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THE FIRST ORDER PREDICATE CALCULUS BASED ON THE LOGIC OF QUANTUM MECHANICS

The fundamentals of the logic of quantum mechanics were discovered by Birkhoff and von Neumann [1]. But the definitive formulation of this logic can be obtained only after reconstructing the whole quantum mechanics on grounds of this logic. In spite of many successes [2], [3], [4], such a reconstruction has not been so far fulfilled. We are also compelled to take into account that the difference between the logical and the nonlogical is not distinctive. Under the circumstances it is reasonable to elaborate a logical calculus which is in line with the usual canons adopted for logical calculus, on the one hand, and which contains the largest possible part of the quantum mechanical laws on the other hand. Calculus Q described below is proposed as a solution to the question we have put.

The alphabet of a quantum logic contains the signs $\neg, \vee, \rightarrow, \exists$ (negation, alternative, implication, existential quantifier), a set $V = \{x, y, z, ...\}$ of individual variables, a set P of predicate symbols and parentheses. Formulae are formed in the usual way and constitute the algebra $F = \langle F^{\circ}, \neg, \vee, \rightarrow \rangle$. We agree that we may omit parentheses, keeping in mind that $\neg$ and $\exists$ bind stronger then $\vee$ and $\rightarrow$ binds weaker then $\vee$.

The rules of transformation contains thirteen axioms and three rules of inference:

A1. $X \vee Y \rightarrow Y \vee X$,

A2. $X \vee (Y \vee Z) \rightarrow (X \vee Y) \vee Z$,

A3. $X \rightarrow \neg\neg X$,

A4. $X \vee \neg X$,

A5. $X \rightarrow X \vee Y$,

A6. $X \vee \neg(\neg X \vee Y) \rightarrow X$,

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A7. $\neg(X \to Y) \vee (\neg(Y \to Z) \vee (X \to Z))$,

A8. $\neg(X \to Y) \vee (Z \vee X \to Z \vee Y)$,

A9. $(X \to Y) \to \neg(X \vee Y) \vee Y$,

A10. $\neg(X \vee Y) \vee Y \to (X \to Y)$,

A11. $X \to \neg(\neg(X \vee Y) \vee Y) \vee Y$,

A12. $\neg(X \to Y) \vee (\neg X \vee Y)$,

A13. $Xy \to \exists x Xx$,

B1. $X, \neg X \vee Y \vdash Y$,

B2. $X \to Y \vdash \neg Y \to \neg X$,

B3. $Xy \to Y \vdash \exists x Xx \to Y$.

Here $X, Y, Z \in F^{\circ}$; $x, y \in V$. Xy means a formula X with marked variable y and Xx means the result of substitution of x instead of all free occurrences of y in this formula. The conclusion in the rule B3 does not contain y freely. The sign $\vdash$ means the derivability (in one step here). We denote the described formal system by Q.

Let $\Gamma \subset F^{\circ}$. We shall speak, X is derivable from Γ and read $\Gamma \vdash X$ if there is such a finite sequence of formulae ended by X, that any formula of this sequence satisfies one of the following conditions:

1. it belongs to Γ;
2. it is one of the axioms A1—A13;
3. it is obtained with the help of the rules of inference B1—B3 from precursors.

$A, B, \ldots \vdash X$ will mean $\{A, B, \ldots\} \vdash X$; $\vdash X$ means $\varnothing \vdash X$; Γ, X means $\Gamma \cup \{X\}$.

There are few examples:

P1. $\vdash X \vee X \to X$,

P2. $\vdash (X \vee Y) \vee Z \to X \vee (Y \vee Z)$,

P3. $\vdash X \to X \vee \neg(\neg X \vee Y)$,

P4. $X \to Y \vdash Z \vee X \to Z \vee Y$,

P5. $X \to Y \vdash X \vee Z \to Y \vee Z$,

P6. $X \to Y \vdash (Y \to Z) \to (X \to Z)$,

P7. $X \to Y,\ Y \to X \vdash (Z \to X) \to (Z \to Y)$,

P8. $\vdash \neg\neg X \to X$,

P9. $\vdash X \vee Y \to \neg(\neg(X \vee Y) \vee Y) \vee Y$,

P10. $\vdash \neg(\neg(X \vee Y) \vee Y) \vee Y \to X \vee Y$,

P11. $\vdash X \vee (Y \vee \neg Y) \to Y \vee \neg Y$,

P12. $X \vee \neg X \to Y \vdash Y$.

