$, $\xi \in \tilde{\mathfrak{A}}$; or in terms of elements x in $\tilde{\mathfrak{M}}$: $\tilde{\sigma}_t(\tilde{\pi}(x)) = \tilde{\pi}(\tau_t x)$. We have $\tilde{\varphi} \circ \tilde{\sigma}_t(\tilde{\pi}^\ell(\xi)^*\tilde{\pi}^\ell(\xi)) = \tilde{\varphi}(\tilde{\pi}^\ell(T_t\xi)^*\tilde{\pi}^\ell(T_t\xi)) = \|T_t\xi\|^2 = \|\xi\|^2 = \tilde{\varphi}(\tilde{\pi}^\ell(\xi)^*\tilde{\pi}^\ell(\xi))$, thus $\tilde{\varphi}$ is $\tilde{\sigma}_t$-invariant. To complete the proof that $\{\tilde{\sigma}_t\}$ is the modular automorphism group of $\tilde{\varphi}$ we must verify the KMS-condition. For this take $x,y \in \tilde{\mathfrak{M}}$ and consider the functions:

$$\tilde{G}_1 : t \mapsto \tilde{\varphi}(\tilde{\sigma}_t(\tilde{\pi}(x))\tilde{\pi}(y)) = \varphi((\tau_t x * y)(e))$$

and

$$\tilde{G}_2 : t \mapsto \tilde{\varphi}(\tilde{\pi}(y)\tilde{\sigma}_t(\tilde{\pi}(x))) = \varphi((y * \tau_t x)(e))$$

We have:

$$\tilde{G}_1(t) = \varphi\left(\int_{\mathfrak{G}} \alpha_g((\tau_t x)(g))y(g^{-1})dg\right)$$

$$= \int_{\mathfrak{G}} \delta(g)^{it}\varphi(\alpha_g(u_t^{g,e}\sigma_t(x(g)))y(g^{-1}))dg$$

$$= \int_{\mathfrak{G}} \delta(g)^{it}\varphi_g(u_t^{g,e}\sigma_t(x(g))\alpha_{g^{-1}}(y(g^{-1})))dg$$

and:

$$\tilde{G}_2(t) = \varphi\left(\int_{\mathfrak{G}} \alpha_g(y(g))(\tau_t x)(g^{-1})dg\right)$$

$$= \int_{\mathfrak{G}} \delta(g)^{-it}\varphi(\alpha_g(y(g))u_t^{g^{-1},e}\sigma_t(x(g^{-1})))dg$$

$$= \int_{\mathfrak{G}} \delta(g)^{it-1}\varphi(\alpha_{g^{-1}}(y(g^{-1}))u_t^{g,e}\sigma_t(x(g)))dg$$

We know (Proposition 2.2) that for each $g \in \mathfrak{G}$ there is a KMS-function F_g such that:

$$F_g(t) = \varphi_g(u_t^{g,e}\sigma_t(x(g))\alpha_{g^{-1}}(y(g^{-1})))$$

and

$$F_g(t + i) = \varphi(\alpha_g^{-1}(y(g^{-1}))u_t^{g,e}\sigma_t(x(g))) .$$

We note that:

$$|F_g(t)| \leq [\varphi(y(g^{-1})^* y(g^{-1})]^{1/2}[\varphi_g(x(g)x(g)^*)]^{1/2} = M_1(g)$$

and

$$|F_g(t + i)| \leq [\varphi(x(g)^* x(g))]^{1/2}[\varphi_{g^{-1}}(y(g^{-1})y(g^{-1})^*)]^{1/2} = M_2(g) ,$$

for all $t \in \mathbb{R}$, thus setting $M = \max\{M_1, M_2\}$, we have $\max\{|F_g(t)|, |F_g(t + i)|\} \leq M(g)$ for all $t \in \mathbb{R}$, where M is an integrable Borel function with compact support. By the Cauchy formula and the boundedness of F_g we have for all z in the interior of the strip $B = \{z \in \mathbb{C};\ 0 \leq \operatorname{Im} z \leq 1\}$:

$$F_g(z) = \int_\Gamma \frac{F_g(\zeta)}{\zeta - z}\, d\zeta ,$$

where Γ is the boundary of the strip; thus, since the integrand is Borel as a function of g, the function $g \mapsto F_g(z)$ is Borel for all $z \in B$. For each $g \in \mathfrak{G}$ we define a KMS-function G_g by $G_g(z) = \delta(g)^{iz}F_g(z)$. By the Phragmen-Lindelöf theorem we have:

$$|G_g(z)| = \delta(g)^{-s}|F_g(z)| \leq \delta(g)^{-s} \sup_{z\in\Gamma} |F_g(z)|$$

(*)
$$= \delta(g)^{-s} \max\{\sup_{t\in\mathbb{R}} |F_g(t)|, \sup_{t\in\mathbb{R}} |F_g(t+i)|\} \leq \delta(g)^{-s}M(g), \quad (s=\operatorname{Im} z)$$

hence $g \mapsto G_g(z)$ is integrable Borel for all z in the strip. We thus get a well defined function on the strip by setting:

$$\tilde{G}(z) = \int_{\mathfrak{G}} G_g(z)dg .$$

Since each G_g is Borel on the strip, so is $\tilde{G}$, and from the first part of the proof (continuity of $t \mapsto T_t$) $\tilde{G}$ is continuous on the boundary Γ. The two-variable function $(g,z) \in \mathfrak{G} \times B \to G_g(z)$ is Borel in each of the variables, hence it is Borel as a function on $\mathfrak{G} \times B$ ($\mathfrak{G}$ being separable). Thus if γ is a Jordan curve in the interior of the strip, the function: $(g,z) \in \mathfrak{G} \times \gamma \mapsto G_g(z)$ is integrable (by (*) above), and Fubini applies:

$$\int_\gamma \tilde{G}(z)dz = \int_\gamma \left(\int_{\mathfrak{G}} G_g(z)dg\right)dz = \int_{\mathfrak{G}} \left(\int_\gamma G_g(z)dz\right)dg = 0 ,$$

each G_g being analytic. By Morera's theorem $\tilde{G}$ is then analytic in the interior of the strip. Moreover:

$$|\tilde{G}(z)| \leq \int_{\mathfrak{G}} |G_g(z)| dg \leq \int_{\mathfrak{G}} \delta(g)^{-s} M(g) dg \leq c\|M\|_1 ,$$

where $c = \sup\{\delta(g)^{-s};\ g \in \text{support } M,\ 0 \leq s \leq 1\}$, thus $\tilde{G}$ is bounded on the strip. Since $\tilde{G}(t) = \tilde{G}_1(t)$ and $\tilde{G}(t+i) = \tilde{G}_2(t)$, the proof of the KMS-condition is complete. Finally, the identity $\tilde{\Delta}^{it} = T_t$ follows from the fact that $\tilde{\pi}^{\ell}(\tilde{\Delta}^{it}\xi) = \tilde{\pi}^{\ell}(T_t\xi) = \tilde{\sigma}_t(\tilde{\pi}^{\ell}(\xi))$ for all $\xi \in \tilde{\mathfrak{A}}$. Q.E.D.

<u>Corollary 3.9</u>. Setting $H_g = \delta(g)\Delta_e^g$ and letting F be a Borel function on the multiplicative group $R^+ = \langle 0,\infty\rangle$, the modular operator $\tilde{\Delta}$ of $\tilde{\varphi}$ is determined by the following:

A vector $\xi \in L^2(\mathfrak{G};\ \mathfrak{H})$ is in the domain of $F(\tilde{\Delta})$ if and only if $\xi(g)$ is in the domain of $F(H_g)$ for almost all $g \in \mathfrak{G}$ and the function $g \mapsto F(H_g)\xi(g)$ is in $L^2(\mathfrak{G};\ \mathfrak{H})$. If this is the case then:

$$(F(\tilde{\Delta})\xi)(g) = F(H_g)\xi(g) \quad \text{a.e.};$$

in particular:

$$(\tilde{\Delta}\xi)(g) = H_g\xi(g) = \delta(g)\Delta_e^g\xi(g) \quad \text{a.e. for all } \xi \in \tilde{\mathfrak{A}} \subseteq \text{dom}(\tilde{\Delta}) .$$

<u>Proof</u>. It follows from the above that $g \mapsto H_g^{it}$ is measurable field of operators on $L^2(\mathfrak{G};\ \mathfrak{H})$ for all $t \in \mathbf{R}$, hence the field $g \mapsto F(H_g)$ is measurable for all trigonometric polynomials F on the multiplicative group R^+, hence for all bounded Borel functions

F on $\mathbb{R}^+$.

Let $\{E(\lambda)\}$ be the spectral resolution of $\tilde{\Delta}$ and let $\xi, \eta \in L^2(\mathfrak{G}; \mathcal{H})$. Then by the previous theorem the following two bounded linear functionals on $L^\infty(\mathbb{R}^+)$:

$$F \in L^\infty(\mathbb{R}^+) \mapsto \langle F(\tilde{\Delta})\xi, \eta\rangle = \int_{\mathbb{R}^+} F(\lambda) d\langle E(\lambda)\xi, \eta\rangle$$

and

$$F \in L^\infty(\mathbb{R}^+) \mapsto \int_{\mathfrak{G}} \langle F(H_g)\xi(g), \eta(g)\rangle dg$$

coincide on trigonometric polynomials, hence on all of $L^\infty(\mathbb{R}^+)$; that is, $(F(\tilde{\Delta})\xi)(g) = F(H_g)\xi(g)$ a.e. for $F \in L^\infty(\mathbb{R}^+)$, $\xi \in L^2(\mathfrak{G}; \mathcal{H})$.

Now let F be any Borel function on $\mathbb{R}^+$ and set $F_n = F \cdot \chi_n$ where χ_n is the characteristic function of the set $\{\lambda \in \mathbb{R}^+; |F(\lambda)| \leq n\}$, $n \in \mathbb{Z}^+$. Then if $\xi \in \mathrm{dom}(F(\tilde{\Delta}))$, we have $F(\tilde{\Delta})\xi = \lim_n F_n(\tilde{\Delta})\xi$ by [10; p. 1196]; thus, by passing to a subsequence if necessary:

$$(F(\tilde{\Delta})\xi)(g) = \lim(F_n(\tilde{\Delta})\xi)(g) = \lim F_n(H_g)\xi(g)$$

$$= F(H_g)\xi(g) \quad \text{a.e.}$$

It follows that $\xi(g) \in \mathrm{dom}(F(H_g))$ a.e. and that $g \mapsto F(H_g)\xi(g)$ is in $L^2(\mathfrak{G}; \mathcal{H})$.

Conversely, suppose $\xi(g) \in \mathrm{dom}(F(H_g))$ a.e. and that $g \mapsto F(H_g)\xi(g)$ is in $L^2(\mathfrak{G}; \mathcal{H})$. Then by the monotone convergence theorem:

$$\int_{\mathbb{R}^+} |F(\lambda)|^2 d\|E(\lambda)\xi\|^2 = \int_{\mathbb{R}^+} \lim F_n(\lambda)^2 d\|E(\lambda)\xi\|^2$$

$$= \lim \int_{\mathbb{R}^+} |F_n(\lambda)|^2 d\|E(\lambda)\xi\|^2$$

$$= \lim \int_{\mathfrak{G}} \|F_n(H_g)\xi(g)\|^2 dg$$

$$= \int_{\mathfrak{G}} \lim\|F_n(H_g)\xi(g)\|^2 dg$$

$$= \int_{\mathfrak{G}} \|F(H_g)\xi(g)\|^2 dg < \infty \ ,$$

hence $\xi \in \mathrm{dom}(F(\widetilde{\Delta}))$.

The last assertion is now obvious. Q.E.D.

Corollary 3.10. For $a \in \mathfrak{M}$, $g \in \mathfrak{G}$ we have:

(i) $\widetilde{\sigma}_t(\pi_\alpha(a)) = \pi_\alpha(\sigma_t(a))$, $t \in \mathbb{R}$

(ii) $\widetilde{\sigma}_t(\widetilde{\lambda}(g)) = \delta(g)^{it}\widetilde{\lambda}(g)\pi_\alpha(u_t^{g,e})$, $t \in \mathbb{R}$.

