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RELATIVE OPERATOR ENTROPY IN NONCOMMUTATIVE INFORMATION THEORY

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Abstract. Relative entropies for positive linear forms on operator algebras have been discussed by various researchers. Following after the operator entropy due to Nakamura and Umegaki, a relative operator entropy for positive operators on a Hilbert space is defined. If A and B are invertible then it is expressed by a 'mean' $A\ s\ B = A^{1/2}(\log A^{-1/2}BA^{-1/2})A^{1/2}$ whose representing function is $\log t$, in the Kubo-Ando theory. It is shown that it satisfies principal properties for relative entropies.

1. Umegaki's ralative entropy is introduced in [17] as a noncommutaitive version of the Kullback-Leibler entropy, which is given by the trace of

$$(1) \qquad A \log A - A \log B.$$

where A and B are positive operators affiliated with a semifinite von Newmann algebra Afterwards, many authors have considered the relative entropy [2, 3, 4 11 13, 14, 16].

In this note, we shall restrict ourselves on bounded positive operators on a Hilbert space and introduce a relative operator entropy $S(A \mid B)$ as the solidarity $A\ s\ B$ whose representing function is log t. The solidarity is somewhat a generalization of means in the Kubo-Ando theory. In the case where A and B are invertible, we have the exact form

$$(2) \qquad S(A \mid B) = A^{1/2}(\log A^{-1/2} BA^{-1/2})A^{1/2}.$$

As a matter of fact, the representing function $f(t) = \log t$ is appeared as $S(1 \mid t)$. Furthermore, the adjoint F of f is the entropy function $-t \log t$, cf. [7] and so

$$F(A) = -A \log A = S(A \mid 1).$$

We should note that $-A \log A$ is the operator entropy considered by Nakamura-Umegaki

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[13]. This means that the relative operator entropy (2) is a relative version of the operator entropy. If A and B commute, then (1) and (2) coincide up to sign.

In § 2, we discuss variational forms of the relative operator entropy which makes easy to define it for noninvertible operators. In § 3, we investigate the relative operator entropy has principal properites on relative entropy. Finally we refer relations between the relative operator entropy and an operator version of Uhlmann's relative entropy.

2. For the sake of convenience, we define the *selfadjoint quotient* of positive A for B by

$$B/A = A^{-\frac{1}{2}} BA^{-\frac{1}{2}}. \tag{3}$$

Then $\|B/A\| = \inf\{a>0\ ;\ B \leqslant aA\}$ and the BKW metric [9] is given by

$$d(A, B) = \log \|A/B\| \cdot \|B/A\|.$$

So, if A is a contraction dominated by B, then

$$\|S(A \mid B)\| \leqslant d(A, B).$$

It is known in [9] that the set of all positive invertible operators with norm 1 is a complete space, on which Kato-Watatani gave a generalized polar decomposition.

Now the notation of solidarity reminds us of Kato's idea in about 10 years ago when the study of means was on its way. It is just the same as (2) if one replaces $\log t$ by any operator monotone function as mentioned in the proof of [5 ; Theorem 3] and its next paragraph. This suggests that one can construct a mean m from an operator monotone functions f by

$$A\ m\ B = A^{\frac{1}{2}} f(B/A) A^{\frac{1}{2}} = f(B/A)/A^{-1}. \tag{4}$$

Afterwards, Kubo and Ando [12] made a complete investigation into it including in the reverse discussion. The principal result is that there exists an affine order-isomorphism between the class of all means and that of nonnegative operator monotone functions on $[0, \infty)$ whose correspondence is given by (4).

Anyway, coming back to Kato, one may consider a binary operation s among positive operators induced by an operator monotone function f on $(0, \infty)$ as in (4), and we call it a solidarity. Solidarity might have an analogy to means in the Kubo-Ando theory. Here we don't discuss on a solidarity itself, though.

Now, we shall give variational forms of (2):

Lemma 1. *If A and B are positive invertible, then*

$$\begin{aligned} S(A \mid B) &= -A^{\frac{1}{2}} \log(A^{\frac{1}{2}} B^{-1} A^{\frac{1}{2}}) A^{\frac{1}{2}} \\ &= -\log(B^{-1}/A^{-1})/A^{-1}, \end{aligned} \tag{5}$$

and

$$S(A \mid B) = B^{\frac{1}{2}} F(A/B) B^{\frac{1}{2}} = F(A/B)/B^{-1}, \tag{6}$$

where $F(t) = -t \log t$.