The proofs are below; the reasons are indicated to the right of lines.

i)

1. $\neg X \vee \neg(\neg\neg X \vee Y) \to \neg X$ (A6)
2. $\neg\neg X \to \neg(\neg X \vee \neg(\neg\neg X \vee Y))$, (1, B2)
3. $\neg\big(\neg\neg X \to \neg(\neg X \vee \neg(\neg\neg X \vee Y))\big) \vee \big(X \vee \neg\neg X \to X \vee \neg(\neg X \vee \neg(\neg\neg X \vee Y))\big)$, (A8)
4. $X \vee \neg\neg X \to X \vee \neg(\neg X \vee \neg(\neg\neg X \vee Y))$, (2, 3, B1)
5. $X \vee \neg(\neg X \vee \neg(\neg\neg X \vee Y)) \to X$, (A6)
6. $\neg\big(X \vee \neg\neg X \to X \vee \neg(\neg X \vee \neg(\neg\neg X \vee Y))\big) \vee \big(\neg(X \vee \neg(\neg X \vee \neg(\neg\neg X \vee Y) \to X) \vee (X \vee \neg\neg X \to X))\big)$, (A7)
7. $\neg(X \vee \neg(\neg X \vee \neg(\neg\neg X \vee Y) \to X) \vee (X \vee \neg\neg X \to X))$, (4, 6, B1)
8. $X \vee \neg\neg X \to X$, (5, 7, B1)
9. $X \to \neg\neg X$, (A3)
10. $\neg(X \to \neg\neg X) \vee (X \vee X \to X \vee \neg\neg X)$, (A8)
11. $X \vee X \to X \vee \neg\neg X$, (9, 10, B1)
12. $\neg(X \vee X \to X \vee \neg\neg X) \vee (\neg(X \vee \neg\neg X \to X) \vee (X \vee X \to X))$, (A7)
13. $\neg(X \vee \neg\neg X \to X) \vee (X \vee X \to X)$, (11, 12, B1)
14. $X \vee X \to X$. (8, 13, B1)

ii) $(X \vee Y) \vee Z \to Z \vee (X \vee Y)$, (A1)

$X \vee Y \to Y \vee X$, (A1)

$Z \vee (X \vee Y) \to Z \vee (Y \vee X)$, (A8, B1)

$Z \vee (Y \vee X) \to (Z \vee Y) \vee X$, (A2)

$(Z \vee Y) \vee X \to X \vee (Z \vee Y)$, (A1)

$Z \vee Y \to Y \vee Z$, (A1)

$X \vee (Z \vee Y) \to X \vee (Y \vee Z)$, (A8, B1)

$(X \vee Y) \vee Z \to X \vee (Y \vee Z)$. (A7, B1)

iii) is based on A5

iv) is based on A8 and B1

v) $X \to Y$, (ass.)

$Z \vee X \to Z \vee Y$, (P4)

$X \vee Z \to Z \vee X$, (A1)

$Z \vee Y \to Y \vee Z$, (A1)

$X \vee Z \to Y \vee Z$. (A7, B1)

vi) $X \to Y$, (ass.)

$X \vee Z \to Y \vee Z$, (P5)

$\neg(Y \vee Z) \to \neg(X \vee Z)$, (B2)

$\neg(Y \vee Z) \vee Z \to \neg(X \vee Z) \vee Z,$ (P5)
$\neg(Y \vee Z) \vee Z \to (X \to Z),$ (A10, A7, B1)
$(Y \to Z) \to (X \to Z).$ (A9, A7, B1)

vii) $X \to Y,$ (ass.)
$Y \to X,$ (ass.)
$Z \vee Y \to Z \vee X,$ (P4)
$\neg(Z \vee X) \to \neg(Z \vee Y),$ (B2)
$\neg(Z \vee X) \vee X \to \neg(Z \vee Y) \vee X,$ (P5)
$\neg(Z \vee Y) \vee X \to \neg(Z \vee Y) \vee Y,$ (P4)
$\neg(Z \vee X) \vee X \to \neg(Z \vee Y) \vee Y,$ (A7, B1)
$(Z \to X) \to (Z \to Y).$ (A9, A10, A7, B1)