Proof. Let $a \in \mathfrak{M}$ and $\xi \in L^2(\mathfrak{G}; \mathcal{H})$; we get:

(i) $$(\widetilde{\sigma}_t(\pi_\alpha(a))\xi)(g) = (\widetilde{\Delta}^{it}\pi_\alpha(a)\widetilde{\Delta}^{-it}\xi)(g)$$

$$= \delta(g)^{it}(\Delta_e^g)^{it}\pi(\alpha_g^{-1}(a))\delta(g)^{-it}(\Delta_e^g)^{-it}\xi(g)$$

$$= \pi(\sigma_t^g \circ \alpha_g^{-1}(a))\xi(g)$$

$$= \pi(\alpha_g^{-1} \circ \sigma_t(a))\xi(g)$$

$$= (\pi_\alpha(\sigma_t(a))\xi)(g), \quad g \in \mathfrak{G}.$$

(ii) $$(\widetilde{\sigma}_t(\widetilde{\lambda}(g))\xi)(h) = (\widetilde{\Delta}^{it}\widetilde{\lambda}(g)\widetilde{\Delta}^{-it}\xi)(h)$$

$$= \delta(h)^{it}(\Delta_e^h)^{it}(\tilde{\Delta}^{-it}\xi)(g^{-1}h)$$

$$= \delta(h)^{it}(\Delta_e^h)^{it}\delta(g^{-1}h)^{-it}(\Delta_e^{g^{-1}h})^{it}\xi(g^{-1}h)$$

$$= \delta(g)^{it}[(\Delta_e^h)^{it}\Delta^{-it}][\Delta^{it}(\Delta_e^{g^{-1}h})^{-it}]\xi(g^{-1}h)$$

$$= \delta(g)^{it}\pi(u_t^{h,e}u_t^{e,g^{-1}h})\xi(g^{-1}h)$$

$$= \delta(g)^{it}\pi(u_t^{h,g^{-1}h})\xi(g^{-1}h)$$

$$= \delta(g)^{it}\pi(\alpha_h^{-1}(u_t^{e,g^{-1}}))\xi(g^{-1}h)$$

$$= (\delta(g)^{it}\pi_\alpha(u_t^{e,g^{-1}})\,(g)\xi)(h)$$

$$= (\delta(g)^{it}\lambda(g)\pi_\alpha(u_t^{g,e})\xi)(h), \quad g,h \in \mathfrak{G}. \qquad \text{Q.E.D.}$$

Before we compute the unitary involution $\tilde{J}$ associated with $\tilde{\varphi}$; we establish some relations between the unitary operators appearing in the expression for $\tilde{S}$. We recall that $(\tilde{S}\xi)(g) = \delta(g)^{-1}S_{e,g}U_{g,e}\xi(g^{-1})$, $\xi \in \tilde{\mathfrak{A}}$, where $U_{g,e} : \mathcal{H} \to \mathcal{H}_g$ is defined by $U_{g,e}\Lambda(a) = \Lambda_g(\alpha_g^{-1}(a))$, $a \in \mathfrak{N}$.

<u>Lemma 3.11</u>. Let $W(h)$ be the canonical unitary implementation of α_h in the representation π, and let $V_{g,e} : \mathcal{H} \to \mathcal{H}_g$ be the unique conepreserving unitary such that $V_{g,e}\pi(a)V_{g,e}^* = \pi_g(a)$, $a \in \mathfrak{M}$ (see §2). Then with $U_{g,e}$ as above we have:

(i) $J_{e,g}U_{g,e} = U_{e,g^{-1}}J_{g^{-1},e}$

(ii) $U_{g,e} = V_{g,e}W(g)$.

Proof.

(i) We have $S_{e,g}U_{g,e} = U_{e,g^{-1}}S_{g^{-1},e}$. Thus, since $U_{e,g^{-1}} = U_{g,e}^{-1}$, the assertion follows from the uniqueness of the polar decomposition $S_{e,g} = J_{e,g}(\Delta_g^e)^{1/2}$ of $S_{e,g}$.

(ii) By [11; Theorem 2.18, proof] $U_{g,e}$ preserves the natural cones associated with φ_e (= φ) and φ_g, and since all three expressions implement the same isomorphism (namely: $\pi(x) \in \pi(\mathfrak{M}) \mapsto \pi_g(\alpha_g^{-1}(x)) \in \pi_\varphi(\mathfrak{M}),\ x \in \mathfrak{M}$), the assertion follows from the uniqueness of the conepreserving unitary implementation (V and W being conepreserving by definition).

Corollary 3.12. The unitary involution $\tilde{J}$ of $\tilde{\varphi}$ is given by:

$$(\tilde{J}\xi)(g) = \delta(g)^{-1/2}W(g^{-1})J\xi(g^{-1}) = \delta(g)^{-1/2}JW(g^{-1})\xi(g^{-1}),\ \xi \in L^2(\mathfrak{G}; \mathcal{H}_e).$$

Proof. By Corollary 3.9, the above lemma and the fact that $J_{g,e} = V_{g,e}J = J_gV_{g,e}$ (Proposition 2.6), we have for $\xi \in \tilde{\mathfrak{A}}$:

$$\begin{aligned}(\tilde{J}\xi)(g) &= (\tilde{\Delta}^{1/2}\tilde{S}\xi)(g)\\ &= \delta(g)^{1/2}(\Delta_e^g)^{1/2}\delta(g)^{-1}S_{e,g}U_{g,e}\xi(g^{-1})\\ &= \delta(g)^{-1/2}J_{e,g}V_{g,e}W(g^{-1})\xi(g^{-1})\\ &= \delta(g)^{-1/2}JW(g^{-1})\xi(g^{-1}) = \delta(g)^{-1/2}W(g^{-1})J\xi(g^{-1})\ .\ \text{Q.E.D.}\end{aligned}$$

We define a representation π' of the commutant $\pi(\mathfrak{M})'$ of $\pi(\mathfrak{M})$ on $L^2(\mathfrak{G}; \mathcal{H})$ by:

$$(\pi'(a)\xi)(g) = a\xi(g), \quad \xi \in L^2(\mathfrak{G};\, \mathcal{H}) .$$

Likewise we define a second representation $\tilde{\rho}_\alpha$ of $\mathfrak{G}$ on $L^2(\mathfrak{G};\, \mathcal{H})$ by:

$$(\tilde{\rho}_\alpha(g)\xi)(h) = \delta(g)^{1/2}W(g)\xi(hg), \quad \xi \in L^2(\mathfrak{G};\, \mathcal{H}) .$$

Identifying $L^2(\mathfrak{G};\, \mathcal{H})$ with $L^2(\mathfrak{G}) \otimes \mathcal{H}$ under the canonical isomorphism we have $\pi'(a) = I \otimes a$ and $\tilde{\rho}_\alpha(g) = \rho(g) \otimes W(g)$ where ρ is the right-regular representation of $\mathfrak{G}$ on $L^2(\mathfrak{G})$, i.e. $(\rho(g)F)(h) = \delta(g)^{1/2}F(hg)$. As an immediate consequence of the last corollary we then get:

<u>Corollary 3.13</u>. The commutant of $\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})$ on $L^2(\mathfrak{G};\, \mathcal{H})$ is generated by the set $\{\pi'(a), \tilde{\rho}_\alpha(g);\ a \in \pi(\mathfrak{M})',\ g \in \mathfrak{G}\}$.

<u>Proof</u>. Since $\{\pi_\alpha(a), \tilde{\lambda}(g);\ a \in \mathfrak{M},\ g \in \mathfrak{G}\}$ generates $\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})$, $\{\tilde{J}\pi_\alpha(a)\tilde{J}, \tilde{J}\tilde{\lambda}(g)\tilde{J};\ a \in \mathfrak{M},\ g \in \mathfrak{G}\}$ generates $\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})'$. For $\xi \in L^2(\mathfrak{G};\, \mathcal{H})$ and $a \in \mathfrak{M}$ we have:

$$\begin{aligned}(\tilde{J}\pi_\alpha(a)\tilde{J}\xi)(g) &= \delta(g)^{-1/2}JW(g^{-1})(\pi_\alpha(a)\tilde{J}\xi)(g^{-1}) \\ &= \delta(g)^{-1}JW(g^{-1})\pi(\alpha_g(a))(\tilde{J}\xi)(g^{-1}) \\ &= \delta(g)^{-1}JW(g^{-1})W(g)\pi(a)W(g^{-1})\delta(g)^{1/2}W(g)J\xi(g) \\ &= J\pi\ (a)J\xi(g) = (\pi'(J\pi(a)J)\xi)(g),\end{aligned}$$

and:

$$(\tilde{J}\tilde{\lambda}(h)\tilde{J}\xi)(g) = \delta(g)^{-1/2}JW(g^{-1})(\tilde{\lambda}(h)\tilde{J}\xi)(g^{-1})$$

$$= \delta(g)^{-1/2} JW(g^{-1})(\tilde{J}\xi)(h^{-1}g^{-1})$$

$$= \delta(g)^{-1/2} JW(g^{-1})\delta(h^{-1}g^{-1})^{-1/2} JW(gh)\xi(gh)$$

$$= \delta(h)^{1/2} W(h)\xi(gh) = (\tilde{\rho}_\alpha(h)\xi)(g) ,$$

thus $\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})' = \{\pi'(J\pi(a)J), \tilde{\rho}_\alpha(h);\ a \in \mathfrak{M},\ h \in \mathfrak{G}\}'' = \{\pi'(b), \tilde{\rho}_\alpha(h);$ $b \in \pi(\mathfrak{M})',\ h \in \mathfrak{G}\}''$. Q.E.D.

The above corollary expresses the commutant of the crossed product on the judicially chosen Hilbert space $L^2(\mathfrak{G}; \mathfrak{H})$ as a crossed product of the commutant $\pi(\mathfrak{M})'$ of $\pi(\mathfrak{M})$. We would like to have a similar commutation theorem for the crossed product based on an arbitrary covariant representation of $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$. This will be accomplished in the next theorem.

<u>Theorem 3.14</u> (the commutation theorem for crossed products). Let (Φ, Γ) be a covariant representation of the covariant system $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ on a Hilbert space $\mathfrak{K}$. Then the commutant $\tilde{\Phi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})'$ of $\tilde{\Phi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})$ on $L^2(\mathfrak{G}; \mathfrak{K})$ is generated by the set $\mathfrak{S}(\pi, \Gamma) =$ $\{I \otimes a,\ \rho(g) \otimes \Gamma(g);\ a \in \Phi(\mathfrak{M})',\ g \in \mathfrak{G}\}$, where $L^2(\mathfrak{G}; \mathfrak{K})$ is identified with $L^2(\mathfrak{G}) \otimes \mathfrak{K}$.

<u>Proof</u>. We first observe that if Γ_1 is another unitary representation of $\mathfrak{G}$ on $\mathfrak{K}$ such that (Φ, Γ_1) is a covariant representation of $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$, then $\mathfrak{S}(\Phi, \Gamma) = \mathfrak{S}(\Phi, \Gamma_1)$ because $\Gamma(g)\Gamma_1(g)^{-1} \in \Phi(\mathfrak{M})'$ for all $g \in \mathfrak{G}$. Letting (π, W) be as above, we denote by $\tilde{\pi}$ and $\tilde{\Phi}$ (resp. π_α and Φ_α) the representations of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ (resp. of $\mathfrak{M}$) on $L^2(\mathfrak{G}; \mathfrak{H})$ and $L^2(\mathfrak{G}; \mathfrak{K})$ corresponding to π and Φ, respectively.

There are three cases to consider, namely according to as whether $\Phi \circ \pi^{-1}$ is a spatial isomorphism, an amplification or an induction [8; Ch. I, §4, Th. 3].

If $\Phi \circ \pi^{-1}$ is a spatial isomorphism implemented by the unitary operator $R : \mathfrak{H} \to \mathfrak{K}$, say, the unitary operator $I \otimes R : L^2(\mathfrak{G}) \otimes \mathfrak{H} \to L^2(\mathfrak{G}) \otimes \mathfrak{H}$ sets up a spatial isomorphism between $\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})$ and $\tilde{\Phi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})$, and the assertion for this case thus follows from the observation at the beginning of the proof.

If $\Phi \circ \pi^{-1}$ is an amplification, i.e. there is a Hilbert space $\mathfrak{L}$ such that $\mathfrak{K} = \mathfrak{H} \otimes \mathfrak{L}$ and $\Phi(a) = \pi(a) \otimes I_{\mathfrak{L}}$, $a \in \mathfrak{M}$, we get:

$$\Phi_\alpha(a) = \pi_\alpha(a) \otimes I_{\mathfrak{L}}, \qquad a \in \mathfrak{M}$$

$$\tilde{\lambda}_\Phi(g) = \tilde{\lambda}(g) \otimes I_{\mathfrak{L}}, \qquad g \in \mathfrak{G},$$

on $L^2(\mathfrak{G};\, \mathfrak{H}) \otimes \mathfrak{L}$, where $\tilde{\lambda}_\Phi(g) = \lambda(g) \otimes I_{\mathfrak{K}}$ on $L^2(\mathfrak{G}) \otimes \mathfrak{K}$, and thus:

$$\tilde{\Phi}(\mathfrak{M} \otimes_\alpha \mathfrak{G}) = \tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G}) \otimes \mathbb{C}_{\mathfrak{L}} .$$

This gives:

$$\begin{aligned}
\tilde{\Phi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})' &= \tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})' \otimes \mathfrak{B}(\mathfrak{L}) \\
&= [(\mathbb{C} \otimes \pi(\mathfrak{M})') \vee (\rho \otimes W)(\mathfrak{G})''] \otimes \mathfrak{B}(\mathfrak{L}) \\
&= [\mathbb{C} \otimes \pi(\mathfrak{M})' \otimes \mathfrak{B}(\mathfrak{L})] \vee [(\rho \otimes W)(\mathfrak{G})'' \otimes \mathbb{C}_{\mathfrak{L}}] \\
&= [\mathbb{C} \otimes \Phi(\mathfrak{M})'] \vee [(\rho \otimes W \otimes I)(\mathfrak{G})]'' \\
&= [\mathbb{C} \otimes \Phi(\mathfrak{M})'] \vee [(\rho \otimes \Gamma)(\mathfrak{G})]'' \\
&= \mathfrak{S}(\Phi,\Gamma) ,
\end{aligned}$$

where we used the fact that $W \otimes I$ and Γ both implement α in the representation $\Phi = \pi \otimes I$. (The tensor-product $U_1 \otimes U_2$ of two representations U_1 and U_2 of $\mathfrak{G}$ is defined by $(U_1 \otimes U_2)(g) = U_1(g) \otimes U_2(g)$).