Proof. The formula (5) is obtained by $\log(1/t) = -\log t$. Since $Xf(X^*X) = f(XX^*)X$ holds in general, applying it for $X = A^{\frac{1}{2}} B^{-\frac{1}{2}}$, we have

$$\begin{aligned} S(A \mid B) &= -\log(XX^*)/A^{-1} = -A^{\frac{1}{2}} \log(XX^*) XB^{\frac{1}{2}} \\ &= -A^{\frac{1}{2}} X \log(X^*X) B^{\frac{1}{2}} = -X^*X \log(X^*X)/B^{-1} \\ &= F(X^*X)/B^{-1} = F(A/B)/B^{-1}. \end{aligned}$$

The above lemma says that if B is invertible, then one can define $S(A \mid B)$ by (5) or (6) even if A is not invertible. Thus, considering the operator monotonity with respect to B, we redefine the relative operator entropy as follows :

Definition 2. For positigve operators A and B, the *relative operator entropy* $S(A \mid B)$ is defined by

$$S(A \mid B) = -\ \underset{\varepsilon \to +0}{\text{s-lim}}\ A^{\frac{1}{2}} \log(A^{\frac{1}{2}} (B+\varepsilon)^{-1} A^{\frac{1}{2}}) A^{\frac{1}{2}}, \tag{7}$$

if the strong limit exists.

Although, by such a technical reason, we rephrased $S(A \mid B)$ as the above, we had better stand on (2). As for the condition of the existence of the limit, one of them is that ran $A^{\frac{1}{2}}$ includes ran $B^{\frac{1}{2}}$ For commuting operators A and B, we have

$$S(A \mid B) = -A \cdot \log A + \underset{\varepsilon \to +0}{\text{s-lim}}\ A \cdot \log(B+\varepsilon), \tag{7'}$$

and then, if the limit exists then supp $A \leqslant$ supp B where supp C is the support projection of C.

3. In this section, we shall show that the relative operator entropy has the desireable properties.

Theorem 3. *The relative entropy $S(A \mid B)$ has the following properties :*

(operator monotonity) *$B \leqslant C$ implies $S(A \mid B) \leqslant S(A \mid C)$.*

(termwise concavity) $S(A\,|\,B)$ *is operator concave with respect to* A, B.

(homogeneity) $S(\alpha A\,|\,\alpha B) = \alpha S(A\,|\,B)$ *for a nonnegative number* α.

Proof. Since $f(t) = \log t$ is operator monotone, the formula (4) implies the required monotonity. Moreover, the operator concavity of $f(t) = \log t$ implies the operator concavity for the term B. As for the term A, the operator concavity of $F(t) = -t \cdot \log t$ follows from the formula (6). The homogeneity is clear by (4).

In (7), the existence of the storong limit depends on the lower boundedness of the set $\{A^{1/2} \log(A^{1/2}(B+\epsilon)^{-1}A^{1/2})A^{1/2}\}$. Contrastively, the upper boundedness is always guaranteed :

Theorem 4. *The relative operator entropy is upper bounded :*

$$S(A\,|\,B) \leqslant -A \cdot \log A + A \cdot \log \|B\|, \text{ and} \tag{8}$$

$$S(A\,|\,B) \leqslant B - A. \tag{9}$$

Proof. By operator monotonity and (7'), we have

$$S(A\,|\,B) \leqslant S(A\,|\,\|B\|) = -A \cdot \log A + A \cdot \log \|B\|.$$

It follows from the Klein inequality $\log t \leqslant t - 1$ that

$$S(A\,|\,B) = \log(B/A)/A^{-1} \leqslant (B/A - 1)/A^{-1} = B - A.$$

By (9), we have a nonpositive condition :

Corollary 4. 1. *If* $A \geqslant B$, *then* $S(A\,|\,B) \leqslant 0$.

Formally, it follows from (8) that $S(A\,|\,B) \leqslant 0$ if $A \leqslant \|B\|$. But the meaningful siuation would be $A \geqslant B$ as in the above corollary. In this situation, $S(A\,|\,B)$ is negative when $A \neq B$:

Corollary 4. 2. *For operators* A, B *with* $A \geqslant B$, $S(A\,|\,B) = 0$ *if and only if* $A = B$.

