Locally bounded linear topological spaces

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A. Kolmogoroff* has shown that a linear topological space is a normed space if and only if it is locally convex** and locally bounded. In most of the existing literature on linear topological spaces, local convexity is usually assumed. The novelty of the present paper lies in the fact that although local boundedness is often assumed, it is never found necessary to require the space to be locally convex. Hence all the theorems of this paper apply to spaces more general than normed spaces (see the discussion after theorem 3 below), although local boundedness is a severe restriction.

Section 1 is devoted to postulates, definitions, and a few fundamental properties of linear topological spaces in general. The

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* See [7]. The numbers in brackets refer to the bibliography at the end of the paper.

** "locally convex" means that every neighborhood of x contains a convex neighborhood; "locally bounded" means that every neighborhood of x contains a bounded neighborhood. Locally bounded spaces were introduced in [4].

postulates are adapted from v. Neumann's in [2], but unlike the latter, they are equivalent to Kolmogoroff's (see [7] or [8]). The concept of boundedness is discussed in section 2, and four equivalent definitions are given

In section 3 it is shown how an "absolute value" may be defined in a locally bounded space. Since the space is not required to be locally convex, the absolute value does not satisfy the triangular inequality. This absolute value does satisfy Frechet's postulates for the "length" of a vector in a "family of abstract vectors" (see [10] pp. 125–6).

Linear funtions* are discussed in section 4. In sections 5 and 6 it is shown how the Riesz theory of completely continuous linear functional equations could be carried over to locally bounded spaces. Only a few of the proofs are indicated. since it is only necessary to show that Riesz's results do not depend on the triangular property of the norm.

In section 5 one of Riesz's theorems is shown to hold for *any* linear topological space. It would be interesting to know how of the Riesz theory could be carried over without requiring local boundedness. Another unsolved question is whether or not it is possible to define an absolute value in a locally bounded space which is not only upper semi-continuous, but continuous.

1. *Linear topological spaces in general.*

By a linear topological (l. t.) space, we mean a linear

* For a discussion of linear functions in spaces which are locally convex but not locally bounded see [11].

space* which has a Hausdorff topology**, with respect to which the fundamental operations $x+y$ and αx are continuous. The Hausdorff axioms are not the most convenient ones however, and we use instead the postulates*** given below, which we shall prove equivalent (see theorem 1).

Throughout the paper we use the following notations: L being a linear space, x_0 a point of L, S, T sets in L, α a real number and A a set of real numbers, x_0+S denotes the set of all x_0+y with $y \in S$; $S+T$ denotes the set of all $x+y$ with $x \in S$; $y \in T$; αS stands for the set of αx with $x \in S$; $S \cdot T$ is the intersection of S and T; AT denotes set of βx with $\beta \in A$, $x \in T$.

Let L be a linear space in which there exists a family U of "fundamental" sets U subject to the following postulates:

T 1: The intersection of all the sets $U \in U$ is the one element set consisting of the origin θ.

T 2: If $U \in U$, $V \in U$, there exists a set $W \in U$ with $W \subset U \cdot V$.

T 3: If $U \in U$ there exists a $V \in U$ such that $V+V \subset U$.

T 4: If $U \in U$ there exists a set $V \in U$ such that $\alpha V \subset U$ for all real α satisfying $-1 \leqq \alpha \leqq 1$.

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* See [1], p. 26 for the definition of a linera space.

** I. e. the space satisfies postulates (A), (B), (C), (5), pp. 228–229 of [6].

*** These are a slight generalization of those given by v. Neumann in 2.

T 5: For any given $x \in L$ and $U \in \mathcal{U}$ there exists a real number α such that $x \in \alpha U$.

Theorem 1: *If L is a linear space satisfying T 1 – T 5, then L is a linear topological space. Conversely in any linear topological space there is a family $\mathcal{U}$ of sets satisfying T 1 – T 5.*

Proof: For any set $S \subset L$, we denote by S_i (interior of S) the set of points x such that there exists $U \in \mathcal{U}$ with $x + U \subset S$. Then, following the proof of v. Neumann in [2] one can show that L is a Hausdorff space, with the sets $x + U_i$ $(U \in \mathcal{U})$, as neighborhoods of the point x, and that the operations αx and $x+y$ are continuous, so that L is a l. t. space.