viii) $\neg X \vee \neg X \to \neg X,$ (P1)
$\neg\neg X \to \neg(\neg X \vee \neg X),$ (B2)
$\neg(\neg X \vee \neg X) \to X \vee \neg(\neg X \vee \neg X),$ (A5, A1, A7, B1)
$X \vee \neg(\neg X \vee \neg X) \to X,$ (A6)
$\neg\neg X \to X.$ (A7, B1)

ix) $X \to \neg(\neg(X \vee Y) \vee Y) \vee Y,$ (A11)
$X \vee Y \to \left(\neg(\neg(X \vee Y) \vee Y) \vee Y\right) \vee Y,$ (P5)
$X \vee Y \to \neg(\neg(X \vee Y) \vee Y) \vee (Y \vee Y),$ (P2, A7, B1)
$\neg(\neg(X \vee Y) \vee Y) \vee (Y \vee Y) \to \neg(\neg(X \vee Y) \vee Y) \vee Y,$ (P1, A8, A7, B1)
$X \vee Y \to \neg(\neg(X \vee Y) \vee Y) \vee Y.$ (A7, B1)

x) $\neg(X \vee Y) \to \neg(X \vee Y) \vee Y,$ (A5)
$\neg(\neg(X \vee Y) \vee Y) \to \neg\neg(X \vee Y),$ (B2)
$\neg(\neg(X \vee Y) \vee Y) \to X \vee Y,$ (P8, A7, B1)
$\neg(\neg(X \vee Y) \vee Y) \vee Y \to (X \vee Y) \vee Y,$ (P5, A7, B1)
$\neg(\neg(X \vee Y) \vee Y) \vee Y \to X \vee Y.$ (P2, A7, P1, B1)

xi) $Y \vee \neg Y \to Z \vee (Y \vee \neg Y),$ (A5, A1, A7, B1)
$\neg(Y \vee \neg Y) \vee (Z \vee (Y \vee \neg Y)),$ (A12, B1)
$Z \vee (Y \vee \neg Y),$ (A4, B1)
$\neg\left((X \vee (Y \vee \neg Y)) \vee (Y \vee \neg Y)\right) \vee (Y \vee \neg Y),$ (subst.)
$X \vee (Y \vee \neg Y) \to Y \vee \neg Y.$ (A10, A12, B1)

xii) $X \vee \neg X \to Y,$ (ass.)
$\neg(X \vee \neg X) \vee Y,$ (A12, B1)
$Y.$ (B1)

The system Q gives a new definition of a quantum logic. The old algebraic definition of it [5] is based on lattices. By the orthomodular lattice we mean an algebra $L^* = \langle L^{*\circ}, \perp, \cup, \cap \rangle$, such that the following conditions are fulfilled for any $\alpha, \beta, \gamma \in L^{*\circ}$:

L1. $\alpha \cup \beta = \beta \cup \alpha$,

L2. $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$,

L3. $\alpha^{\perp\perp} = \alpha$,

L4. $\alpha \cup (\beta \cup \beta^{\perp}) = \beta \cup \beta^{\perp}$,

L5. $\alpha \cup (\alpha \cap \beta) = \alpha$,

L6. $\alpha \cup \beta = \big((\alpha \cup \beta) \cap \beta^{\perp}\big) \cup \beta$,

L7. $\alpha \cap \beta = (\alpha^{\perp} \cup \beta^{\perp})^{\perp}$.

One may also regard L^* as an algebra with only two operation $^{\perp}$ and $\cup$. Then L7 will be an abbreviative definition of $\cap$. On the other hand, against the opinion of some investigators (see as an example [5], page 441), the implication may by considered with the help of an abbreviative definition

D1. $\alpha \supset \beta \underset{\text{df}}{=} (\alpha \cup \beta)^{\perp} \cup \beta.$

In that way we get an algebra $L = \langle L^{\circ}, ^{\perp}, \cup, \supset \rangle$, which we call the quantum lattice. By the way, the operation $\cup$ may by eliminated from by means of definition.