Proof. Suppose $S(A\,|\,B) = 0$. Then, it follow from (9) that $0 = S(A\,|\,B) \leqslant B - A \leqslant 0$, which implies $A = B$. Conversely, we have $S(A\,|\,A) = A \cdot \log(\text{supp}\,A) = 0$.

Remark. For invertible operators, it is easy to see that the assumption $A \geqslant B$ in the above can be removed.

Since the operator concave function $F(t) = -t \cdot \log(t)$ satisfies $F(0) = 0 \geqslant 0$, we can make use of the Jensen inequality due to Hansen and Pedersen [4 : Theorem 2. 1], see also M. Fujii [7] :

$$F(X^*AX + Y^*BY) \geqslant X^*F(A)X + Y^*F(B)Y \quad \text{for } X^*X + Y^*Y \leqslant 1, \tag{10}$$

in order to show the joint concavity. But their inequality means directly that the relative operator entropy is subadditive :

Lemma 5. *The relative operator entropy is subadditive :*

$$S(A+B\,|\,C+D) \geqslant S(A\,|\,C) + S(B\,|\,D).$$

Proof. We may assume that both C and D are invertible. Put $X = C^{1/2}(C+D)^{-1/2}$ and $Y = D^{1/2}(C+D)^{-1/2}$: Since $X^*X + Y^*Y = 1$, it follows from (10) that

$$\begin{aligned} S(A+B\,|\,C+D) &= F((A+B)/(C+D))/(C+D)^{-1} \\ &= F(X^*(A/C)X + Y^*(B/D)Y)/(C+D)^{-1} \\ &\geqslant \{X^*F(A/C)X + Y^*F(B/D)Y\}/(C+D)^{-1} \\ &= \{F(A/C)/C^{-1} + F(B/D)/D^{-1}\} \\ &= S(A\,|\,C) + S(B\,|\,D). \end{aligned}$$

Theorem 6. *The relative operator entropy is jointly concave : If* $A = \alpha A_1 + (1-\alpha)A_2$ *and* $B = \alpha B_1 + (1-\alpha)B_2$ *for* $0 < \alpha < 1$, *then*

$$S(A\,|\,B) \geqslant \alpha S(A_1\,|\,B_1) + (1-\alpha)S(A_2\,|\,B_2).$$

Proof. By Lemma 5 and the homogenuity, we have

$$\begin{aligned} S(A\,|\,B) &= S(\alpha A_1 + (1-\alpha)A_2\,|\,\alpha B_1 + (1-\alpha)B_2) \\ &\geqslant S(\alpha A_1\,|\,\alpha B_1) + S((1-\alpha)A_2\,|\,(1-\alpha)B_2) \\ &= \alpha S(A_1\,|\,B_1) + (1-\alpha)S(A_2\,|\,B_2). \end{aligned}$$

By operator monotonity, the above condition for B can be relaxed into $B \geqslant \alpha B_1 + (1-\alpha)B_2$. Nest we show the generalized Pierls-Bogoliubov inequality by making use of Davis' inequality [1 ; Theorem VI. 1]

$$\Psi(F(A)) \leqslant F(\Psi(A))$$

for a unital positive linear map Ψ :

Theorem 7. *Let* Φ *be a normal positive linear map from a* W^**-algebra containing* A *and* B *to a suitable* W^**-algebra. If* $\Phi(1)$ *is invertible, then,*

$$(11) \qquad \Phi(S(A \mid B)) \leqslant S(\Phi(A) \mid \Phi(B)).$$

Proof. Let B be invertible, then so does $\Phi(B)$. Put a unital map $\Phi_B(X) = \Phi(X/B^{-1})/\Phi(B)$. Then, it follows from (6) that

$$\begin{aligned}\Phi(S(A \mid B)) &= \Phi(F(A/B)/B^{-1}) = \Phi_B(F(A/B))/\Phi(B)^{-1}\\ &\leqslant F(\Phi_B(A/B))/\Phi(B)^{-1}\\ &= F(\Phi(A)/\Phi(B))/\Phi(B)^{-1} = S(\Phi(A) \mid \Phi(B)).\end{aligned}$$

Since Φ is normal, (11) holds even if B is not invertible.