Conversely, having given a l. t. space, let $\mathcal{U}$ be the family of all open sets which contain the origin θ. Now $\mathcal{U}$ is a complete neighborhood system of θ and from the Hausdorff postulates (cf. [6] postulates (A), (B), (C), (5) pp. 228–229) it is evident that postulates T 1 and T 2 are satisfied. Moreover, since $x+y$ and αx are continuous at the origin, it follows that T 3 and T 5 are satisfied. It remains only to prove that T 4 is satisfied. In order to do this we first prove the following lemma, which is of interest in itself.

Lemma 1: *Let L be a l.t. space, A a non-vacuus open set of real numbers and G an open set of L. If either A does not contain 0, or if G does contain the origin θ, then the set AG is open.*

Proof: First suppose that A does not contain 0. Then, since A and G are open, for any $\alpha \in A$ and any $x \in G$ there is a

$\delta < 0$ such that $\beta \epsilon A$ for $|\beta - \alpha| < \delta$, and a Hausdorff neighborhood $U(x)$ with $U(x) \subset G$. Now $\alpha \neq 0$, so γy is continuous at $\gamma = 1/\alpha$, $y = \alpha x$. Hence there exists a $U(\alpha x)$ such that $1/\beta U(\alpha x) \subset U(x)$ for sufficiently small $|\beta - \alpha|$. Hence for all β in a sufficiently small neighborhood $I(\alpha)$ of α we have $U(\alpha x) \subset I(\alpha) U(x) \subset AG$, which shows that AG is open.

Next suppose that $\theta \epsilon G$, and take any $\alpha \epsilon A$ and $x \epsilon G$. If $\alpha \neq 0$, the above proof applies. If $\alpha = 0$, $\alpha x = \theta$, and we must show that there exists a neighborhood $U(\theta) \subset AG$. Take $\beta \epsilon A$, $\beta \neq 0$. Since $\theta \epsilon G$, there is a neighborhood $V(\theta) \subset G$. From the continuity of αx, there exists a $U(\theta)$ with $1/\beta U(\theta) \subset V(\theta) \subset G$, or $U(\theta) \subset \beta V(\theta) \subset AG$. This completes the proof of the lemma.

To prove that T 4 is satisfied by the family U of open sets containing θ, observe that since αx is continuous, there is for any $U \epsilon U$ a $W \epsilon U$ and a real interval $I: -\delta \leqq \alpha \leqq S$ such that $IW \subset U$. Since $\theta \epsilon W$, lemma 1 applies, so that IW is open. Putting $V = IW$, we have $\alpha V \subset U$ for all α satisfying $-1 \leqq \alpha \leqq 1$.

q.e.d.

Having demonstrated theorem 1, we may regard T 1–T 5 as a set of postulates defining a l.t. space. Any family U' of sets satisfying postulates T 1 – T 5 will be said to be equivalent to U if for each $U \epsilon U$ there is a $U' \epsilon U'$ with $U' \subset U$, and if for each $V' \in U'$ there is a $V \in U$ with $V' \subset V$. Equivalent families define the same l.t. space, since they define the same points as limit points of a given set. We shall find the following lemma useful in a later paragraph.

LEMMA 2*: Any l.t. space is a regular Hausdorff space, i. e., any open neighborhood U of the origin θ contain a neighborhood V of θ whose closure $\bar{V}$ is contained in U.

In what follows we shall need to make use of the concept of relativization. Namely if M is any linear subset of a *l. t.* space L then for every $U \in$ U we can define a fundamental set $U_M = M \cdot L$, and the family U_M of such sets will satisfy postulates T1 – T5. Thus we may consider M as a *l t.* space by relativization. If G is an open relative to M, *i. e.* will be an open set of the *l. t.* space M.