D2. $\alpha \cup \beta \underset{\text{df}}{=} (\alpha \supset \beta) \supset \beta$

Indeed, in consequence of D1 $(\alpha \supset \beta) \supset \beta = \Big(\big((\alpha \cup \beta)^{\perp} \cup \beta\big) \cup \beta\Big)^{\perp} \cup \beta$, and the last is equal to $\alpha \cup \beta$ in consequence of L6. At last, if we introduce a new constant $\mathbf{0}$, then we may eliminate the operation $^{\perp}$ also, taking an abbreviative definition

D3. $\alpha^{\perp} \underset{\text{df}}{=} \alpha \supset 0.$

It is well known, a partial order in any lattice can by defined with the help of

D4. $\alpha \leqslant \beta \underset{\text{df}}{=} \alpha \cup \beta = \beta$.

$\mathbf{0}$ and 1 are the zero and the unit elements of L.

Lemma 1. *The conditions $\alpha \leqslant \beta$ and $\alpha \supset \beta = 1$ are equivalent in any quantum lattice.*

Indeed, let $\alpha \supset \beta = 1$. Then we have $(\alpha \cup \beta)^{\perp} \cup \beta = 1$ in consequence of D1, or $(\alpha \cup \beta) \cap \beta^{\perp} = 0$. Therefore $\big((\alpha \cup \beta) \cap \beta^{\perp}\big) \cup \beta = \beta$ and this is equivalent to $\alpha \cup \beta = \beta$ on account of L6. Conversely, let $\alpha \leqslant \beta$. This means $\alpha \cup \beta = \beta$ and then $\alpha \supset \beta = (\alpha \cup \beta)^{\perp} \cup \beta = \beta^{\perp} \cup \beta = 1$.

Let C be any set of individuals. Any mapping $i\colon V \mapsto C$ gives rise to a morphism $\hat{\imath}\colon F \mapsto F^c$, from the algebra of formulae F into the algebra of "quasiformulae" F^c. Here $\hat{\imath}(X)$ is the result of substitution of $i(x)$ in a formula X instead of all free occurrences of any variable x.

Definition 1. *We call the triplet* $\mathcal{A} = \langle C, L, h \rangle$ *a model of a set of formulae* Γ, *if* C *is a non-empty set,* L *is a quantum lattice,* $h\colon F^c \mapsto L$ *is a morphism of the algebra of quasiformulae* F^c *into* L, *preserving the operations* $\neg, \vee, \rightarrow$ *(turning into* $^\perp, \cup, \supset$*), and the following conditions* H *and* G *are fulfilled:*

H. $$h(\exists x\, Qx) = \bigcup_{a \in C} h(Qa),$$

where $\exists x Qx$ *is any quasiformula, beginning with* $\exists$ *and* $\bigcup$ *denotes the infinite join.*

G. $$h(\hat{\imath}(X)) = 1$$

for any i *and* $X \in F^\circ$.

Definition 2. *We call a formula* $X \in F^\circ$ *true in model* $\mathcal{A}$ *if for any* i, $h(\hat{\imath}(X)) = 1$

Theorem 1 (consistency). *If* $\Gamma \vdash X$ *then* X *is true in any model of* Γ.

This theorem follows from lemmas 2 and 3 below.

Lemma 2. *Any axiom* A1—A13 *is true in any model* $\mathcal{A}$.

Let us put $a = i(x)$, $b = i(y)$, $\alpha = h(\hat{\imath}(X))$, $\beta = h(\hat{\imath}(Y))$, $\gamma = h(\hat{\imath}(Z))$. In all cases we may replace the proof of $\alpha \supset \beta = 1$ by the proof of $\alpha \leqslant \beta$ in consequence of lemma 1. We verify:

i)—v) The verification for A1—A5 is easy.

vi) $\alpha \cup (\alpha^\perp \cup \beta)^\perp \leqslant \alpha$ is true in consequence of L7, L3 and L5.