Remark that the above inequality holds for a general operator concave functions. Though this remark suggests the inequality of solidarity, we refer to it no more. Back to the original situation, we have the Peierls-Bogoliubov inequality for the relative operator entropy :

Corollary 7. 1 (Peierls-Bogoliubov inequality). *Let* ϕ *be a normal positive linear functional on a* W^**-algebra containing* A *and* B*, then,*

$$\phi(S(A \mid B)) \leqslant \phi(A)\ \{\log(\phi(A)) - \log(\phi(B))\}.$$

Applying Theorem 7 for a map $\phi(X) = PXP$ from M to the reduced von Neuman algebra PMP, we have so-called monotonity of the relative entropy :

Corollary 7. 2 (monotonity). *If* P *is a projection, then*

$$S(PAP \mid PBP) \geqslant PS(A \mid B)P.$$

4. For $0 < r < 1$, let m_r be the operator mean corresponding to the representing function $f_r(t) = t^r$. By (4), for invertible A and B, $A\ m_r\ B = (B/A)^r/A^{-1}$. If A commutes with B, then $A\ m_r\ B = A^{1-r}B^r$. As Uhlmann himself remarked in [16], the quadratic interpolation $QI_r(p, q)$ for seminorms $p(x) = <Ax, x>^{1/2}$ and $q(x) = <Bx, x>^{1/2}$ is the seminorm defined by $A\ m_r\ B$ for commuting A and B :

$$(12) \qquad QI_r(p, q)(x) = <A\ m_r\ B\ x, x>^{1/2}.$$

The equation (12) holds even if A does not commutes with B, by which various properties of the quadratic interpolation of positive linear forms are led via operator means

by Kubo and Ando [12]. For instance, the operator version of the Wigner-Yanase-Dyson-Lieb concavity (cf. [10, 16]) is derived from the concavity of operator means. As a matter of fact, if $A \geqslant \alpha A_1 + \beta A_2$ and $B \geqslant \alpha B_1 + \beta B_2$ for $\alpha, \beta > 0$ with $\alpha + \beta = 1$, then we have

$$\begin{aligned}A\ m_r\ B &\geqslant (\alpha A_1 + \beta A_2)\ m_r\ (\alpha B_1 + \beta B_2)\\ &\geqslant \alpha A_1\ m_r\ \alpha B_1 + \beta A_2\ m_r\ \beta B_2\\ &= \alpha(A_1\ m_r\ B_1) + \beta(A_2\ m_r\ B_2).\end{aligned}$$

On the other hand, the Uhlmann's relative entropy [16] is based on the interpolation theory of positive linear forms. The argument of positive sesquilinear forms is reduced to that of operators via the Pusz-Woronowicz theory [4, 16], see also [6] : For positive linear forms ϕ, ψ on a unital $*$-algebra A, put a sesquilinear form

$$(13) \qquad <x, y> = \phi(xy^*) + \psi(y^*x).$$

Let $x \to \tilde{x}$ be the usual map from A to H which is the Hilbert space with the inner product corresponding to (13). Then there exists derivatives A, B on H with

$$(14) \qquad <A\tilde{x}, \tilde{y}> = \phi(xy^*) \text{ and } <B\tilde{x}, \tilde{y}> = \psi(y^*x).$$

It follows from (12) that A and B are commuting positive contraction with $A + B = 1$.

Now, in this situation, the Uhlmann's relative entropy $S(\phi \mid \psi)_U$ is expressed by

$$(15) \qquad S(\phi \mid \psi)_U = -\liminf_{r \to +0} <\{(A\ m_r\ B - A)/r\}\tilde{1}, \tilde{1}>.$$

Considering that the Uhlmann's relative entropy is upper semicontinuous with respect to the term ψ, its operator version would be defined by

$$(16) \qquad S(A \mid B)_U = -\operatorname*{s\text{-}lim}_{\epsilon \to +0}\ \operatorname*{s\text{-}lim}_{r \to +0}\ (A\ m_r\ (B + \epsilon) - A)/r,$$

if the strong limit exists. Fortunately, two relative operator entropy coincide up to sign. In fact, for invertible B, we have

$$\begin{aligned}-S(A \mid B)_U &= \operatorname*{s\text{-}lim}_{r \to +0}\ (A\ m_r\ B - A)/r\\ &= \operatorname*{s\text{-}lim}_{r \to +0}\ B^{1/2}(A/B\ m_r\ 1 - A/B)B^{1/2}/r\\ &= \{\operatorname*{s\text{-}lim}_{r \to +0}\ ((A/B)^{1-r} - A/B)/r\}/B^{-1}\end{aligned}$$

$$= -(A/B)\log(A/B)/B^{-1} = S(A\,|\,B).$$

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