2 *Boundedness*

Several equivalent definitions of boundedness in *l. t.* spaces have been given by various authors. For the sake of completeness they are included here.

(1) (Banach and Kolmogoroff) A set $S \subset L$ will be called bounded if for any sequence $x_\nu \in S$ and any real sequence α_ν converging to 0, the sequence $\alpha_\nu x_\nu$ converges to the zero element θ.

(2) (v. Neumann) A set $S \subset L$ will be called bounded if for any

* For the proof, see [7].

neighborhood U of the origin there is a number α such $S \subset \alpha U$.
(3) (Michal and Paxson)*. A set $S \subset L$ will be called bounded if for any $x_0 \in S$ there is a positive number S such that αx_0 is not in S for $|\alpha| > S$.

(4) A set $S \subset L$ will be called bounded if given $U \in U$ there is an integer ν such that $|\alpha| < 1/\nu$ implies $\alpha S \subset U$.

THEOREM 2: The above four definitions of boundedness are equivalent.

Proof: Michal and Paxon (see [3]) have shown that definition (3) is equivalent to (2), and it has been shown (see [4], theorem 1) that definition (4) is also equivalent to (2). It remains to show that definitions (1) and (2) are equivalent.* Let $S \subset L$ be bounded according to definition (1) and assume that S is not bounded according to definition (2). Then for some U and each α there is an $x \in S$ with $1/\alpha\, x$ is not in U. Let α_i be a sequence of number which approach ∞ with i. Then there is a sequence $x_i \in S$ such that $1/\alpha\, x$ is not in U. Hence $1/\alpha_i\, x_i$ does not converge to θ, although $1/\alpha_i \to 0$, which contradicts (1).

Conversely let S be a set of L bounded according to (2). Let U be any neighborhood of the origin and let α_i be a sequence of real numbers which approach 0. Let V be chosen in accordance

* The statement of this definition in [3] is incorrect.

* This equivalence was stated in Bull. Am Match. Soc., vol. 43 (1937), abstract N° 228. The proof is published here for the first time.

with postulat T 4, so that $\alpha V \subset U$ for $|\alpha| < 1$. By definition (2) there is a number β such that $S \subset \beta V$ or $1/\beta\, V \subset S$. Choose ν so large that $|\alpha_i \beta| \leqq 1$ for $i > \nu$. Then for $i > \nu$, and for any sequence $x_i \in S$ we have $\alpha_i x_i = \alpha_i \beta \left(\frac{1}{\beta}\right) x_i \in \alpha_i \beta \left(\frac{1}{\beta}\right) S \subset \alpha_i \beta V \subset U$. That is, $\alpha_i x_i$ converges to θ.

3. *Locally bounded spaces; absolute values*

DEFINITION: A *l. t.* space will be called *locally bounded* if there exists a fundamental set $U \in U$ which is bounded.

THEOREM 3: * *To each element x of a locally bounded l. t. space L it is possible to order a real number $|x|$, called the absolute value of x, with the following properties:*

(i) $|x| \geqq 0$; $|x| = 0$ implies $x = \theta$.

(ii) $|\alpha x| = |\alpha|\,|x|$ for any real α.

(iii) For every $n > 0$ there is a $\delta > 0$ such that $|x + y| < n$ whenever $|x| < S$ and $|y| < S$

(iv) The sets U': $|x| < \alpha$, $\alpha > 0$ form a family U' equivalent to the original family U of fundamental sets.

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* A somewhat less general theorem was proved in [4]. In the present paper the term "absolute value" is used instead of "pseudo-norm" to avoid confusion with v. Neuman's "pseudo-metric" introduced in [2].

(v) For every $x \in L$ and $n > 0$ there is a $\delta > 0$ such that $|y| - |x| < n$ for $|y - x| < \delta$.

Conversely, a linear space in which there is defined an absoluet value with the properties (i), (ii), (iii) is a locally bounded l. t. space.