vii)
$$\begin{aligned}
&(\alpha \supset \beta)^\perp \cup ((\beta \supset \gamma)^\perp \cup (\alpha \supset \beta)) = && \text{(D1)}\\
&= ((\alpha \cup \beta) \cap \beta^\perp) \cup ((\beta \cup \gamma) \cap \gamma^\perp) \cup (\alpha \cup \gamma)^\perp \cup \gamma = && \text{(L6)}\\
&= ((\alpha \cup \beta) \cap \beta^\perp) \cup (\beta \cup \gamma) \cup (\alpha \cup \gamma)^\perp = && \text{(L6)}\\
&= (\alpha \cup \beta) \cup \gamma \cup (\alpha \cup \gamma)^\perp = && \text{(L4)}\\
&= \beta \cup (\alpha \cup \gamma) \cup (\alpha \cup \gamma)^\perp = 1
\end{aligned}$$

viii)
$$\begin{aligned}
&(\alpha \supset \beta)^\perp \cup (\gamma \cup \alpha \supset \gamma \cup \beta) = && \text{(D1)}\\
&= ((\alpha \cup \beta) \cap \beta^\perp) \cup (\gamma \cup \alpha \cup \gamma \cup \beta)^\perp \cup \gamma \cup \beta = && \text{(L6, L4)}\\
&= (\alpha \cup \beta) \cup (\gamma \cup \alpha \cup \beta)^\perp \cup \gamma = 1
\end{aligned}$$

ix), x) are true in consequence of D1.

xi)
$$\begin{aligned}
&\alpha \leqslant ((\alpha \cup \beta)^\perp \cup \beta)^\perp \cup \beta\\
&\alpha \leqslant \alpha \cup \beta && \text{(L6)}
\end{aligned}$$

xii) $(\alpha \supset \beta)^{\perp} \cup \alpha^{\perp} \cup \beta =$ (D1)

$= ((\alpha \cup \beta) \cap \beta^{\perp}) \cup \beta \cup \alpha^{\perp} =$ (L6)

$= \alpha \cup \beta \cup \alpha^{\perp} = 1$

xiii) $\alpha(b) \leqslant \bigcup_{a \in C} \alpha(a)$ is true according to the definition of the infinite join.

Lemma 3. *The set of formulae, true in a model $\mathcal{A}$ is closed under the rules of inference* B1—B3.

We verify:

i) If $\alpha = 1$ and $\alpha^{\perp} \cup \beta = 1$, then $\beta = 1$. This is evident.

ii) If $\alpha \leqslant \beta$, then $\beta^{\perp} \leqslant \alpha^{\perp}$. This is true in consequence of L1—L7.

iii) If for any $b \in C$ $\alpha(b) \leqslant \beta$, then $\bigcup_{a \in C} \alpha(a) \leqslant \beta$. This is evident in consequence of the definition of the infinite join.

Passing to the establishment of the completeness of Q we define the relation $\approx$ in the set of formulae F°:

E. $X \approx Y \underset{\mathrm{df}}{=} (\Gamma \vdash X \to Y \text{ and } \Gamma \vdash Y \to X)$.

Lemma 4. *$\approx$ is a relation of congruence in the algebra F.*

Indeed, this relation is symmetric and, in consequence of A7 and B1, it is transitive. Hence, it is a relation of equivalence. This relation is also consistent with operations of F in consequence of B2, P4 and P5, P6 and P7.

Lemma 5. *The Lindenbaum-Tarski's algebra* [6] *$F/\approx$ is a quantum lattice. This means,* L1—L6 *and* D1 *are true for $\neg/\approx$, $\vee/\approx$ and $\to/\approx$ in the capacity of $^{\perp}$, $\cup$ and $\supset$.*

The proof is received by referring to A1, A2 and P2, A3 and P8, A5 and P11, A5 and A6, P9 and P10, A9 and A10.

The natural morphism $f\colon F \mapsto F/\approx$ possess some important properties.

Lemma 6. *If $\Gamma \vdash X \to Y$, then $f(X) \leqslant f(Y)$.*

Indeed, if $\Gamma \vdash X \to Y$, then $\Gamma \vdash X \vee Y \to Y$, and, certainly, $\Gamma \vdash Y \to X \vee Y$. Therefore $X \vee Y \approx Y$. This means $f(X) \cup f(Y) = f(Y)$ or $f(X) \leqslant f(Y)$.

Lemma 7. *$\mathcal{E} = \langle V, F/\approx, f\rangle$ is a model of Γ.*

In consequence of lemma 5 it is enough to verify the conditions H and G. G is evident.

For proving H it is enough to show for any $X, Y \in F^{\circ}$: for any $y \in V$ $f(Xy) \leqslant f(\exists x\, Xx)$, and if for any $y \in V$ $f(Xy) \leqslant \beta$ then $f(\exists x\, Xx) \leqslant \beta$. In consequence of lemma 6 it is enough to show: for any $y \in V$ $\Gamma \vdash Xy \rightarrow \exists x Xx$, and if for any $y \in V$ $\Gamma \vdash Xy \rightarrow Y$, then $\Gamma \vdash \exists x Xx \rightarrow Y$. But A13 and B13 express just the last.