Finally, if $\Phi \circ \pi^{-1}$ is an induction, i.e. there is a projection E in $\pi(\mathfrak{M})'$ such that $\Phi(a) = \pi(a)_E$, $a \in \mathfrak{M}$, we have:

$$\Phi_\alpha(a) = \pi_\alpha(a)_{I\otimes E}, \quad a \in \mathfrak{M}$$

and

$$\tilde{\lambda}_\Phi(g) = \tilde{\lambda}(g)_{I\otimes E}, \quad g \in \mathfrak{G},$$

and thus:

$$\tilde{\Phi}(\mathfrak{M} \otimes_\alpha \mathfrak{G}) = \tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})_{I\otimes E} .$$

Since $I \otimes E$ lies in $\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})'$ we get:

$$\begin{aligned}\tilde{\Phi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})' &= \tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})'_{I\otimes E} \\ &= [(I \otimes \pi(\mathfrak{M})') \cup (\rho \otimes W)(\mathfrak{G})]''_{I\otimes E} .\end{aligned}$$

A finite product of elements from the set $(I \otimes \pi(\mathfrak{M})') \cup (\rho \otimes W)(\mathfrak{G})$ is of the form $\rho(g) \otimes bW(g)$ for a $g \in \mathfrak{G}$ and a $b \in \pi(\mathfrak{M})'$ (namely, if $g_1,\ldots,g_n \in \mathfrak{G}$ and $b_1,\ldots,b_n \in \pi(\mathfrak{M})'$ and setting $b = b_1W(g_1)b_2W(g_2) \cdots W(g_{n-1})b_n(W(g_1) \cdots W(g_{n-1}))^*$, we have $b \in \pi(\mathfrak{M})'$ (since $\mathrm{ad}(W)$ leaves $\pi(\mathfrak{M})'$ invariant) and $(I \otimes b_1)(\rho(g_1) \otimes W(g_1)) \cdots (I \otimes b_n)(\rho(g_n) \otimes W(g_n)) = \rho(g) \otimes bW(g)$). Since the set $\{\rho(g) \otimes bW(g);\ b \in \pi(\mathfrak{M})',\ g \in \mathfrak{G}\}$ is closed under products and involution, we have by [2; Chapter I, §2, Proposition 1]:

$$\tilde{\Phi}(\mathfrak{M} \otimes_{\alpha} \mathfrak{G})' = \{\rho(g) \otimes bW(g);\ b \in \pi(\mathfrak{M})',\ g \in \mathfrak{G}\}''_{I\otimes E}$$

$$= (\{\rho(g) \otimes bW(g);\ b \in \pi(\mathfrak{M})',\ g \in \mathfrak{G}\}_{I\otimes E})''$$

$$= \{\rho(g) \otimes (bW(g))_E;\ b \in \pi(\mathfrak{M})',\ g \in \mathfrak{G}\}''$$

To finish the proof it therefore remains to show that:

$$\{\rho(g) \otimes a\Gamma(g);\ a \in \Phi(\mathfrak{M}),\ g \in \mathfrak{G}\}' = \{\rho(g) \otimes b_E\Gamma(g);\ b \in \pi(\mathfrak{M})',\ g \in \mathfrak{G}\}$$

$$\subseteq \{\rho(g) \otimes (cW(g))_E;\ c \in \pi(\mathfrak{M})',\ g \in \mathfrak{G}\}'$$

(the other inclusion being obvious). This, of course, will follow immediately if we can show that for each $c \in \pi(\mathfrak{M})'$ and $g \in \mathfrak{G}$, there is a $b \in \pi(\mathfrak{M})'$ such that:

$$(*) \qquad b_E\Gamma(g) = (cW(g))_E .$$

Identifying $\pi(\mathfrak{M})_E$ (resp. $\pi(\mathfrak{M})'_E$) with $\pi(\mathfrak{M})E = \pi(a)E;\ a \in \mathfrak{M}$ (resp. with $E\pi(\mathfrak{M})'E$) and treating the $\Gamma(g)$'s as partial isometries in $\mathfrak{H}$ with $\Gamma(g)^*\Gamma(g) = \Gamma(g)\Gamma(g)^* = E$, (*) becomes:

$$EbE\Gamma(g) = EcW(g)E ,$$

or

$$EbE = EcW(g)\Gamma(g)^* ,$$

that is, we must show that $EcW(g)E$ commutes with $\pi(\mathfrak{M})E$ for all $c \in \pi(\mathfrak{M})'$, $g \in \mathfrak{G}$. But from the consistency relation:

$$\Gamma(g)\pi(a)_E\Gamma(g)^* = (W(g)\pi(x)W(g)^*)_E, \quad a \in \mathfrak{M}, \quad g \in \mathfrak{G},$$

we get, with the same identification as above:

$$W(g)^*\Gamma(g)aE = aW(g)^*\Gamma(g)$$

or:

$$W(g)\Gamma(g)^*aE = aW(g)\Gamma(g)^*, \quad \text{all} \quad a \in \pi(\mathfrak{M}),\ g \in \mathfrak{G} .$$

Thus, for $a \in \pi(\mathfrak{M})$, $c \in \pi(\mathfrak{M})'$, $g \in \mathfrak{G}$:

$$EcW(g)\Gamma(g)^*aE = EcaW(g)\Gamma(g)^* = EacW(g)\Gamma(g)^*$$

$$= aE \cdot EcW(g)\Gamma(g)^* . \qquad \text{Q.E.D.}$$

Remark 3.15. The statement that "the commutant of the crossed product is the crossed product of the commutant" means more precisely the following:

If $\{\Phi,\Gamma\}$ is a covariant representation of $\{\mathfrak{M},\mathfrak{G},\alpha\}$ on a Hilbert space $\mathcal{K}$, we obtain an action α' of $\mathfrak{G}$ on $\Phi(\mathfrak{M})'$ by setting

$$\alpha'_g(a) = \Gamma(g)a\Gamma(g)^*, \quad a \in \Phi(\mathfrak{M})'$$

Denoting by U the unitary operator on $L^2(\mathfrak{G};\ \mathcal{K})$ given by:

$$(U\xi)(g) = \Gamma(g)\xi(g), \quad \xi \in L^2(\mathfrak{G};\ \mathcal{K}) ,$$

we get:

$$U\Phi_\alpha(a)U^* = I \otimes \Phi(a), \qquad a \in \mathfrak{M}$$

$$U\tilde{\lambda}(k)U^* = \lambda(k) \otimes \Gamma(k) = \tilde{\lambda}_\alpha(k), \qquad k \in \mathfrak{G}$$

$$U\Phi'(a)U^* = U(I \otimes a)U^* = \Phi'_{\alpha'}(a), \quad a \in \Phi(\mathfrak{M})'$$

$$U\tilde{\rho}_{\alpha}(k)U^* = \rho(k) \otimes I = \tilde{\rho}(k), \qquad k \in \mathfrak{G},$$

where

$$(\Phi'_{\alpha'}(a)\xi)(g) = \alpha'_g(a)\xi(g), \quad a \in \Phi(\mathfrak{M})' .$$

Thus defining the <u>right-handed</u> crossed product of the covariant system $\{\mathfrak{M},\mathfrak{G},\alpha\}$, based on the representation Φ, as the von Neumann algebra $\mathfrak{R}_r(\mathfrak{M},\alpha,\Phi)$ generated by the covariant representation $\{\Phi'_{\alpha},\tilde{\rho}\}$ of $\{\mathfrak{M},\mathfrak{G},\alpha\}$ on $L^2(\mathfrak{G}; \mathfrak{H})$, where:

$$(\Phi'_{\alpha}(a)\xi)(g) = \Phi(\alpha_g(a))\xi(g), \quad a \in \mathfrak{M}$$

$$(\tilde{\rho}(k)\xi)(g) = \delta(k)^{\frac{1}{2}}\, \xi(gk), \qquad k \in \mathfrak{G} ,$$

we see that the commutant of the crossed product $\mathfrak{R}(\mathfrak{M},\alpha,\Phi)$ is canonically isomorphic to the right-handed crossed product $\mathfrak{R}_r(\Phi(\mathfrak{M})',\alpha',\text{id.})$ of $\Phi(\mathfrak{M})'$ by α' if Φ is covariant; i.e., $U\mathfrak{R}(\mathfrak{M},\alpha,\Phi)'U^* = \mathfrak{R}_r(\Phi(\mathfrak{M})',\alpha',\text{id.})$.

If Φ is an arbitrary normal representation of $\mathfrak{M}$ (not necessarily covariant), it follows from the proof of Theorem 3.14 that the commutant of the crossed product $\mathfrak{R}(\mathfrak{M},\alpha,\Phi)$ still is a reduced algebra of the right-handed crossed product of the commutant of $\mathfrak{M}$ in some covariant representation.

<u>Remark 3.16</u>. We note that Theorem 3.8 gives the following sufficient condition for the semifiniteness of the crossed product:

If there exists a f.n.s.f. trace φ on $\mathfrak{M}$ such that $\varphi \circ \alpha_g =$

$\delta(g)^{-1}\varphi$, where δ is the modular function of $\mathfrak{G}$, then the crossed product $\mathfrak{M} \otimes_{\alpha} \mathfrak{G}$ is semifinite.

Namely, in this case we have $\sigma_t^{\varphi} = \mathrm{id}$ and $u_t^{g,e} = (D(\varphi \circ \alpha_g) : D\varphi)_t = \delta(g)^{-it}$ thus by Theorem 3.8 and formula (3.3) $\sigma_t^{\tilde{\varphi}} = \text{identity}$, for all $t \in \mathbb{R}$. It follows that the dual weight $\tilde{\varphi}$ of φ is a f.n.s.f. trace on $\mathfrak{M} \otimes_{\alpha} \mathfrak{G}$, hence $\mathfrak{M} \otimes_{\alpha} \mathfrak{G}$ is semifinite.

§4. Second dual weight and the "twisted" Plancherel theorem

In this section the group $\mathfrak{G}$ will be assumed to be abelian, and φ will be a fixed f.n.s.f. weight on $\mathfrak{M}$. We shall investigate the relationship between the second dual weight $\tilde{\tilde{\varphi}}$ of φ and the tensor product weight $\varphi \otimes \mathrm{Tr}$ on $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$, where Tr is the natural trace on $\mathfrak{B}(L^2(\mathfrak{G}))$; or, more precisely, we shall determine the amount by which the image of $\tilde{\tilde{\varphi}}$ on $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$, under the natural isomorphism, is twisted away from $\varphi \otimes \mathrm{Tr}$, as measured by the Radon-Nikodym cocycle $(D\tilde{\tilde{\varphi}} : D(\varphi \otimes \mathrm{Tr}))$.

In what follows the W^*-algebra $\mathfrak{M}$ will be identified with its image in the canonical representation associated with φ (i.e., π_φ = identity), and the Hilbert space of φ will be denoted by $\mathfrak{H}$. We recall the definition of the dual action $\hat{\alpha}$ of α (our definition is conjugate to that of [15] since we use the conjugate definition of the Fourier-transform):

For each $p \in \hat{\mathfrak{G}}$, the dual group of $\mathfrak{G}$, we denote by $\mu(p)$ the unitary operator on $L^2(\mathfrak{G})$ given by:

$$(\mu(p)\xi)(g) = \langle g,p\rangle\xi(g) \qquad \xi \in L^2(\mathfrak{G}),\ p \in \hat{\mathfrak{G}}\ ,$$

where $\langle g,p\rangle = p(g)$ denotes the value of the character p at the point g. Setting $\tilde{\mu}(p) = I \otimes \mu(p)$ on $\mathfrak{H} \otimes L^2(\mathfrak{G})$ we have:

$$\tilde{\mu}(p)\pi_\alpha(a)\tilde{\mu}(p)^* = \pi_\alpha(a), \quad a \in \mathfrak{M}$$

and

$$\tilde{\mu}(p)\tilde{\lambda}(g)\tilde{\mu}(p)^* = \langle g,p\rangle\tilde{\lambda}(g), \quad g \in \mathfrak{G} .$$

Thus the mapping: $p \in \hat{\mathfrak{G}} \mapsto \mathrm{Ad}(\tilde{\mu}(p))$ defines an action of $\hat{\mathfrak{G}}$ on the crossed product $\mathfrak{R}(\mathfrak{M},\alpha)$. We set $\hat{\alpha}(p) = \mathrm{Ad}(\tilde{\mu}(p))$, and $\hat{\alpha}$ is called the dual action of α. Clearly the definition of $\hat{\alpha}$ is independent of (i.e., depends functorially on) the particular representation chosen for $\mathfrak{M}$, thus we have a well-defined action of $\hat{\mathfrak{G}}$ on the abstract crossed product $\mathfrak{M} \otimes_\alpha \mathfrak{G}$. Denoting by γ the regular representation of $\hat{\mathfrak{G}}$ on $L^2(\hat{\mathfrak{G}})$ and setting $\tilde{\gamma} = I \otimes I \otimes \gamma$ on $\mathfrak{H} \otimes L^2(\mathfrak{G}) \otimes L^2(\hat{\mathfrak{G}})$, the crossed product $(\mathfrak{M} \otimes_\alpha \mathfrak{G}) \otimes_{\hat{\alpha}} \hat{\mathfrak{G}}$ of $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ by $\hat{\alpha}$ is realized on the Hilbert space $L^2(\hat{\mathfrak{G}};\ L^2(\mathfrak{G};\ \mathfrak{H})) = L^2(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{H}) = \mathfrak{H} \otimes L^2(\mathfrak{G}) \otimes L^2(\hat{\mathfrak{G}})$ as the von Neumann algebra $\mathfrak{R}(\mathfrak{M},\alpha,\hat{\alpha}) = \mathfrak{R}(\mathfrak{R}(\mathfrak{M},\alpha),\hat{\alpha})$ generated by the following operators

$$(\pi_{\hat{\alpha}}(\pi_\alpha(a))\xi)(g,p) = \alpha_g^{-1}(a)\xi(g,p), \qquad a \in \mathfrak{M}$$

$$(\pi_{\hat{\alpha}}(\tilde{\lambda}(h))\xi)(g,p) = \overline{\langle h,p\rangle}\xi(g-h,p), \quad k \in \mathfrak{G}$$

$$(\tilde{\gamma}(q)\xi)(g,p) = \xi(g,p-q), \qquad q \in \hat{\mathfrak{G}}$$

We set $\Phi(a) = \pi_{\hat{\alpha}}(\pi_\alpha(a))$, $a \in \mathfrak{M}$, $v(h) = \pi_{\hat{\alpha}}(\tilde{\lambda}(h))$, $h \in \mathfrak{G}$ and $w(q) = \tilde{\gamma}(q)$, $q \in \hat{\mathfrak{G}}$. In order to see more clearly what the isomorphism between $\mathfrak{R}(\mathfrak{M},\alpha,\hat{\alpha})$ and $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$ looks like we transform $\mathfrak{R}(\mathfrak{M},\alpha,\hat{\alpha})$ to the Hilbert space $L^2(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{H})$ in two steps as follows:

First, letting $\mathfrak{F} : L^2(\mathfrak{G}) \to L^2(\hat{\mathfrak{G}})$ be the Fourier transform, i.e.:

$$(\mathfrak{F}\xi)(p) = \int_{\mathfrak{G}} \overline{\langle g,p\rangle}\xi(g)dg, \quad \xi \in \mathfrak{K}(\mathfrak{G}) ,$$

with inverse $\mathfrak{F}^*$:

$$(\mathfrak{F}^*\xi)(g) = \int_{\mathfrak{G}} \langle g,p\rangle\xi(p)dp, \quad \xi \in \mathcal{K}(\hat{\mathfrak{G}}) ,$$

we set $\tilde{\mathfrak{F}} = I \otimes I \otimes \mathfrak{F} : \mathfrak{H} \otimes L^2(\mathfrak{G}) \otimes L^2(\mathfrak{G}) \to \mathfrak{H} \otimes L^2(\mathfrak{G}) \otimes L^2(\hat{\mathfrak{G}})$. The von Neumann algebra $\tilde{\mathfrak{F}}^*\mathfrak{R}(\mathfrak{m},\alpha,\hat{\alpha})\tilde{\mathfrak{F}}$ is then generated by the operators $\Phi_1(a) = \tilde{\mathfrak{F}}^*\Phi(a)\tilde{\mathfrak{F}}$, $a \in \mathfrak{m}$; $v_1(k) = \tilde{\mathfrak{F}}^*v(k)\tilde{\mathfrak{F}}$, $h \in \mathfrak{G}$; $w_1(q) = \tilde{\mathfrak{F}}^*w(q)\tilde{\mathfrak{F}}$, $q \in \hat{\mathfrak{G}}$; we have:

$$(\Phi_1(a)\xi)(g,h) = \alpha_g^{-1}(a)\xi(g,h), \qquad a \in \mathfrak{m}$$

$$(v_1(k)\xi)(g,h) = \xi(g - k,h - k), \quad k \in \mathfrak{G}$$

$$(w_1(p)\xi)(g,h) = \langle h,p\rangle\xi(g,h), \qquad p \in \hat{\mathfrak{G}} .$$

Next we consider the unitary operator V on $L^2(\mathfrak{G} \times \mathfrak{G}; \mathfrak{H})$ given by:

$$(V\xi)(g,h) = \xi(g + h,h), \quad \xi \in L^2(\mathfrak{G} \times \mathfrak{G}; \mathfrak{H}) ,$$

and set $\mathfrak{P} = V\tilde{\mathfrak{F}}^*\mathfrak{R}(\mathfrak{m},\alpha,\hat{\alpha})\tilde{\mathfrak{F}}V^*$. $\mathfrak{P}$ is then generated by $\Phi_2(a) = V\Phi_1(a)V^*$, $a \in \mathfrak{m}$; $v_2(k) = Vv_1(k)V^*$, $k \in \mathfrak{G}$; $w_2(p) = Vw_1(p)V^*$, $p \in \hat{\mathfrak{G}}$; we get:

$$(\Phi_2(a)\xi)(g,h) = \alpha_{g+h}^{-1}(a)\xi(g,h), \quad a \in \mathfrak{m}$$

(4.1) $$(v_2(k)\xi)(g,h) = \xi(g,h - k), \qquad k \in \mathfrak{G}$$

$$(w_2(p)\xi)(g,h) = \langle h,p\rangle\xi(g,h), \qquad p \in \hat{\mathfrak{G}} .$$

This stage of the isomorphism between $\mathfrak{R}(\mathfrak{m},\alpha,\hat{\alpha})$ and $\mathfrak{m} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$ will be referred to as the V-representation. We note that $v_2(k) = I \otimes I \otimes \lambda(k)$ and $w_2(p) = I \otimes I \otimes \mu(p)$ on $\mathfrak{H} \otimes L^2(\mathfrak{G}) \otimes L^2(\mathfrak{G})$. Since the $\mu(p)$'s generate the maximal abelian von Neumann algebra $L^\infty(\mathfrak{G})$ on $L^2(\mathfrak{G})$, an operator in $\mathfrak{B}(L^2(\mathfrak{G}))$ commutes with

$\{\lambda(g),\mu(p);\ g \in \mathfrak{G},\ p \in \hat{\mathfrak{G}}\}$ if and only if it lies in $L^\infty(\mathfrak{G})$ and is invariant under left translation, i.e. if it is a scalar. It follows that $\mathcal{B} = \{v_2(k),\ w_2(p);\ k \in \mathfrak{G},\ p \in \hat{\mathfrak{G}}\}'' = \mathbb{C} \otimes \mathbb{C} \otimes \mathcal{B}(L^2(\mathfrak{G})) \cong \mathcal{B}(L^2(\mathfrak{G}))$ and consequently $\mathcal{P} \cong (\mathcal{P} \cap \mathcal{B}') \otimes \mathcal{B}$. Returning to the representation Φ_1 above, it is clear that $\Phi_1(\mathfrak{M})$ commutes with $\mathcal{B}$, and by the commutation theorem 3 we also have $\Phi_1(\mathfrak{M}) \subseteq \mathcal{P}$, thus $\Phi_1(\mathfrak{M}) \subseteq \mathcal{P} \cap \mathcal{B}'$; by [15; Lemma 4.4] $\Phi_1(\mathfrak{M})$ and $\mathcal{B}$ generate $\mathcal{P}$, hence $\mathcal{P} \cong \Phi_1(\mathfrak{M}) \otimes \mathcal{B}$. The final step in the isomorphism between $\mathcal{R}(\mathfrak{M},\alpha,\hat{\alpha})$ and $\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))$ is now obtained by defining an isomorphism $\Psi : \mathcal{P} \to \mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))$ as follows:

$$\Psi(\Phi_1(a)) = a \otimes I = \Phi_3(a), \qquad a \in \mathfrak{M}$$

(4.2) $$\Psi(v_2(k)) = I \otimes \lambda(k) = \tilde{\lambda}(k), \quad k \in \mathfrak{G}$$

$$\Psi(w_2(p)) = I \otimes \mu(p) = \tilde{\mu}(p), \quad p \in \hat{\mathfrak{G}}$$

We observe that the inverse of Ψ is nothing but $\pi_\alpha \otimes I :$ $\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G})) \to \mathcal{P}$, thus for an element $x \in L^\infty(\mathfrak{G};\ \mathfrak{M}) \subseteq \mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))$ we have:

(4.3) $$\begin{aligned}(\Psi^{-1}(x)\xi)(g,h) &= ((\pi_\alpha \otimes I)(x)\xi)(g,h) \\ &= \alpha_g^{-1}(x(h))\xi(g,h)\ .\end{aligned}$$

The second dual weight $\tilde{\tilde{\varphi}}$ is constructed from $\tilde{\varphi}$ in the same way $\tilde{\varphi}$ was obtained from φ. The following proposition shows that no additional relative modular objects are introduced by passing from $\tilde{\varphi}$ to $\tilde{\tilde{\varphi}}$:

Proposition 4.1. The dual weight $\tilde{\varphi}$ is invariant under the dual action $\hat{\alpha}$.

Proof. It is clear that the $\tilde{\mu}(p)$'s leave the left Hilbert algebra $\tilde{\mathfrak{A}} \subseteq L^2(\mathfrak{G}; \mathcal{H})$ invariant, and denoting again by $\tilde{\mu}(p)$ the operators on $\tilde{\mathfrak{M}}$ obtained by pulling $\tilde{\mu}(p)$ back to $\tilde{\mathfrak{M}}$ via $\tilde{\Lambda}$, we have for $x, y \in \tilde{\mathfrak{M}}$:

$$((\tilde{\mu}(p)x) * (\tilde{\mu}(p)y))(g) = \int_{\mathfrak{G}} \alpha_h(\langle g + h, p\rangle x(g + h))\langle -h, p\rangle y(-h)dh$$

$$= \langle g,p\rangle \int_{\mathfrak{G}} \alpha_h(x(g + h))y(-h)dh$$

$$= (\tilde{\mu}(p)(x * y))(g)$$

and:

$$((\tilde{\mu}(p)x)^{\#})(g) = \alpha_g^{-1}(\langle g,p\rangle x(-g)^*) = \langle g,p\rangle x^{\#}(g)$$

$$= (\tilde{\mu}(p)x^{\#})(g) ,$$

i.e., the $\tilde{\mu}(p)$'s are #-automorphisms of $\tilde{\mathfrak{M}}$ and $\tilde{\mathfrak{A}}$. Thus for $\xi \in \tilde{\mathfrak{A}}$ we have:

$$\tilde{\varphi} \circ \hat{\alpha}_p(\tilde{\pi}^{\ell}(\xi)^*\tilde{\pi}^{\ell}(\xi)) = \tilde{\varphi}(\tilde{\pi}^{\ell}(\tilde{\mu}(p)\xi)^*\tilde{\pi}^{\ell}(\tilde{\mu}(p)\xi))$$

$$= \|\mu(p)\xi\|^2 = \|\xi\|^2 = \tilde{\varphi}(\tilde{\pi}^{\ell}(\xi)^*\tilde{\pi}^{\ell}(\xi)) \ . \qquad \text{Q.E.D.}$$

We also note in passing that the formula in Corollary 3.6 for the value of $\tilde{\varphi}$ on certain elements of $\mathfrak{R}(\mathfrak{m},\alpha)$ holds more generally: namely, it follows from the proof of Proposition 3.2 that $\mathcal{K}(\mathfrak{G}; \mathfrak{m})$

itself is a #-algebra under the operations of Definition 3.1 and the representation $\tilde{\pi}$ of $\tilde{\mathfrak{M}}$ (Theorem 3.4) may, of course, be extended to all of $\mathcal{K}(\mathfrak{G};\ \mathfrak{m})$. By inequality (3.1) the linear space $\tilde{\mathfrak{N}} = \{x \in \mathcal{K}(\mathfrak{G};\ \mathfrak{m}),\ \int_{\mathfrak{G}} \varphi(x(g)^* x(g)) < \infty\}$ is a left-ideal in $\mathcal{K}(\mathfrak{G};\ \mathfrak{m})$ (the subalgebra $\tilde{\mathfrak{M}}$ from Definition 3.1 is then nothing but $\tilde{\mathfrak{N}}^{\#} \cap \tilde{\mathfrak{N}}$), and Corollary 3.6 holds for elements of $\tilde{\mathfrak{N}}$ in the following form:

$$(4.4) \qquad \tilde{\varphi}(\tilde{\pi}(y^{\#} * x)) = \varphi((y^{\#} * x)(0)), \quad x,y \in \tilde{\mathfrak{N}} ,$$

thus $\tilde{\pi}(\tilde{\mathfrak{N}}) \subseteq \mathfrak{N}_{\tilde{\varphi}}$.