'PROOF: Let $U \in \mathcal{U}$ be bounded. From postulate T 4 there exists a $V \in \mathcal{U}$ such that $\alpha V \subset U$ for $-1 \leqq \alpha \leqq 1$. Denote by I the open interval $-1 < \alpha < 1$ and put $W = IV_i$, whete V_i is the interior of V. Since V_i is open, W is also open by lemma 1. Clearly $W \subset U$ so that W is bounded. By construction, we have $\alpha W \subset W$ for $-1 < \alpha < 1$ Hence, since W is open, it is easily shown that $(-1) W = W$, so that $\alpha W \subset W$ for $-1 \leqq \alpha \leqq 1$.

Consider the family $\mathcal{U}'$ of sets ρW, where ρ takes on all positive values. This family is equivalent to the original family $\mathcal{U}$ of fundamental sets. For, let there be given any $V \in \mathcal{U}$. Then since W is bounded, there is a $\rho > 0$ such that $\rho W \subset V$. Conversely, since each ρW is an open set * containing θ, there exists a $V \in \mathcal{U}$ such that $V_i \subset \rho W$. By lemma 2, the space is regular, so there exists a $U \in \mathcal{U}$ with $\overline{U} \subset V_i$. Hence $U \subset \rho W$. It follows without difficulty that the family $\mathcal{U}'$ satisfies the postulates T 1 – T 5.

For any given $x \in L$, define $|x|$ as the greatest lower bound of positive ρ satisfying $x \in \rho W$. Since the family $\mathcal{U}': \{\rho W\}$

* This follows readily from the continuity of scalar multiplication and the fact that W is open.

satisfies postulate T 1, it is clear that $|x|$ satisfies property (i). Property (ii) follows follows from definition of $|x|$ and the fact that $\rho W = (-\rho) W$. To prove (iv) notice that if $x \epsilon W$, then $|x| < 1$, and conversely if $|x| < 1$ then $x \epsilon \alpha W$, for some α satifying $0 < \alpha < 1$, so that $x \epsilon W$. That is, the set $|x| < 1$ is identical with W. But, from (ii), this means that the set $|x| < \rho$ is identical with the set ρW, that the sets $|x| < \rho$ $\rho > 0$ form a family U' equivalent to U. Property (iii) now follows at once, since ths family U' is known to satisfy postulate T 3. In order to prove (v), let $x \epsilon L$ and $n > 0$ be given. For $\rho > |x|$ we have $x \epsilon \rho W$, so that $x \epsilon (|x| + n) W$. But since $(|x| + n) W$ is an open set, there is a $\delta > 0$ such that $y \epsilon (|x| + n) W$ for $y - x \epsilon S W$. That is $|y| < |x| + n$ for $|y - x| < S$.

The proof of the converse is left to the reader.

Corollary 1: *In a locally bounded space a set S is bounded if and only if the set of real numbers $|x|$, $x \epsilon S$, is bounded.*

Corolary 2: *If $\{x_n\}$ cnd $\{y_n\}$ are sequences in a locally bounded space L such that $|x_n|$ and $|y_n|$ are bounded, then so is $|x_n + y_n|$.*

To prove corollary 1, note that if S is bounded, $S \subset \alpha W$ for same $\alpha > 0$, i. e. $|x| < \alpha$ for $x \epsilon S$, conversely $|x| < \alpha$ for $x \epsilon S$ implies $S \subset \alpha W$. Corollary 2 follows from corollary 1 and the fact that if S_1, S_2 are bounded sets, so also is $S_1 + S_2$.

Remark: The above absolute value is more general than a norm, since it does not necessarily satisfy the triangular inequality, but only the weaker conditions (iii) and (v). Condition (iii) states that $|x + y|$ is continuous in x, y at the origin, while (v) states that $|y|$ is upper semi-continuos at $y = x$. We were able to prove (v) from the fact that W was open. Conversely if we are given a linear space subject to (i), (ii), (iii), (v) then the set $|x| < 1$ is open, as the reader may easily demonstrate by making use of (v).