Lemma 8. *If* $f(X) = 1$, *then* $\Gamma \vdash X$.

On account of P11 $f(Y \vee \neg Y) = 1$, therefore, if $f(X) = 1$, then $X \approx Y \vee \neg Y$. It means $\Gamma \vdash Y \vee \neg Y \rightarrow X$ and in consequence of A13, A7 and B1, $\Gamma \vdash X$.

Theorem 2 (completeness). *If a formula X is true in any model of* Γ, *then* $\Gamma \vdash X$.

Indeed, so far as $\mathcal{E}$ is a model of Γ, we have $f(X) = 1$ in $\mathcal{E}$ and, in consequence of lemma 8, $\Gamma \vdash X$.

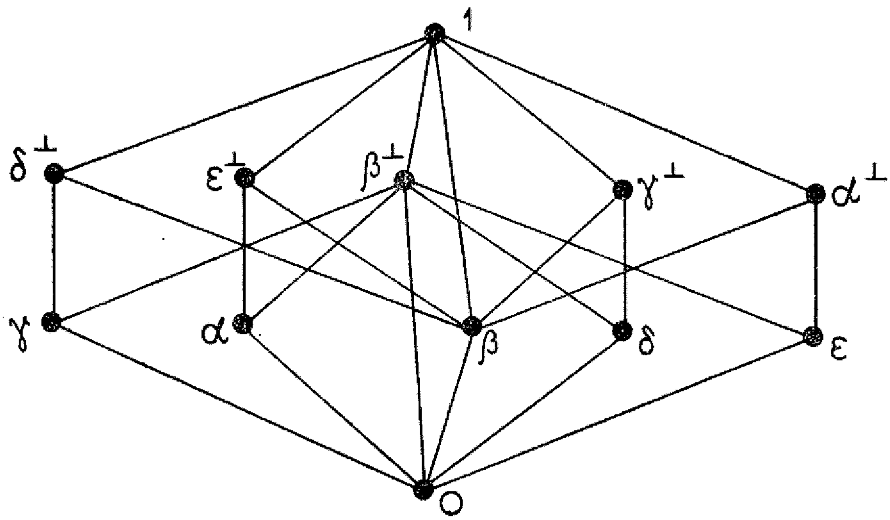

Figure 1

Note 1. Piron [5] proposed the fullest list of axioms for a set of sentences of the quantum theory. Our calculus Q corresponds roughly to the axioms of the Pirons groups O, T, C and P, but does not reflect the axioms of the Pirons group A. The last group contains two axioms (see [5], page 448):

A_1. If $L \ni a \neq 0$, then there exists an atom ω such that $\omega \leqslant a$.

A_2. For any atom ω and any $a, \beta \in L$ if $a \leqslant \beta \leqslant a \cup \omega$ then $\beta = a$ or $\beta = a \cup \omega$.

(Atoms are the minimal elements of $L - \{0\}$).

In our opinion, there is no simple logical calculus of the first order, which corresponds to the axioms of Pirons group A too, because these axioms there are of typical "second order" character.

Note 2. The deduction theorem, taken in the form "If $\Gamma, X \vdash Y$ then $\Gamma \vdash \neg X \vee Y$" is not true. Indeed, to obtain a contradiction let us assume that the deduction theorem is true. We have evidently $\neg Z \vee X$, $\neg Z \vee \neg X \vee Y, Z \vdash Y$. Then we conclude from the deduction theorem that $\neg Z \vee X$, $\neg Z \vee \neg X \vee Y \vdash \neg Z \vee Y$. In consequence of theorem 1 this means $\gamma^{\perp} \cup \beta = 1$ for any quantum lattice in which $\gamma^{\perp} \cup \alpha = 1$ and $\gamma^{\perp} \cup \alpha^{\perp} \cup \beta = 1$. But it isn't true for the twelve element lattice given in Figure 1 above.

Of course, the stronger deduction theorem "If $\Gamma, X \vdash Y$ then $\Gamma \vdash X \rightarrow Y$" is also false.

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