To obtain a left Hilbert algebra which generates $\mathcal{R}(\mathfrak{m},\alpha,\hat{\alpha})$ we consider $\mathcal{K}(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{m})$ and set $\tilde{\tilde{\mathfrak{N}}} = \{x \in \mathcal{K}(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{m});$ $\int_{\mathfrak{G}\times\hat{\mathfrak{G}}} (x(g,p)^* x(g,p))dgdp < \infty\}$. For $x \in \mathcal{K}(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{m})$ we denote by $\tilde{x}$ the element of $\mathcal{K}(\hat{\mathfrak{G}};\ \mathcal{R}(\mathfrak{m},\alpha))$ given by $\tilde{x}(p) = \tilde{\pi}(x_p)$, $p \in \hat{\mathfrak{G}}$, where x_p denotes the element of $\mathcal{K}(\mathfrak{G};\ \mathfrak{m})$ given by $x_p(g) = x(g,p)$, $g \in \mathfrak{G}$, $p \in \hat{\mathfrak{G}}$. For $x,y \in \mathcal{K}(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{m})$ the product $\tilde{x} * \tilde{y}$ should, according to Definition 3.1, be defined by:

$$(\tilde{x} * \tilde{y})(p) = \int_{\hat{\mathfrak{G}}} \hat{\alpha}_q(\tilde{x}(p + q))\tilde{y}(-q)\ dq$$

and the #-operation:

$$\tilde{x}^{\#}(p) = \hat{\alpha}_{-p}(\tilde{x}(-p)^*) .$$

Pulling these operations back to $\mathcal{K}(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{m})$ and evaluating at a point $g \in \mathfrak{G}$ we get:

$$(x * y)(g,p) = \int_{\mathfrak{G}\times\hat{\mathfrak{G}}} \langle g + h,q\rangle \alpha_h(x(g + h,p + q))y(-h,-q)dhdq$$

and

$$x^{\#}(g,p) = \overline{\langle g,p\rangle}\alpha_g^{-1}(x(-g,-p)^*)$$

As before one sees that with these operations $\tilde{\tilde{\mathfrak{N}}}$ is a left-ideal in $\mathcal{K}(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{m})$ and setting $\tilde{\tilde{\mathfrak{M}}} = \tilde{\tilde{\mathfrak{N}}}^{\#} \cap \tilde{\tilde{\mathfrak{N}}}$ and $\tilde{\tilde{\mathfrak{A}}} = \tilde{\tilde{\Lambda}}\tilde{\tilde{\mathfrak{M}}}$, where $(\tilde{\tilde{\Lambda}}x)(g,p) = \Lambda x(g,p)$, one shows as in Theorem 3.4 that $\tilde{\tilde{\mathfrak{A}}}$ is a left Hilbert algebra whose left von Neumann algebra coincides with the second crossed product $\mathfrak{R}(\mathfrak{m},\alpha,\hat{\alpha})$. The representation $\tilde{\tilde{\pi}}$ of $\mathcal{K}(\mathfrak{G} \times \hat{\mathfrak{G}};\ \mathfrak{m})$ such that $\tilde{\tilde{\pi}} = \tilde{\tilde{\pi}}^{\ell} \circ \tilde{\tilde{\Lambda}}$ on $\tilde{\tilde{\mathfrak{N}}}$ is now given by:

$$\begin{aligned}
\tilde{\tilde{\pi}}(x) = \tilde{\tilde{\pi}}(\tilde{x}) &= \int_{\hat{\mathfrak{G}}} w(q)\pi_{\hat{\alpha}}(\tilde{x}(q))dq \\
&= \int_{\hat{\mathfrak{G}}} w(q)\pi_{\hat{\alpha}}(\tilde{\pi}(x_q))dq \\
&= \int_{\hat{\mathfrak{G}}} w(q)\pi_{\hat{\alpha}}\left(\int_{\mathfrak{G}} \tilde{\lambda}(h)\pi_{\alpha}(x_q(h))dh\right)dq \\
&= \int_{\mathfrak{G}\times\hat{\mathfrak{G}}} w(q)\pi_{\hat{\alpha}}(\tilde{\lambda}(h))\pi_{\hat{\alpha}} \circ \pi_{\alpha}(x(h,q))dhdq \\
&= \int_{\mathfrak{G}\times\hat{\mathfrak{G}}} w(q)v(h)\Phi(x(h,q))dhdq\ ,
\end{aligned}$$

thus for $x,y \in \tilde{\tilde{\mathfrak{N}}}$ we have:

$$\begin{aligned}
\tilde{\tilde{\varphi}}(\tilde{\tilde{\pi}}(y^{\#} * x)) &= \int_{\mathfrak{G}\times\hat{\mathfrak{G}}} \varphi(y^*(g,h)x(g,h))dgdh \\
&= \varphi((y^{\#} * x)(0,0))\ .
\end{aligned}$$

Since we are mainly going to work in the V-representation of $\mathfrak{R}(\mathfrak{M},\alpha,\hat{\alpha})$ we transform the above structure to $\mathcal{K}(\mathfrak{G}\times\mathfrak{G};\mathfrak{M})$ via the operators $\tilde{\mathfrak{F}}^*$ and V (we use the same symbols $\tilde{\mathfrak{F}}^*$ and V for the mappings from $\mathcal{K}(\mathfrak{G}\times\hat{\mathfrak{G}};\mathfrak{M})$ and $\mathcal{K}(\mathfrak{G}\times\mathfrak{G};\mathfrak{M})$ corresponding to $\tilde{\mathfrak{F}}^*$ and V on $\mathcal{K}(\mathfrak{G}\times\hat{\mathfrak{G}};\mathfrak{H})$ and $\mathcal{K}(\mathfrak{G}\times\mathfrak{G};\mathfrak{H})$, respectively). Transforming first by $\tilde{\mathfrak{F}}^*$ we obtain the following structure on $(\tilde{\mathfrak{F}}^*\mathcal{K}(\mathfrak{G}\times\hat{\mathfrak{G}};\mathfrak{M})\cap\mathcal{K}(\mathfrak{G}\times\mathfrak{G};\mathfrak{M})$ (see [15]):

$$(x * y)(g,h) = (\tilde{\mathfrak{F}}^*((\tilde{\mathfrak{F}}x) * (\tilde{\mathfrak{F}}y)))(g,h)$$
$$= \int_{\mathfrak{G}} \alpha_k^{-1}(x(g - k,h))y(k,h + k - g)dk$$

and

$$x^{\#}(g,h) = (\tilde{\mathfrak{F}}^*(\tilde{\mathfrak{F}}x)^{\#})(g,h) = \alpha_g^{-1}(x(-g,h - g)^*)$$

Extending these operations to all of $\mathcal{K}(\mathfrak{G}\times\mathfrak{G};\mathfrak{M})$ and transforming the structure once more by the operator V, we finally end up with the following structure on $\mathcal{K}(\mathfrak{G}\times\mathfrak{G};\mathfrak{M})$:

$$(x * y)(g,h) = (V((V^*x) * (V^*y)))(g,h)$$
$$= ((V^*x) * (V^*y))(g + h,h)$$
$$= \int_{\mathfrak{G}} \alpha_k^{-1}((V^*x)(g + h - k,h))(V^*y)(k,h + k - g - h)dk$$
$$= \int_{\mathfrak{G}} \alpha_k^{-1}(x(g - k,h))y(g,k - g)dk$$

(4.5) and:

$$x^{\#}(g,h) = (V(V^*x)^{\#})(g,h) = ((V^*x)^{\#})(g + h,h)$$
$$= \alpha_{g+h}^{-1}((V^*x)(-g - h,-g)^*) =$$

$$= \alpha_{g+h}^{-1}(x(-h,-g)^*)$$

Denoting again by $\approx\!\pi$ and $\approx\!\varphi$ the images of $\approx\!\pi$ and $\approx\!\varphi$ under these transformations we have for $x \in \mathcal{K}(\mathfrak{G} \times \mathfrak{G}; \mathfrak{M})$:

$$(4.6) \quad \tilde{\tilde{\pi}}(x) = V\tilde{\mathfrak{F}}^* \tilde{\tilde{\pi}}(\tilde{\mathfrak{F}}V^* x)\tilde{\mathfrak{F}}V^*$$

$$= \int_{\mathfrak{G}\times\mathfrak{G}\times\hat{\mathfrak{G}}} \overline{\langle h,p\rangle} w_2(p)v_2(g)\Phi_2(x(g-h,h))dgdpdh$$

and for $x,y \in (V\tilde{\mathfrak{F}}^*\tilde{\tilde{\mathfrak{N}}}) \cap \mathcal{K}(\mathfrak{G} \times \mathfrak{G}; \mathfrak{M})$ we get:

$$(4.7) \qquad \tilde{\tilde{\varphi}}(\tilde{\tilde{\pi}}(y^\# * x)) = \varphi((\tilde{\mathfrak{F}}V^*(y^\# * x))(0,0)) .$$

With these preparations we now prove:

<u>Theorem 4.2</u>. Denoting again by $\tilde{\tilde{\varphi}}$ the image of the second dual weight on $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$, under the above isomorphism, we have:

$$(4.8) \quad ((D\tilde{\tilde{\varphi}} : D(\varphi \otimes \mathrm{Tr}))_t\xi)(g) = (D(\varphi \circ \alpha_g) : D\varphi)_t\xi(g) ,$$

$$\xi \in L^2(\mathfrak{G}; \mathcal{H}), \ t \in \mathbb{R} .$$

<u>Proof</u>. We have to show that the conditions of Proposition 2.2 are satisfied. To avoid cumbersome notation we shall use the same symbol $\tilde{\tilde{\varphi}}$ for the second dual weight, regardless of which representation we are working in. For $t \in \mathbb{R}$, we denote by U_t the operator on the right hand side of (4.8), namely, U_t is the element of $L^\infty(\mathfrak{G}; \mathfrak{M})$ given by $U_t(g) = u_t^{g,0} = (D(\varphi \circ \alpha_g) : D\varphi)_t$. Since $\sigma^{\varphi\otimes\mathrm{Tr}} = \sigma \otimes I$, we have $(\sigma_t^{\varphi\otimes\mathrm{Tr}}(x))(g) = \sigma_t(x(g))$ for $x \in L^\infty(\mathfrak{G}; \mathfrak{M})$,

thus:

$$(U_{t_1}\sigma_{t_1}^{\varphi\otimes\mathrm{Tr}}(U_{t_2}))(g) = u_{t_1}^{g,0}\sigma_{t_1}(u_{t_2}^{g,0}) = u_{t_1+t_2}^{g,0} = U_{t_1+t_2}(g) \ ,$$

which shows the cocycle identity (2.16). Furthermore, on $\mathfrak{R}(\mathfrak{M},\alpha,\hat{\alpha})$, since $\tilde{\varphi}$ is invariant under $\hat{\alpha}$, we have by Corollary 3.10:

$$\sigma_t^{\approx\varphi}(\Phi(a)) = \sigma_t^{\approx\varphi}(\pi_{\hat{\alpha}} \circ \pi_{\alpha}(a)) = \pi_{\hat{\alpha}}(\sigma_t^{\tilde{\varphi}}(\pi_{\alpha}(a)))$$

$$= \pi_{\hat{\alpha}} \circ \pi_{\alpha}(\sigma_t(a)) = \Phi(\sigma_t(a)), \qquad a \in \mathfrak{M}$$

$$\sigma_t^{\approx\varphi}(v(g)) = \sigma_t^{\approx\varphi}(\pi_{\hat{\alpha}}(\tilde{\lambda}(g))) = \pi_{\hat{\alpha}}(\sigma_t^{\tilde{\varphi}}(\tilde{\lambda}(g)))$$

$$= \pi_{\hat{\alpha}}(\tilde{\lambda}(g)\pi_{\alpha}(u_t^{g,0})) = v(g)\Phi(u_t^{g,0}), \quad g \in \mathfrak{G}$$

and

$$\sigma_t^{\approx\varphi}(w(p)) = w(p), \qquad p \in \hat{\mathfrak{G}}$$

Denoting by $\theta : \mathfrak{R}(\mathfrak{M},\alpha,\hat{\alpha}) \to \mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$ the canonical isomorphism, we have by (4.3): $(\theta(\Phi(a)))(h) = (\pi_{\alpha}(a))(h) = \alpha_h^{-1}(a)$, $a \in \mathfrak{M}$, and $(\theta(\Phi(u_t^{g,0})))(h) = (\pi_{\alpha}(u_t^{g,0}))(h) = \alpha_h^{-1}(u_t^{g,0})$; also $\theta(v(g)) = \tilde{\lambda}(g)$ and $\theta(w(p)) = \tilde{\mu}(p)$, $g \in \mathfrak{G}$, $p \in \hat{\mathfrak{G}}$. Thus on $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$ we have:

$$(\sigma_t^{\approx\varphi}(\pi_{\alpha}(a)))(h) = \alpha_h^{-1}(\sigma_t(a)), \quad a \in \mathfrak{M}$$

$$\sigma_t^{\approx\varphi}(\tilde{\lambda}(g)) = \tilde{\lambda}(g)\pi_{\alpha}(u_t^{g,0}), \qquad g \in \mathfrak{G}$$

$$\sigma_t^{\approx\varphi}(\tilde{\mu}(p)) = \tilde{\mu}(p), \qquad p \in \mathfrak{G} \ .$$

On the other hand:

$$(U_t\sigma^{\varphi\otimes Tr}(\pi_\alpha(a))U_t^*)(h) = u_t^{h,0}\sigma_t((\pi_\alpha(a))(h))u_t^{0,h}$$

$$= u_t^{h,0}\sigma_t(\alpha_h^{-1}(a))u_t^{0,h}$$

$$= \sigma_t^h(\alpha_h^{-1}(a)) = \alpha_h^{-1}(\sigma_t(a)), \qquad a \in \mathfrak{M}$$

$$(U_t\sigma_t^{\varphi\otimes Tr}(\tilde{\lambda}(g))U_t^*\xi)(h) = (U_t\tilde{\lambda}(g)U_t^*\xi)(h)$$

$$= u_t^{h,0}(U_t^*\xi)(h - g) = u_t^{h,0}u_t^{0,h-g}\xi(h - g)$$

$$= \alpha_{h-g}^{-1}(u_t^{g,0})\xi(h - g)$$

$$= (\tilde{\lambda}(g)\pi_\alpha(u_t^{g,0})\xi)(h), \qquad g \in \mathfrak{G}$$

$$(U_t\sigma_t^{\varphi\otimes Tr}(\tilde{\mu}(p))U_t^*\xi)(h) = (\tilde{\mu}(p)\xi)(h), \qquad p \in \hat{\mathfrak{G}},$$

thus $\sigma_t^{\tilde{\tilde{\varphi}}}$ and $Ad(U_t) \circ \sigma_t^{\varphi\otimes Tr}$ coincide on generators and condition (2.17) is satisfied.