Two different absolute values $|x|$ and $\|x\|$ for a given space will be called *equivalent* if there exist positive numbers μ and ν such that.

$$|x| \leqq \mu \|x\| \quad \text{and} \quad \|x\| \leqq \nu |x|$$

for all $x \in L$. It will be seen that these conditions are necessary and sufficient for the equivalence of the family U consisting of the sets $|x| < \rho$, $\rho > 0$ and the family U' consisting of the sets $\|x\| < \rho$, $\rho > 0$.

The question immediately arises whether or not it would always be possible in a locally bounded space to define an absolute value satifying the triangular inequality, i. e. a norm, which would be equivalent to the absolute value defined above. Tychonoff (see [8]) has answered this question in the negative by the example $H_{1/2}$, the space of all infinitely dimensional vectors $x = (x_1, x_2, x_3, \ldots)$ such that $\sum_{i=1}^{\infty} |x_i|^{1/2}$ converges, (each x_i being a real number). For, putting $|x| = \left\{ \sum_{i=1}^{\infty} |x_i|^{1/2} \right\}^2$ we obtain on absolute value satisfying (i), (ii), (iii) and (v), whence $H_{1/2}$

is a locally bounded *l. t.* space. However $H_{1/2}$ is not locally convex, as Tychonoff demonstrates, and hence it is *not* possible to define an equivalent norm, by Kolmogoroff's necessary and sufficient condition for normability (see introduction or [7]). Although the absolute value $|x|$ in $H_{1/2}$ does not satisfy the triangular inequality it does have the weaker property * that $|x + y| < 2(|x| + |y|)$

We have shown in theorem 3 that $|x|$ is an upper semi-continuous function of x. The question arises whether $|x|$ is always continuous. The following example shows that is not. Consider the space of complex numbers $z = \alpha + \beta i$. Put $|z| = \frac{1}{2}\sqrt{\alpha^2 + \beta^2}$ for $\beta \neq 0$ and $|z| = \sqrt{\alpha^2}$ for $\beta = 0$. Then $|z|$ is certainly not a continuous function of z although the set $|z| < 1$ satisfies all the properties that W was supposed to have in the proof of theorem 3, so that (i), (ii), (iii), (v) are satisfied by $|z|$. In this simple example, the ordinary modulus of z, which *is* continuous is equivalent to the above absolute value. The general question of whether in a given locally bounded space it is possible to define an absolute value with properties (i) – (v) and which is in addition a continuous function of z is still left unanswered.

4. *Linear functions in locally bounded spaces*

Let L and L_1 be locally bounded *l. t.* spaces. A function $F(z)$

* See [8]. This property is convenient in proving that $|x|$ satisfies (v).

on L to L_1 will be called linear if it is additive and continuous. In case L_1 is the space of real numbers, a function on L to L_1 is of course called a functional. The following properties of linear function may be demonstrated exactly as in the case of normed spaces (see for example [1]).

(I) A linear function $F(x)$ is homogeneous, i. e. $F(\alpha x) = \alpha F(x)$.

(II) Given a linear function $F(x)$ there exists a positive number m such that $| F(x) | \leqq m | x |$ for all $x \in L$. The least such number will be called the modulus of F and will be denoted by $| F |$ or $| F |_L$, if it is desired to emphasize the domain L of F. Conversely if an additive function F has a modulus, then F is continuous.

THEOREM 4: *Let* L *be the linear space of all linear functions on* L *to* L_1. *Then the modulus* $| F |$ *which is defined on* L *has the properties* (i), (ii), (iii), (v) *of the absolute value of theorem 3. Thus* L *is a locally bounded l. t. space.*

PROOF: Ad (i): If $| F | = 0$ then $| F(x) | \leqq 0 | x |$ for all $x \in L$

i. e $F(x) = \theta$ identically.

Ad (ii); Clear from property (I) of linear functions.

Ad (iii): Let x be arbitrary in the set $| x | = 1$. Then since L satisfies (iii) we have $| F(x) + G(x) | < n_1 < n$ for $| F(x) | < S$ and $| G(x) | < S$, *i. e.* $| F + G | < n$ for $| F | < S$, $| G | < S$.

