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MATHEMATICS

ON NORMED RINGS

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Definition 1. The set R of elements $x, y, \ldots$ is called a normed ring, if: a) it is a complete normed space with multiplication by complex numbers; b) together with the elements x and y exists their product xy possessing the usual algebraical properties and continuous at least with respect to one of its factors; c) it contains a unit element e.

It may be shown that in R exists a norm equivalent to a given one and possessing the following properties:

$$\|xy\| \leqslant \|x\| \cdot \|y\|, \quad \|e\| = 1.$$

In what follows we consider only commutative rings.

Definition 2. A set of elements $x, y, \ldots \subset R$ is called an ideal I, if it possesses the following properties: a) if $x \in I$, $y \in I$, then $\alpha x + \beta y \in I$, where α and β are arbitrary complex numbers; b) if $x \in I$ and z is an arbitrary element of the ring R, then $xz \in I$.

The whole ring and the nul ideal, i. e. the ideal consisting of only one element 0, are trivial ideals. Every ideal different from these two we shall call non-trivial.

Definition 3. A non-trivial ideal not contained in any other non-trivial ideal we call a maximal ideal.

Theorem 1. *Every maximal ideal is closed* (*i. e. the set of elements contained in this ideal is closed in* R).

Theorem 2. *Every non-trivial ideal is contained in a maximal ideal.*

The ring of residues R/I, where I is a closed ideal, may be again made a normed ring. As is known, the elements of R/I are classes X, Y, $\ldots$, consisting of such elements x', x'', $\ldots$ that $x' - x'' \in I$. The norm of the class X is defined in the following way: $\|X\| = \inf |x|$, where $x \in X$. It may be shown that in the ring of residues all axioms of a normed ring are satisfied.

Theorem 3. *The ring of residues to a maximal ideal is the corpus of complex numbers.*

Thus, to every element $x \in R$ and every maximal ideal M there corresponds a complex number $x(M)$, namely the number, which is correlated to x by the homomorphism $R \sim R/M$. Hence follows

Theorem 4. *To every element $x \in R$ there corresponds a function*

$x(M)$ defined on the set $\mathfrak{M}$ of all maximal ideals. To the sum of elements corresponds the sum of functions, to the product of elements — the product of functions.

The function $x(M)$ is bounded. In fact, $|x(M)| \leqslant \|x\|$. It may be shown that $\sup |x(M)| = \lim_{n\to\infty} \sqrt[n]{\|x^n\|}$. Hence we obtain

Theorem 5. *If in the ring R there are no elements different from zero, for which $\sqrt[n]{\|x^n\|} \to 0$, then our ring is isomorphic to the ring of functions defined on the set of maximal ideals.*

We shall show that these functions $x(M)$ are continuous. To this end we must topologize the set $\mathfrak{M}$ of maximal ideals. We shall define the topology by a system of neighbourhoods.

Definition 4. A set K of elements of the ring R is called a set of generators of the ring R, if the smallest closed ring containing K is the whole ring R.

Definition 5. By the neighbourhood of the maximal ideal M_0 we shall mean the set of maximal ideals satisfying the inequalities

$$|x_i(M) - x_i(M_0)| < \varepsilon, \quad (i = 1, 2, \ldots, n),$$

where ε and n are arbitrary and $x, \ldots, x_n$ are arbitrary elements of the set K of generators of the ring R.

The so obtained topology in $\mathfrak{M}$ does not depend on the choice of K. The set $\mathfrak{M}$ of maximal ideals in this topology proves to be a bicompact Hausdorff space and the functions $x(M)$ are continuous on it. As the result we obtain the following fundamental

Theorem 6. *Every abstract normed ring R may be homomorphically mapped in the ring of continuous functions defined on the Hausdorff bicompact space of maximal ideals of the ring R. For the isomorphism it is necessary and sufficient that the conditions of Theorem 5 should be satisfied.*

In some cases the topological properties of the set $\mathfrak{M}$ may be stated more precisely, namely:

1. If the ring R is separable, then the set $\mathfrak{M}$ is metrisable.
2. If the ring R is generated by a system of n generators, then the set $\mathfrak{M}$ is homeomorphic to a compact subspace of the n-dimensional complex space.

Theorem 7. *If in the ring R for each element x there is a complex-conjugated element, i. e. such an element y that $y(M) = \overline{x(M)}$, whatever be M from $\mathfrak{M}$, then each function continuous on the set $\mathfrak{M}$ is the limit of a uniformly convergent sequence of functions corresponding to the elements of the ring R*.*

Hence it follows that if in the ring R a norm is introduced in such a way that from the uniform convergence of the functions $x_n(M)$ follows the convergence of the norms of the elements x_n (for which it is, for instance, sufficient to demand that $\|x^2\| = \|x\|^2$), then the ring R consists of all functions continuous on $\mathfrak{M}$ (we assume that the conditions of theorem 7 are satisfied in the ring R).

Thus, in many cases the aggregate of functions $x(M)$ uniquely determines the ring. This follows from

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* This theorem was proved by the author together with G. Šilov. The proof will be given in their common paper.

T h e o r e m 8. *Let there be given two normed rings R_1 and R_2. Let the intersection of all maximal ideals in each of them be the nul ideal. Then from the algebraic isomorphism of the rings R_1 and R_2 follows their continuous isomorphism.*

The applications of the above results will be given elsewhere. A detailed exposition will be published in the «Recueil Mathématique de Moscou».

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