It remains to verify the KMS-condition. For this we consider the following elements of $\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))$: Let $a_i, b_i \in \mathfrak{N}_\varphi$, $i = 1,2$, and let $F_i, G_i \in \mathcal{K}(\mathfrak{G})$ be such that $\hat{G}_i$ has compact support, $i = 1,2$. Then $\lambda(F_i) = \int_{\mathfrak{G}} \lambda(g)F_i(g)dg$ = convolution by F_i on $L^2(\mathfrak{G})$, $\mu(G_i) = \int_{\hat{\mathfrak{G}}} \mu(p)\hat{G}_i(p)dp$ = multiplication by G_i on $L^2(\mathfrak{G})$ and $A_i = \mu(G_i)\lambda(F_i)$ is a Hilbert-Schmidt operator $(A_i\xi)(g) = \int_{\mathfrak{G}} \xi(h)L_i(g,h)dh$ on $L^2(\mathfrak{G})$ with kernel $L_i(g,h) = F_i(g - h)G_i(g)$, $i = 1,2$. Thus setting $\tilde{A}_i = I \otimes A_i$, $\tilde{\mu}(G_i) = I \otimes \mu(G_i)$, $\tilde{\lambda}(F_i) = I \otimes \lambda(F_i)$, $i = 1,2$, we have $(a_i \otimes I)\tilde{A}_i = \Phi_3(a_i)\tilde{A}_i \in \mathfrak{N}_{\varphi\otimes Tr}$, $i = 1,2$. Also, in the V-representation, the functions $y_i(g,h) = F_i(g + h)G_i(h)b_i$, $i = 1,2$, lie in $\tilde{\tilde{\mathfrak{N}}}$; we have $\Psi(\tilde{\tilde{\pi}}(y_i)) =$

$\Psi(w(G_i)v(F_i)\Phi_2(b_i)) = \tilde{\mu}(G_i)\tilde{\lambda}(F_i)\pi_\alpha(b_i) \in \mathfrak{N}_{\approx\varphi}$, thus $\pi_\alpha(b_i)^*\tilde{A}_i^*\Phi_3(a_i) \in \mathfrak{N}_{\approx\varphi}^* \cap \mathfrak{N}_{\varphi\otimes Tr}$. It suffices to check the KMS-condition on elements of this form, so we set out to find a KMS-function $\tilde{K}$ such that:

$$\tilde{K}(t) = \tilde{\tilde{\varphi}}(U_t\sigma_t^{\varphi\otimes Tr}(\pi_\alpha(b_1)^*\tilde{A}_1^*\Phi_3(a_1))\Phi_3(a_2)^*\tilde{A}_2\pi_\alpha(b_2))$$

and

$$\tilde{K}(t + i) = (\varphi \otimes Tr)(\Phi_3(a_2)^*\tilde{A}_2\pi_\alpha(b_2)U_t\sigma_t^{\varphi\otimes Tr}(\pi_\alpha(b_1)^*\tilde{A}_1^*\Phi_3(a_1))) .$$

For the second expression we have:

$$(\varphi \otimes Tr)(\Phi_3(a_2)^*\tilde{A}_2\pi_\alpha(b_2)U_t\sigma_t^{\varphi\otimes Tr}(\pi_\alpha(b_1)^*\tilde{A}_1^*\Phi_3(a_1)))$$
$$= (\varphi \otimes Tr)(\Phi_3(a_2)^*\tilde{A}_2[\pi_\alpha(b_2)U_t\sigma_t^{\varphi\otimes Tr}(\pi_\alpha(b_1)^*)]\tilde{A}_1^*\Phi_3(\sigma_t(a_1))) .$$

If $z \in L^\infty(\mathfrak{G}; \mathfrak{m})$ is of the form $z = c \otimes \mu(H)$ for a $c \in \mathfrak{m}$ and $H \in L^\infty(\mathfrak{G})$, we have:

$$(\varphi \otimes Tr)(\Phi_3(a_2)^*\tilde{A}_2 z\tilde{A}_1^*\Phi_3(\sigma_t(a_1)))$$

(4.9)
$$= \varphi(a_2^* c\sigma_t(a_1))\ Tr(A_2\mu(H)A_1^*)$$
$$= \int_{\mathfrak{G}\times\mathfrak{G}} H(h)F_2(g - h)G_2(g)\overline{F_1(g - h)}\overline{G_1(g)}\varphi(a_2^* c\sigma_t(a_1))dgdh$$
$$= \int_{\mathfrak{G}\times\mathfrak{G}} F_2(g - h)G_2(g)\overline{F_1(g - h)}\overline{G_1(g)}\varphi(a_2^* z(h)\sigma_t(a_1))dgdh ,$$

because the trace of the trace-class operator $A_2\mu(H)A_1^*$ with kernel $L(g,h) = \int_{\mathfrak{G}} H(k)L_2(g,k)\overline{L_1(h,k)}dk$ is equal to $\int_{\mathfrak{G}} L(g,g)dg = \int_{\mathfrak{G}\times\mathfrak{G}} H(k)L_2(g,k)\overline{L_1(g,k)}dkdg = \int_{\mathfrak{G}\times\mathfrak{G}} H(k)F_2(g - k)G_2(g)\overline{F_1(g - k)}\overline{G_1(g)}dkdg.$

Since $a_i \in \mathfrak{N}_\varphi$ and $A_i \in \mathfrak{N}_{Tr}$, $i = 1,2$, both sides of the above equation (4.9) are σ-weakly continuous linear functionals on $L^\infty(\mathfrak{G};\mathfrak{m})$, hence (4.9) holds for all $z \in L^\infty(\mathfrak{G};\mathfrak{m})$. Thus with $z = \pi_\alpha(b_2)U_t\sigma_t^{\varphi\otimes Tr}(\pi_\alpha(b_1)^*)$, we get:

$$(\varphi \otimes Tr)(\Phi_3(a_2)^*\tilde{A}_2\pi_\alpha(b_2)U_t\sigma_t^{\varphi\otimes Tr}(\pi_\alpha(b_1^*)\tilde{A}_1^*\Phi_3(a_1)))$$

(4.10)

$$= \int_{\mathfrak{G}\times\mathfrak{G}} F_2(g-h)G_2(g)\overline{F_1(g-h)}\overline{G_1(g)}\varphi(a_2^*\alpha_h^{-1}(b_2)u_t^{h,0}\sigma_t(\alpha_h^{-1}(b_1)^*a_1))dgdh.$$

To compute the expression involving $\tilde{\tilde{\varphi}}$ we pull the operators back to the V-representation (4.1), (4.5) and (4.6). In order to apply formula (4.7) for $\tilde{\tilde{\varphi}}$ we shall have to approximate $\Psi^{-1}(U_t)$ and $\Psi^{-1}(\Phi_3(a_i)) = \Phi_1(a_i)$, $i = 1,2$, by operators of the form $\tilde{\tilde{\pi}}(x)$, $x \in \mathcal{K}(\mathfrak{G} \times \mathfrak{G};\ \mathfrak{m})$. For this we observe that if $z \in L^\infty(\mathfrak{G};\ \mathfrak{m})$, $M, N \in \mathcal{K}(\mathfrak{G})$ and $x(g,h) = M(g + h)N(h)\alpha_h(z(h))$, then $\tilde{\tilde{\pi}}(X)$ tends weakly to $\Psi^{-1}(z)$ as M tends, as a bounded approximate unit in $L^1(\mathfrak{G})$, to the Dirac function at 0 and N tends uniformly on compact sets to the constant identity function. To see this, let $y \in \tilde{\tilde{\mathfrak{N}}}$ and $\eta \in L^2(\mathfrak{G} \times \mathfrak{G};\ \mathcal{H})$; we have:

$$\langle\tilde{\tilde{\pi}}(x)\tilde{\tilde{\Lambda}}(y), \eta\rangle = \langle\tilde{\tilde{\Lambda}}(x * y), \eta\rangle$$

$$= \int_{\mathfrak{G}\times\mathfrak{G}} \langle\Lambda[(x * y)(g,h)], \eta(g,h)\rangle dgdh$$

(4.5)

$$= \int_{\mathfrak{G}\times\mathfrak{G}} \left\langle\Lambda\left(\int M(g-k+h)N(h)\alpha_{k-h}^{-1}(z(h))y(g,k-g)dk\right), \eta(g,h)\right\rangle dgdh$$

$$= \int_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}} M(g-k+h)N(h)\langle\alpha_{k-h}^{-1}(z(h))\ \Lambda(y(g,k-g)), \eta(g,h)\rangle dkdgdh.$$

Now when M converges to the Dirac function at $0 \in \mathfrak{G}$, as a bounded approximate unit in $L^1(\mathfrak{G})$, the above integral converges to:

$$\int_{\mathfrak{G}\times\mathfrak{G}} N(h)\langle \alpha_g^{-1}(z(h))\Lambda(y(g,h)), \eta(g,h)\rangle dg dh .$$

Next when N converges to the constant identity function, the last integral converges to:

$$\int_{\mathfrak{G}\times\mathfrak{G}} \langle \alpha_g^{-1}(z(h))\Lambda(y(g,h)), \eta(g,h)\rangle dg dh = \langle \Psi^{-1}(z)\tilde{\tilde{\Lambda}}(y), \eta\rangle .$$

This holding for all $y \in \tilde{\tilde{\mathfrak{A}}}$, $\eta \in L^2(\mathfrak{G} \times \mathfrak{G}; \mathfrak{H})$, we see that $\tilde{\tilde{\pi}}(x) \to \Psi^{-1}(z)$ weakly when M, N converge as prescribed. Now let $M_i, N_i \in \mathcal{K}(\mathfrak{G})$ be such that $\hat{N}_i \in \mathcal{K}(\hat{\mathfrak{G}})$, $i = 1,2,3$, and set

$$x_i(g,h) = M_i(g + h)N_i(h)\alpha_h(a_i), \quad i = 1,2$$

$$x_3(g,h) = M_3(g + h)N_3(h)\alpha_h(u_t^{h,0})$$

and as before let:

$$y_i(g,h) = F_i(g + h)G_i(h)b_i, \quad i = 1,2 .$$

Noting that by Theorem 3.8 and Proposition 4.1 we have $\tilde{\tilde{\sigma}}_t(\tilde{\tilde{\pi}}(x)) = \tilde{\tilde{\pi}}(\hat{\tau}_t(x))$, $x \in \mathcal{K}(\mathfrak{G} \times \mathfrak{G}; \mathfrak{M})$, where $(\hat{\tau}_t x)(g,h) = u_t^{g+h,0}\sigma_t(x(g,h))$, we get after some computation:

$$(\tau_t(y_1^{\#} * x_1) * x_3 * x_2^{\#} * y_2)(g,h)$$

$$= \int_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}} M_1(m+h+g-n)\overline{M_2(k-\ell)}M_3(n-\ell)N_1(m+h)\overline{N_2(k-g)}N_3(n-g)\overline{F_1(m)}\overline{G_1(m+h)}$$

$$\cdot\ F_2(k)G_2(k-g)\cdot u_t^{g+h,n}\alpha_n^{-1} \circ \sigma_t[\alpha^{-1}_{g-n+h}(b_1^*)\alpha_{h+m}(a_1)]$$

$$\cdot\ \alpha^{-1}_{\ell-n+g}(u_t^{n-g,0})\alpha^{-1}_{g+\ell-k}(a_2^*)b_2\,dk\,d\ell\,dm\,dn\ ,$$

thus:

$$\tilde{\tilde{\varphi}}(\tilde{\tilde{\sigma}}_t(\tilde{\tilde{\pi}}(y_1)^*\tilde{\tilde{\pi}}(x_1))\tilde{\tilde{\pi}}(x_3)\tilde{\tilde{\pi}}(x_2)^*\tilde{\tilde{\pi}}(y_2))$$

$$= \varphi((\mathfrak{F}V^*(\hat{\tau}_t(y_1^\# * x_1) * \ x_3 * x_2^\# * y_2))(0,0))$$

$$= \varphi\Big(\int_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}} M_1(m-n)\overline{M_2(k-\ell)}M_3(n-\ell)N_1(m+h)\overline{N_2(k+h)}N_3(n+h)\overline{F_1(m)}\,\overline{G_1(m+h)}$$

$$\cdot\ F_2(k)G_2(k+h)u_t^{0,n}\alpha_n^{-1} \circ \sigma_t[\alpha_n(b_1^*)\alpha_{h+m}(a_1)]$$

$$\cdot\ \alpha^{-1}_{\ell-n-h}(u_t^{n+h,0})\alpha^{-1}_{\ell-h-k}(a_2^*)b_2\,dk\,d\ell\,dm\,dn\,dh\Big)\ .$$