Ad (v): Proff similar to proceding.

5. *Completely continuous linear functions.*

In this section we shall not require that the spaces be locally bounded. A linear function $f(x)$ on a l. t. space L to a l. t. space L_1 will be called *completely continuous* if there exists a fundamental set U whose f -transform $f(U)$ is compact.* A completely continuous linear function takes bounded sets into compact sets, for if S is bounded, then for some real α we have $S \subset \alpha U$, and $f(S) \subset f(\alpha U) = \alpha f(U)$. Now since $f(U)$ is compact, so are $\alpha f(U)$ and $f(S)$. Hence when L is a linear normed space, the above definition of complete continuity reduces to the usual one. (See [1] p. 96). The following theorem was proved by Riesz (see [5]. or [1], p. 152) for normed spaces.

Theorem 4. *If $y = f(x)$ is a completely continuous linear function on any l. t. space L to L, then the equation $x = f(x)$ has only a finite number of linearly independent solutions.*

Proof: Let M be the set of elements of our space L which satisfy the equation $x = f(x)$. Then M is a linear manifold, for if $x_1 \epsilon M$ and $x_2 \epsilon M$ then $a_1 x_1 + a_2 x_2$ belongs to M. By relativization, may be considered a l. t. space where fundamental sets are taken to be the sets $M \cdot U$, where U is a fundamental set in L (i. e. the sets $M \cdot U$ satisfy postulates T1 – T5). Now M is closed in L, since it is the set of all elements transformed into θ by the continuous function $x - f(x)$

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* A set $S \subset L$ will be called compact if every infinite subset of S has a limit point in L.

Next, let U_0 be the fundamental set such that $f(U_0)$ is compact. Any point of $M \cdot U_0$ is clearly in $f(U_0)$, i e $M \cdot U_0 \subset f(U_0)$ Hences ince $f(U_0)$ is compact, any infinite subset of $M \cdot U_0$ has a limit point in the space L. But since M is closed, this limit point is also in M. Therefore the set $M \cdot U_0$ is compact in the l. t. space M. Hence the interior of $M \cdot U_0$ is compact, so that the l. t. space M contains a compact, open, non-vacous set Hence by theorem 4 of [4], M is finite dimensional. That is there exist an integer n such that every element of M is expressible in the form $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$, where $x_1, x_2, \dots, x_n$ is a set of n fixed elements of M. q. e. d

6. *The Riesz tgeory of linear functional equations in locally bounded spaces.*

In order to further develop the Riesz theory of completely continuous linear transformations for l.t. spaces, it seems necessary to require local boundednes. If we do suppose that the space be locally bounded, so that an absolute value may be defined, the task becomes easy, since the proofs for the most part follow those of Riezs (see [5]). In a locally bounded l.t. space L, a set $S \subset L$ is bounded is bounded if and only if the set of real numbers $|x|$ with $x \in S$ is bounded (corollary 1 to theorem 3). Moreover in such a space completely continuous transformations are precisely those which take bounded sets into compact sets.

The following five lemmas on linear manifolds* are fundamental to Riesz's theory (see [5], pp. 75-79).

* By a linear manifold we shall mean a *closed* linear set.

LEMMA 3: *Let* L *be a loeally bounded l. t. spaco, and* M_1 *a linear manifold which is a proper subset of a linear manifold* $M_2 \subset L$. *Then there exists an element* $x_0 \in M_2$ *such that* $|x_0| = 1$ *and*

$$|x - x_0| \geqq 1/2 \quad \textit{for all} \quad x \in M.$$

LEMMA 4: *If* M *is a linear manifold of finite dimension in a locally bounded l. t. space and* y *an element not in* M *then there is an element* $x_0 \in M$ *such that*

$$|y - x| \geqq |y - x_0| \quad \textit{for all} \quad x \in M.$$

LEMMA 5: *If* M *is a finite dimensional linear manifold in any l. t. space, any bounded subset of* M *is compact.*

LEMMA 6: *If* M *is a linear manifold in any l.t. space, and if every bounded set of* M *ts compact, then* M *is finite dimensional.*

LEMMA 7: *Let* L *be a locally bounded l. t. space, and let* M_1 *and* M_2 *be two linear manifolds in* L *which have no common element other than* θ. *If at least one of* M_1 *and* M_2 *is finite dimensional then there exists a constant* μ *such that*

$$|x| + |y| \leqq \mu \, |x + y| \quad \textit{for all} \quad x \in M_1 \quad \textit{and all} \quad y \in M_2.$$

Lemma 3 may be proved exactly as in Riesz's paper. Lemma 5 is clear from the fact that a linear manifold in a l.t. space is itself a l.t. space by relativization, and the fact that any finite dimensional l.t. space is continuously isomorphic with a Euclidean space (see [8]).

On the basis of lemma 5, we can demonstrate lemma 4 as follows. Let β be the lower bound of $|y-x|$ for $x \varepsilon M$, and let x_n be a sequence in M such that $|y-x_n| \to \beta$. On writing $x_n = (x_n - y) + y$ it follows from corollary 2 of theorem 3 that the sequence $|x_n|$ is bounded, since $|x_n - y|$ and $|y|$ are bounded. Hence by lemma 5, $\{x_n\}$ is a compact set, and has a subsequence $\{x^{(n)}\}$ which converges to an element $x_0 \varepsilon M$. Clearly $|y-x_0| = \beta$, q.e.d.

For the proof of lemma 6 see theorem 4 of [4], regarding M as a l. t. space by relativization.

In order to prove lemma 7, let M_1 be the linear manifold with a finite dimension. If the lemma were not true, there would exist sequences $x_n \varepsilon M_1$ and $y_n \varepsilon M_2$ with $|x_n| + |y_n| > n\,|x_n + y_n|$ for all n. We may suppose without loss of generality that $|x_n| + |y_n| = 1$, for otherwise we could divide by $|x_n| + |y_n|$ and obtain this equality. Hence $\{x_n\}$ is a bounded sequence, and so it is compact by lemma 5, so that there is a subsequence $\{x^{(n)}\}$ of $\{x_n\}$ with $x^{(n)} \to x_0$. Now since $|x^{(n)} + y^{(n)}| < 1/n$, $x^{(n)} + y^{(n)} \to \theta$ so that $y^{(n)} \to -x_0$. By property (v) (Theorem 3) of the absolute value we have $|x^{(n)}| < |x_0| + 1/4$ and $|y^{(n)}| < |x_0| + 1/4$ for all sufficiently large n. Since $|x^{(n)}| + |y^{(n)}| = 1$ we find that

$$2\,|x_0| > 1/2 \qquad \text{or} \qquad |x_0| > 1/4$$

However, since M_1 and M_2 are both closed, and since x_0 is the limit of sequence $x^{(n)} \varepsilon M_1$ and a sequence $-y^{(n)} \varepsilon M_2$, we have $x_0 \varepsilon M_1 \cdot M_2$, whence $x_0 = \theta$, contrary to the inequality $|x_0| > 1/4$. This contradiction proves lemma 7.

From now on we shall understand that our space L is a locally bounded l. t. space, and $u(x)$ will always denote a completely continuous linear function on L to L.

THEOREM 5: *The transformation $t(x)=x-u(x)$ takes* L *into a closed set*

For the proof, see p. 151 of [1]. This proof is independent of the triangular property of the norm.

The theorems below may be proved on the basis of lemmas 3-7 and theorems 4 and 5 above, following Riesz. ([5], p. 80 et sequi)

THEOREM 6: *If the equation $x-u(x)$ has a solution for each $y \varepsilon$ L then this solution is unique.*

THEOREM 7: *The necessary and sufficient condition that the linear transformation $t(x)=x-u(x)$ have a unique linear inverse is hat the homogeneous equation $x-u(x)=\theta$ have a unique solution.*

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