Letting successively M_1, M_2, M_3 tend to the Dirac function ε_0 at $0 \in \mathfrak{G}$, the right hand side of the last equation tends successively to:

as $M_1 \to \varepsilon_0$:

$$\varphi\Big(\int_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}} \overline{M_2(k-\ell)}M_3(n-\ell)N_1(n+h)\overline{N_2(k+h)}N_3(n+h)\overline{F_1(n)}\,\overline{G_1(n+h)}$$

$$\cdot\ F_2(k)G_2(k+h)u_t^{0,n}\alpha_n^{-1} \circ \sigma_t[\alpha_n(b_1^*)\alpha_{h+n}(a_1)]$$

$$\cdot\ \alpha^{-1}_{\ell-n-h}(u_t^{n+h,0})\alpha^{-1}_{\ell-h-k}(a_2^*)b_2\,dk\,d\ell\,dn\,dh\Big)$$

$$\downarrow\ M_2 \to \varepsilon_0$$

$$\varphi\Big(\int_{\mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}} M_3(n-\ell)N_1(n+h)\overline{N_2(\ell+h)}N_3(n+h)\overline{F_1(n)}\,\overline{G_1(n+h)}$$

$$\cdot F_2(\ell)G_2(\ell+h)\cdot u_t^{0,n}\cdot\alpha_n^{-1} \circ \sigma_t[\alpha_n(b_1^*)\alpha_{h+n}(a_1)]$$

$$\cdot\ \alpha^{-1}_{\ell-n-h}(u_t^{n+h,0})\alpha_h(a_2^*)b_2\,d\ell\,dn\,dh\Big)$$

$$\downarrow\ M_3 \to \varepsilon_0$$

$$\varphi\Big(\int_{\mathfrak{G}\times\mathfrak{G}} N_1(\ell+h)\overline{N_2(\ell+h)}N_3(\ell+h)\overline{F_1(\ell)}\,\overline{G_1(\ell+h)}$$

$$\cdot\ F_2(\ell)G_2(\ell+h)u_t^{0,\ell}\alpha_\ell^{-1} \circ \sigma_t[\alpha_\ell(b_1^*)\alpha_{h+\ell}(a_1)]$$

$$\cdot\ \alpha_h(u_t^{\ell+h,0})\alpha_h(a_2^*)b_2\,d\ell\,dh\Big).$$

Finally, letting N_1, N_2, N_3 tend to the constant identity function and changing the dummy-variable from ℓ to g, the last integral becomes:

$$\varphi\Big(\int_{\mathfrak{G}\times\mathfrak{G}} \overline{F_1(g)}\,\overline{G_1(g + h)}F_2(g)G_2(g + h)u_t^{0,g}\cdot\alpha_g^{-1} \circ \sigma_t \circ \alpha_g[b_1^*\alpha_h(a_1)]$$

$$\cdot u_t^{g,-h} \cdot\alpha_h(a_2^*)b_2\,dg\,dh\Big)$$

$$= \varphi\Big(\int_{\mathfrak{G}\times\mathfrak{G}} \overline{F_1(g)}\,\overline{G_1(g+h)}F_2(g)G_2(g+h)\sigma_t[b_1^*\alpha_h(a_1)]u_t^{0,-h}\alpha_h(a_2^*)b_2\,dg\,dh\Big)$$

$$= \int_{\mathfrak{G}\times\mathfrak{G}} \overline{F_1(g)}\,\overline{G_1(g+h)}F_2(g)G_2(g+h)\varphi \circ \alpha_h(\alpha_h^{-1} \circ \sigma_t \circ \alpha_h[\alpha_h^{-1}(b_1^*)a_1]$$

$$\cdot u_t^{h,0}a_2^*\alpha_h^{-1}(b_2))\,dg\,dh$$

$$= \int_{\mathfrak{G}\times\mathfrak{G}} \overline{F_1(g)}\,\overline{G_1(g+h)}F_2(g)G_2(g+h)\varphi_h(u_t^{h,0}\sigma_t[\alpha_h^{-1}(b_1^*)a_1]a_2^*\alpha_h^{-1}(b_2))\,dg\,dh\ .$$

Under the above limiting processes the operators $\tilde{\tilde{\pi}}(x_1)$, $\tilde{\tilde{\pi}}(x_2)$, $\tilde{\tilde{\pi}}(x_3)$ tend weakly to $\Phi_1(a_1)$, $\Phi_1(a_2)$, $\Psi^{-1}(U_t)$, respectively. Since $\tilde{\tilde{\pi}}(y_i) \in \mathfrak{N}_{\tilde{\tilde{\varphi}}}$, $i = 1,2$, the mapping $A \in \mathcal{R}(\mathfrak{M},\alpha,\hat{\alpha}) \mapsto \tilde{\tilde{\varphi}}(\tilde{\tilde{\sigma}}_t(\tilde{\tilde{\pi}}(y_1)^*)A\tilde{\tilde{\pi}}(y_2))$ is a σ-weakly continuous linear functional on $\mathcal{R}(\mathfrak{M},\alpha,\hat{\alpha})$, thus under the above limiting processes we have $\tilde{\tilde{\varphi}}(\tilde{\tilde{\sigma}}_t(\tilde{\tilde{\pi}}(y_1)^{*}\tilde{\tilde{\pi}}(x_1))\tilde{\tilde{\pi}}(x_3)\tilde{\tilde{\pi}}(x_2)^{*}\tilde{\tilde{\pi}}(y_2))$ $\to \tilde{\tilde{\varphi}}(\tilde{\tilde{\sigma}}_t(\tilde{\tilde{\pi}}(y_1)^*\Phi_1(a_1))\Psi^{-1}(U_t)\Phi_1(a_2)^{*}\tilde{\tilde{\pi}}(y_2))$. Transporting the operators back to $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$ again we therefore get:

$$
\begin{aligned}
&\tilde{\tilde{\varphi}}(U_t\sigma_t^{\varphi\otimes\mathrm{Tr}}(\pi_\alpha(b_1)^{*}\tilde{A}_1^{*}\Phi_3(a_1))\Phi_3(a_2)^{*}\tilde{A}_2\pi_\alpha(b_2)) \\
&= \tilde{\tilde{\varphi}}(\tilde{\tilde{\sigma}}_t(\pi_\alpha(b_1)^{*}\tilde{A}_1^{*}\Phi_3(a_1))U_t\Phi_3(a_2)^{*}\tilde{A}_2\pi_\alpha(b_2)) \\
&= \int_{\mathfrak{G}\times\mathfrak{G}} \overline{F_1(g)}\,\overline{G_1(g+h)}F_2(g)G_2(g+h)\varphi_h(u_t^{h,0}\sigma_t[\alpha_h^{-1}(b_1^*)a_1]a_2^*\alpha_h^{-1}(b_2))dgdh \\
&= \int_{\mathfrak{G}\times\mathfrak{G}} \overline{F_1(g-h)}\,\overline{G_1(g)}F_2(g-h)G_2(g)\varphi_h(u_t^{h,0}\sigma_t[\alpha_h^{-1}(b_1^*)a_1]a_2^*\alpha_h^{-1}(b_2))dgdh \\
&= \int_{\mathfrak{G}} F(h)\varphi_h(u_t^{h,0}\sigma_t[\alpha_h^{-1}(b_1^*)a_1]a_2^*\alpha_h^{-1}(b_2))dh \ ,
\end{aligned}
\tag{4.11}
$$

where we have set $F(h) = \int_{\mathfrak{G}} \overline{F_1(g-h)}\,\overline{G_1(g)}F_2(g-h)G_2(g)dg$. From before we have (4.10):

$$
(\varphi \otimes \mathrm{Tr})(\Phi_3(a_2)^*A_2\pi_\alpha(b_2)U_t\sigma_t^{\varphi\otimes\mathrm{Tr}}(\pi_\alpha(b_1)^*A_1^*\Phi_3(a_1)))
$$
$$
= \int_{\mathfrak{G}} F(h)\varphi(a_2^*\alpha_h^{-1}(b_2)u_t^{h,0}\sigma_t[\alpha_h^{-1}(b_1^*)a_1])dh \ .
$$

We know (Proposition 2.2) that for each $h \in \mathfrak{G}$ there is a KMS-function K_h such that:

$$
K_h(t) = \varphi_h(u_t^{h,0}\sigma_t[\alpha_h^{-1}(b_1^*)a_1]a_2^*\alpha_h^{-1}(b_2))
$$

and

$$
K_h(t+i) = \varphi(a_2^*\alpha_h^{-1}(b_2)u_t^{h,0}\sigma_t[\alpha_h^{-1}(b_1^*)a_1]) \ .
$$

As in the proof of Theorem 3.8 a KMS-function for $(\tilde{\tilde{\varphi}}, \varphi\otimes\mathrm{Tr}, U_t)$ is now obtained by setting $\tilde{K}(z) = \int_{\mathfrak{G}} F(h)K_h(z)dh$, $0 \le \mathrm{Im}\, z \le 1$, and the proof is complete. Q.E.D.

We recall that two f.n.s.f weights φ and ψ on a von Neumann algebra $\mathfrak{M}$ are said to commute if ψ is σ_t^{φ}-invariant (or, equivalently, φ is σ_t^{ψ}-invariant). This is equivalent to each of the following conditions (see [5] and [17]):

i) $\sigma_t^{\varphi}(u_s^{\psi,\varphi}) = u_s^{\psi,\varphi}$, $t,s \in \mathbb{R}$

ii) $u_{s+t}^{\psi,\varphi} = u_s^{\psi,\varphi} u_t^{\psi,\varphi}$, $t,s \in \mathbb{R}$

iii) there exists a unique positive, non-singular self adjoint operator K affiliated with the fixed point algebra $\mathfrak{M}_{\varphi}$ of σ_t^{φ} such that $\psi(a) = \varphi(K\cdot a)$, $a \in \mathfrak{M}^+$. In this case we have $u_t^{\psi,\varphi} = K^{it}$.

All the above conditions are, obviously, symmetric in φ and ψ. We now have:

Corollary 4.3. The second dual weight $\tilde{\tilde{\varphi}}$ and the tensor product weight $\varphi \otimes \mathrm{Tr}$ on $\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))$ commute if and only if φ and $\varphi \circ \alpha_g$ commute for all $g \in \mathfrak{G}$. If this is the case, then the unique self adjoint operator K affiliated with $(\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G})))_{\varphi\otimes\mathrm{Tr}}$ such that $\tilde{\tilde{\varphi}}(a) = (\varphi \otimes \mathrm{Tr})(Ka)$, $a \in (\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G})))_+$, is given by:

$$(K\xi)(g) = K_g \xi(g) ,$$

where K_g is the unique self adjoint operator affiliated with $\mathfrak{M}_{\varphi}$ such that $\varphi \circ \alpha_g(a) = \varphi(K_g a)$, $a \in \mathfrak{M}_+$, and where $\mathrm{dom}(K) = \{\xi \in L^2(\mathfrak{G}; \mathcal{H});\ \xi(g) \in \mathrm{dom}(K_g)$ a.e. and $g \mapsto K_g\xi(g)$ is in $L^2(\mathfrak{G}; \mathcal{H})\}$.

Proof. By the above theorem we have $(D\tilde{\tilde{\varphi}} : D(\varphi \otimes \mathrm{Tr}))_t = U_t$ where $U_t(g) = u_t^{g,0}$. We have $(\sigma_t^{\varphi\otimes\mathrm{Tr}}(U_s))(g) = \sigma_t(u_s^{g,0})$, $t,s \in \mathbb{R}$,

$g \in \mathfrak{G}$, thus $\sigma_t^{\varphi\otimes \mathrm{Tr}}(U_s) = U_s$, all $t,s \in \mathbb{R}$, if and only if $\sigma_t(u_s^{g,0}) = u_s^{g,0}$, all $t,s \in \mathbb{R}$, $g \in \mathfrak{G}$, which, by condition i) above, proves the first statement. The second statement now follows from the above theorem and the proof of Corollary 3.9. Q.E.D.

Remark 4.4. Theorem 4.2 may be viewed as a "twisted" Plancherel theorem for weights on covariant systems. We see that the duality becomes complete when φ is α-invariant and has properly infinite centralizer (the last condition ensures the existence of an isomorphism between $\mathfrak{M}$ and $\mathfrak{M} \otimes \mathfrak{B}(L^2(\mathfrak{G}))$ which carries φ over in $\varphi \otimes$ Trace). We also see that $\tilde{\tilde{\varphi}}$ is nothing but the weight obtained from the family $\{\varphi \circ \alpha_g;\ g \in \mathfrak{G}\}$ by a generalized version of the "mixed" weight construction of §2 (the situation in §2 corresponds to the case when $\mathfrak{G}$ is a group of order 2).

§5. Galois correspondence

We shall here write down a couple of applications of the results in §3. The first one is a mere restatement of the commutation Theorem 3.14, and gives an alternative description of the crossed product as the fixed point algebra of a certain action of $\mathfrak{G}$ on $\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))$. As a second application we show the Galois type correspondence between subgroups and sub-crossed products when the group $\mathfrak{G}$ is abelian. For the relation between the two results, see remark at the end of the section.

We first prove:

Theorem 5.1. The crossed product $\mathfrak{M} \otimes_\alpha \mathfrak{G}$ coincides with the fixed point algebra of the action: $g \in \mathfrak{G} \mapsto \alpha_g \otimes \mathrm{Ad}(\rho(g))$ of $\mathfrak{G}$ on $\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))$, where ρ is the right regular representation of $\mathfrak{G}$ on $L^2(\mathfrak{G})$.

Proof. Denoting by $\mathrm{Fix}(\alpha \otimes \mathrm{Ad} \circ \rho)$ the fixed point algebra of $\alpha \otimes \mathrm{Ad} \circ \rho$ we have with notation as in §3:

$$\mathrm{Fix}(\alpha \otimes \mathrm{Ad} \circ \rho) = (\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G}))) \cap \{W(g) \otimes \rho(g);\ g \in \mathfrak{G}\}'$$

$$= [(\mathfrak{M}' \otimes \mathbb{C}_{L^2(\mathfrak{G})}) \vee \{W(g) \otimes \rho(g);\ g \in \mathfrak{G}\}'']'$$

$$= [\tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G})']' = \tilde{\pi}(\mathfrak{M} \otimes_\alpha \mathfrak{G}) . \qquad \text{Q.E.D.}$$

We now assume the group $\mathfrak{G}$ is abelian and show, as another application of the commutation theorem, the Galois type correspondence between subgroups of $\hat{\mathfrak{G}}$ and certain subalgebras of $\mathfrak{M} \otimes_{\alpha} \mathfrak{G}$. For the case when the action α admits a relatively invariant weight, this was shown by Takesaki in [15; Th. 7.1], and the proof given there carries over word by word to our general situation, thanks to the commutation theorem 3.14. However, for the sake of completeness we present a proof here. To that end we first recall from [16] some facts from the theory of induced covariant representations:

For a closed subgroup $\mathfrak{G}_0$ of $\mathfrak{G}$ we denote by α^0 the restriction of α to $\mathfrak{G}_0$ and consider the subsystem $\{\mathfrak{M}, \mathfrak{G}_0, \alpha^0\}$ of $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$. A function on $\mathfrak{G}$ which is constant on $\mathfrak{G}_0$-cosets will be identified with the corresponding function on $\mathfrak{G}/\mathfrak{G}_0$, and we let $g \in \mathfrak{G} \mapsto \dot{g} \in \mathfrak{G}/\mathfrak{G}_0$ denote the quotient map. The Haar measures m and $\dot{m}$ on $\mathfrak{G}_0$ and $\mathfrak{G}/\mathfrak{G}_0$, respectively, are assumed to be adjusted so that for any $\xi \in \mathcal{K}(\mathfrak{G})$ we have:

$$\int_{\mathfrak{G}} \xi(g)dg = \int_{\mathfrak{G}/\mathfrak{G}_0} d\dot{m}(\dot{g}) \int_{\mathfrak{G}_0} \xi(g+h)dm(h)$$

(note that the function $g \in \mathfrak{G} \mapsto \int_{\mathfrak{G}_0} \xi(g+h)dm(h)$ is constant on $\mathfrak{G}_0$-cosets).

Now, if $\{\Phi^0, U^0\}$ is a covariant representation of $\{\mathfrak{M}, \alpha^0, \mathfrak{G}_0\}$ on a Hilbert space $\mathcal{K}_0$, the induced representation $\{\Phi, U\}$ of $\{\Phi^0, U^0\}$ from $\mathfrak{G}_0$ to $\mathfrak{G}$ is the covariant representation of $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ defined by:

the Hilbert space $\mathcal{K}$ of $\{\Phi, U\}$ is the set of measurable functions $\xi : \mathfrak{G} \to \mathcal{K}_0$ such that:

$$\xi(g-h) = U^0(h)\xi(g), \quad g \in \mathfrak{G}, \quad h \in \mathfrak{G}_0$$

and

$$\int_{\mathfrak{G}/\mathfrak{G}_0} \|\xi(g)\|^2 \, d\dot{m}(\dot{g}) < \infty ,$$

where the last integration is justified by the first condition (i.e. $\|\xi(g - h)\| = \|\xi(g)\|$, $h \in \mathfrak{G}_0$, $g \in \mathfrak{G}$). The representations Φ and U are then defined by:

$$(\Phi(a)\xi)(g) = \Phi^0(\alpha_g^{-1}(a))\xi(g), \quad a \in \mathfrak{M}$$

$$(U(h)\xi)(g) = \xi(g - h), \quad h, g \in \mathfrak{G}.$$

It is now easy to verify that the operators $\Phi(a)$, $U(h)$ actually leave $\mathcal{K}$ invariant and that $\{\Phi, U\}$ is a convariant representation of $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ on $\mathcal{K}$.

The canonical system of imprimitivity associated with the induction from $\mathfrak{G}_0$ to $\mathfrak{G}$ is the von Neumann algebra $\mathfrak{A} = \{\tilde{\mu}(F);\ F \in L^\infty(\mathfrak{G}/\mathfrak{G}_0)\}$ of operators on $\mathcal{K}$ where

$$(\tilde{\mu}(F)\xi)(g) = F(\dot{g})\xi(g), \quad F \in L^\infty(\mathfrak{G}/\mathfrak{G}_0), \quad \xi \in \mathcal{K} .$$

Returning to the setting of §3, we identify $\mathfrak{M}$ with its image in the canonical representation on a Hilbert space $\mathfrak{H}$ associated with an arbitrary f.n.s.f. weight on $\mathfrak{M}$. We see then that the covariant representation $\{\pi_\alpha, \tilde{\lambda}\}$ of $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ defined in §1 is nothing but the covariant representation obtained by inducing the trivial covariant representation of $\{\mathfrak{M}, \{0\}, \alpha\}$ on $\mathfrak{H}$ from $\{0\}$ to $\mathfrak{G}$. We denote by $\{\pi_\alpha^0, \tilde{\lambda}^0\}$ the covariant representation of $\{\mathfrak{M}, \mathfrak{G}_0, \alpha^0\}$ obtained by inducing the trivial representation from $\{0\}$ to the subgroup $\mathfrak{G}_0$; the stage theorem for induced representations then ensures that the representation $\{\pi_\alpha, \tilde{\lambda}\}$ of $\{\mathfrak{M}, \mathfrak{G}, \alpha\}$ on $L^2(\mathfrak{G}; \mathcal{H})$ may be identified with the representation obtained by inducing $\{\pi_\alpha^0, \tilde{\lambda}^0\}$ from $\mathfrak{G}_0$ to $\mathfrak{G}$. Considered as induced from $\{\pi_\alpha^0, \tilde{\lambda}^0\}$, the representation $\{\pi_\alpha, \tilde{\lambda}\}$ acts on the Hilbert space $L^2(\mathfrak{G}, \mathfrak{G}_0; \mathcal{H})$ of all measurable functions $\xi : \mathfrak{G} \times \mathfrak{G}_0 \to \mathcal{H}$ such that:

$$\xi(g - k, h) = \xi(g, h - k), \quad g \in \mathfrak{G}, \quad h, k \in \mathfrak{G}_0$$

and

$$\int_{\mathfrak{G}/\mathfrak{G}_0} \left(\int_{\mathfrak{G}_0} \|\xi(g,h)\|^2 dh \right) d\dot{m}(\dot{g}) < \infty ,$$

and $\{\pi_\alpha, \tilde{\lambda}\}$ is here given by:

$$(\pi_\alpha(a)\xi)(g,h) = \alpha_{g+h}^{-1}(a)\xi(g,h), \quad a \in \mathfrak{M}$$

$$(\tilde{\lambda}(k)\xi)(g,h) = \xi(g - k, h), \quad g, k \in \mathfrak{G}, \quad h \in \mathfrak{G}_0 .$$

The natural isomorphism $T : L^2(\mathfrak{G}; \mathcal{H}) \to L^2(\mathfrak{G}, \mathfrak{G}_0; \mathcal{H})$ is given by:

$$(T\xi)(g,h) = \xi(g + h), \quad g \in \mathfrak{G}, \quad h \in \mathfrak{G}_0 .$$

Therefore on $L^2(\mathfrak{G}, \mathfrak{G}_0; \mathcal{H})$, the generators $\{\pi'(a), \tilde{\rho}_\alpha(g);\ a \in \mathfrak{M}, g \in \mathfrak{G}\}$ of the commutant of $\mathfrak{R}(\mathfrak{M}, \mathfrak{G}, \alpha)$ are given by (Corollary 3.13):

$$(\pi'(a)\xi)(g,h) = a\xi(g,h), \quad a \in \mathfrak{M},$$

$$(\tilde{\rho}_\alpha(k)\xi)(g,h) = W(g)\xi(g + k, h) : g, k \in \mathfrak{G}, \ h \in \mathfrak{G}_0 .$$

We set $W^0 = W/\mathfrak{G}_0$ and denote by $\{\pi^{0'}, \tilde{\rho}_\alpha^0\}$ the covariant representation of $\{\mathfrak{M}', \mathfrak{G}_0, \mathrm{Ad}(W^0(g))\}$ which generates the commutant of $\mathfrak{R}(\mathfrak{M},\mathfrak{G}_0,\alpha^0)$ on $L^2(\mathfrak{G}_0; \mathfrak{H})$ (Corollary 3.13). The following is known [16; Theorem 4.3]:

Theorem 5.2. There exists an isomorphism $\gamma : \mathfrak{R}(\mathfrak{M},\mathfrak{G}_0,\alpha^0)' \to \mathfrak{a}' \cap \mathfrak{R}(\mathfrak{M},\mathfrak{G},\alpha)'$ such that:

$$\gamma(\pi^{0'}(a)) = \pi'(a), \quad a \in \mathfrak{M}' ,$$

and

$$\gamma(\tilde{\rho}^0_\alpha(h)) = \tilde{\rho}_\alpha(h), \quad h \in \mathfrak{G}_0 .$$

With these preparations we now prove:

Theorem 5.3. Let $\mathfrak{G}_0$ be a closed subgroup of $\mathfrak{G}$ and let $\hat{\mathfrak{G}}_0$ be the annihilator of $\mathfrak{G}_0$ in $\hat{\mathfrak{G}}$, i.e. $\hat{\mathfrak{G}}_0 = \{p \in \hat{\mathfrak{G}}; \langle g,p\rangle = 1$ for all $g \in \mathfrak{G}_0\}$. Further let $\hat{\alpha}^0$ denote the restriction to $\hat{\mathfrak{G}}_0$ of the dual action $\hat{\alpha}$ of $\hat{\mathfrak{G}}$ on $\mathfrak{R}(\mathfrak{M},\mathfrak{G},\alpha)$. Then the fixed point algebra $\mathfrak{n}$ of $\hat{\alpha}^0$ is generated by $\pi_\alpha(\mathfrak{M})$ and $\{\tilde{\lambda}(h); h \in \mathfrak{G}_0\}$ and is thus canonically isomorphic to $\mathfrak{R}(\mathfrak{M},\mathfrak{G}_0,\alpha^0)$. Conversely, the set of $p \in \hat{\mathfrak{G}}$ such that $\hat{\alpha}_p$ leaves $\pi_\alpha(\mathfrak{M})$ and $\{\tilde{\lambda}(h); h \in \mathfrak{G}_0\}$ elementwise fixed coincides with $\hat{\mathfrak{G}}_0$.

Proof. We have $\hat{\alpha}(p) = \mathrm{Ad}(\tilde{\mu}(p))$ where $(\tilde{\mu}(p)\xi)(g) = \langle g,p\rangle\xi(g)$, $\xi \in L^2(\mathfrak{G}; \mathfrak{H})$, hence $\mathfrak{n} = \{\tilde{\mu}(p); p \in \hat{\mathfrak{G}}_0\}' \cap \mathfrak{R}(\mathfrak{M},\mathfrak{G},\alpha)$. But $\{\tilde{\mu}(p); p \in \mathfrak{G}_0\}''$ coincides with the von Neumann algebra of multiplication by bounded functions which are constant on $\mathfrak{G}_0$-cosets. That is, $\{\tilde{\mu}(p); p \in \mathfrak{G}_0\}'' = \mathfrak{a}$, the canonical system of imprimitivity associated with the induction from $\mathfrak{G}_0$ to $\mathfrak{G}$, so $\mathfrak{n} = \mathfrak{a}' \cap \mathfrak{R}(\mathfrak{M},\mathfrak{G},\alpha)$. Since by Corollary 3.12 $\tilde{J}\mu(p)\tilde{J} = \mu(-p)$, we have $\tilde{J}\mathfrak{a}\tilde{J} = \mathfrak{a}$ and hence $\tilde{J}\mathfrak{a}'\tilde{J} = \mathfrak{a}'$, thus $\tilde{J}\mathfrak{n}\tilde{J} = \mathfrak{a}' \cap \tilde{J}\mathfrak{R}(\mathfrak{M},\mathfrak{G},\alpha)\tilde{J} = \mathfrak{a}' \cap \mathfrak{R}(\mathfrak{M},\mathfrak{G},\alpha)'$. By Theorem 5.2 there is an isomorphism $\gamma : \mathcal{R}(\mathfrak{M},\mathfrak{G}_0,\alpha^0)' \to \mathfrak{C}' \cap \mathcal{R}(\mathfrak{M},\mathfrak{G},\alpha)$ such that $\gamma(\pi^{0'}(a)) = \pi'(a)$, $a \in \mathfrak{M}$, and $\gamma(\tilde{\rho}^0_\alpha(h)) = \tilde{\rho}_\alpha(h)$, $h \in \mathfrak{G}_0$. Thus, since by Corollary 3.13 $\mathcal{R}(\mathfrak{M},\mathfrak{G}_0,\alpha^0)'$ is generated by $\{\pi^{0'}(a),\tilde{\rho}^0_\alpha(g);\ a \in \mathfrak{M},\ g \in \mathfrak{G}_0\}$, $\tilde{J}\mathfrak{n}\tilde{J}$ is generated by $\{\pi'(a),\tilde{\rho}_\alpha(g);\ a \in \mathfrak{M},\ g \in \mathfrak{G}_0\}$, hence by Corollary 3.13, $\mathfrak{n}$ is generated by $\{\pi_\alpha(a),\tilde{\lambda}(g);\ a \in \mathfrak{M},\ g \in \mathfrak{G}_0\}$. - The second statement follows from the fact that $\hat{\alpha}_p(\pi_\alpha(a)) = \pi_\alpha(a)$, $a \in \mathfrak{M}$, and $\hat{\alpha}_p(\tilde{\lambda}(g)) = \langle g,p\rangle\tilde{\lambda}(g)$, $g \in \mathfrak{G}$ (see §4). Q.E.D.

Remark 5.4. When $\mathfrak{G}$ is abelian, the action $g \in \mathfrak{G} \mapsto \alpha_g \otimes \rho(g) \in \operatorname{Aut}(\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G})))$ considered in Theorem 5.1 is nothing but the second dual action of α (see §4 and [15; Theorem 4.6]). Thus in this case Theorem 5.1 follows from Theorem 5.3. Recently A. Connes and M. Takesaki have shown that under certain integrability conditions on the action α, the systems $\{\mathfrak{M},\alpha\}$ and $\{\mathfrak{M} \otimes \mathcal{B}(L^2(\mathfrak{G})),\alpha \otimes \mathrm{Ad} \circ \rho\}$ are isomorphic, and have thus obtained a Galois type correspondence between subgroups of $\mathfrak{G}$ and α-globally invariant subalgebras of $\mathfrak{M}$ containing the fixed point algebra of α, when the group $\mathfrak{G}$ is abelian (work to appear). Since Theorem 5.1 does not require the commutativity of $\mathfrak{G}$, it holds promise that a similar result can be obtained for non-abelian groups, or at least for the compact ones.

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